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SERIES OF DIFFERENCES OF AVERAGES IN WEIGHTED SPACES

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Dedicated to Roberto Macías on his 60th birthday

1. Introduction

The purpose of this note is to announce some results which extend those in [6] about the series of differences of ergodic averages and the series of the differences of differentiation operators along lacunary sequences.

In order to state the problem, we consider a non atomic probability measure space $(X, \mathcal{F}, \mu)$ and an invertible ergodic measure preserving transformation $\tau : X \to X$. One of the aims in Ergodic Theory is the study of the behavior of the ergodic averages

$$R_n f(x) = \frac{1}{n} \sum_{j=0}^{n-1} f(\tau^j x),$$

where f is a measurable function. The basic results about convergence of the averages are von Neumann's Theorem and Birkhoff's Theorem (see [11]). These results assert that for all $f \in L^p(\mu)$, $1 \leq p < \infty$, the sequence $R_n f$ converges in $L^p(\mu)$-norm and a.e. Furthermore, since τ is ergodic, the limit is the constant function $\int_X f\, d\mu$. It is known that there is no rate of convergence (see [7], pages 14–15 or [11], pages 94–98). For instance, we have the following result due to Kakutani and Petersen (see [11], page 94).

Theorem 1.1. *Let $(X, \mathcal{F}, \mu)$ be a non atomic probability measure space and let $\tau : X \to X$ be an invertible ergodic measure preserving transformation. If $b_n \geq 0$ and $\sum_n b_n = \infty$ then there exists $f \in L^\infty(\mu)$ such that $\int_X f = 0$ and*

$$\sup_l \left| \sum_{n=1}^{l} b_n R_n f(x) \right| = \infty \quad \text{a.e..}$$

Therefore,

$$\sum_{n=1}^{\infty} b_n R_n f(x) \quad \text{diverges} \quad \text{a.e.}$$

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We observe that for that function we have that $R_n f(x) \to 0$. Consequently, $R_n f(x)$ is small but it is not small enough compared to b_n since the series $\sum_{n=1}^{\infty} b_n R_n f(x)$ diverges.

Now we notice that the study of the convergence of $R_n f$ is the same as the study of the convergence of the series $\sum_{k=2}^{\infty}(R_k f(x) - R_{k-1}f(x))$ since

$$\sum_{k=2}^{n}(R_k f(x) - R_{k-1}f(x)) = R_n f(x) - f(x).$$

Therefore, in order to obtain information about how the convergence of $R_n f$ occurs we may try to study the properties of convergence of the series

$$\sum_{k=2}^{\infty}(R_k f(x) - R_{k-1}f(x)).$$

We begin studying the absolute convergence of that series. The next example shows that there is not absolute convergence (it is adapted from the example in [6], pages 526–527).

Example 1.1. Let $X = [0,1)$, $\theta \in (0,1)$, θ irrational, $\tau(x) = x + \theta \pmod{1}$. If $f = \chi_{(0,1/2)}$ then the series $\sum_{n=2}^{\infty} |R_n f(x) - R_{n-1}f(x)|$ does not converge a.e.

Observe that

$$|R_n f(x) - R_{n-1}f(x)| = \frac{1}{n}\left|f(\tau^{n-1}x) - R_{n-1}f(x)\right|.$$

Since

$$\lim_{n\to\infty} R_n f(x) = \int_X f d\mu = 1/2$$

then there exists n_0 such that

$$\frac{1}{4} \le R_{n-1}f(x) \le \frac{3}{4}$$

for every $n \ge n_0$. Now observe that $f(\tau^{n-1}x)$ is 0 or 1. Then

$$\left|f(\tau^{n-1}x) - R_{n-1}f(x)\right| \ge \frac{1}{4}$$

for every $n \ge n_0$. Therefore

$$\sum_{n=n_0}^{\infty} |R_n f(x) - R_{n-1}f(x)| = \sum_{n=n_0}^{\infty} \frac{1}{n}\left|f(\tau^{n-1}x) - R_{n-1}f(x)\right| \ge \frac{1}{4}\sum_{n=n_0}^{\infty}\frac{1}{n} = \infty,$$

as we wished to prove.

Now we may wonder about the convergence of

$$\sum_{n=2}^{\infty} v_n(R_n f(x) - R_{n-1}f(x)), \quad \text{where} \quad \sup_n |v_n| < \infty.$$

It is worth noting that if the convergence is understood in the sense of $L^p(\mu)$ then our question is equivalent to ask about the unconditional convergence of the series

$$\sum_{k=2}^{\infty}(R_k f(x) - R_{k-1} f(x)).$$

The answer to this question is negative as the example in [6] (pages 526–527) shows.

Instead of working with the complete sequence of the natural numbers we can think of working with a subsequence. Given a lacunary sequence n_k, i.e., a sequence of natural numbers such that $n_k < n_{k+1}$ and $n_k/n_{k+1} > \rho > 1$ (let us say $n_k = 2^k$), we consider

$$A_k f(x) = \frac{1}{n_k} \sum_{j=0}^{n_k - 1} f(\tau^j x).$$

As before, it is clear that

$$\sum_{k=1}^{\infty}(A_k f(x) - A_{k-1} f(x))$$

converges. An example given by Akcoglu, Jones and Schwartz [1] shows that this series is not absolutely convergent in general. In this way, Jones and Rosenblatt [6] arrived to the problem of the study of the convergence of the series

$$\sum_{k=1}^{\infty} v_k (A_k f(x) - A_{k-1} f(x)), \qquad \sup_n |v_n| < \infty.$$

They proved the following:

- If $1 < p < \infty$ then the series converges a.e. and in the $L^p(\mu)$ norm for all $f \in L^p(\mu)$.
- If $f \in L^1(\mu)$ then the series converges a.e and in measure.

Jones and Rosenblatt obtained their results using transference methods (see [2]). That means that they start solving the problem in the case $X = \mathbb{Z}$, μ the counting measure and $\tau(x) = x + 1$. Observe that in this particular case the averages are

$$A_k f(x) = \frac{1}{n_k} \sum_{j=0}^{n_k - 1} f(x + j).$$

Actually, Jones and Rosenblatt prefer to work in the continuous case, i.e., in the real line, changing sums by integrals. In the continuous case we have that the averages are defined by

$$D_k f(x) = \frac{1}{\varepsilon_k} \int_0^{\varepsilon_k} f(x + t)\, dt$$

and ε_k is lacunary sequence of positive numbers. In what follows we make more precise statements for the problem in $\mathbb{R}$, establish some of the results in [6] and inform about the extensions obtained in [3].

2. The case of the real line

Let $\rho > 1$ and let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a ρ-lacunary sequence of positive numbers, that is,

$$\varepsilon_{k+1}/\varepsilon_k \geq \rho > 1 \quad \text{for all } k \in \mathbb{Z}.$$

That condition implies clearly that $\lim_{k\to-\infty} \varepsilon_k = 0$ and $\lim_{k\to\infty} \varepsilon_k = \infty$. Therefore if $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, and

$$D_k f(x) = \frac{1}{\varepsilon_k}\int_0^{\varepsilon_k} f(x+t)\,dt \tag{2.1}$$

then $\lim_{k\to-\infty} D_k f(x) = f(x)$ and $\lim_{k\to\infty} D_k f(x) = 0$ a.e. As in the ergodic case, in order to give some information about how the convergence occurs, we may consider the series

$$\sum_{k=-\infty}^{\infty} (D_k f(x) - D_{k-1} f(x))$$

which obviously converges a.e. by the above remark. Arguing as before, it is natural to ask about the convergence properties of

$$\sum_{k=-\infty}^{\infty} v_k (D_k f(x) - D_{k-1} f(x)), \tag{2.2}$$

where v_k is a bounded sequence of real or complex numbers. In other words, we wish to study the convergence of the partial sums

$$T_N f(x) = \sum_{k=N_1}^{N_2} v_k (D_k f(x) - D_{k-1} f(x)), \quad N = (N_1, N_2),$$

as $N_1 \to -\infty$ and $N_2 \to \infty$. We observe that T_N is a convolution operator since $T_N f(x) = K_N * f(x)$, where

$$K_N(x) = \sum_{k=N_1}^{N_2} v_k(\varphi_k(x) - \varphi_{k-1}(x)) \quad \text{and} \quad \varphi_k(x) = \frac{1}{\varepsilon_k}\chi_{(-\varepsilon_k,0)}(x)$$

In order to prove the a.e. convergence, we consider the maximal operator

$$T^* f(x) = \sup_N |T_N f(x)|.$$

We have studied the convergence of $T_N f$ and the boundedness of T^* in weighted L^p spaces:

$$L^p(w) = \{f : \|f\|_{L^p(w)} = \left(\int_{\mathbb{R}} |f|^p w\right)^{1/p} < \infty\}.$$

and not only in the case of the Lebesgue measure. That implies that our strategy would be different than the one by Jones and Rossenblatt. In order to explain some of the differences, we shall sketch briefly the argument in [6].

First of all, Jones and Rosenblatt worked with the case $\varepsilon_k = 2^k$. Then they write

$$\begin{aligned} v_k(D_k f(x) - D_{k-1} f(x)) &= v_k(D_k f(x) - E_k f(x)) \\ &+ v_k(E_k f(x) - E_{k-1} f(x)) \\ &+ v_k(E_{k-1} f(x) - D_{k-1} f(x)) \end{aligned}$$

where

$$E_k f(x) = \sum_{|I|=2^k : I \text{ dyadic}} \left(\frac{1}{|I|} \int_I f \right) \chi_I(x).$$

The behavior of the series associated to the second term is well known [14], while the series associated to the other terms are essentially the same. Therefore, the problem is reduced to study the series

$$\sum_k v_k(D_k f(x) - E_k f(x)).$$

Clearly, since E_k is not a convolution operator we have that the partial sums

$$\tilde{T}_N f(x) = \sum_{k=N_1}^{N_2} v_k(D_k f(x) - E_k f(x))$$

are not convolution operators. In order to study the boundedness of the maximal operator

$$\tilde{T}^* f(x) = \sup_N |\tilde{T}_N f(x)|$$

Jones and Rosenblatt obtain the uniform boundedness of the operators $\tilde{T}_N f(x)$. They establish the following steps:

1. The operators $\tilde{T}_N f(x)$ are uniformly bounded in $L^2(dx)$. Since $\tilde{T}_N f(x)$ is not a convolution operator, they do not use the Fourier transform. They apply Cotlar's Lemma and a lemma due to Carbery [4].
2. The operators $\tilde{T}_N f(x)$ are uniformly of weak type $(1,1)$. That is done using standard technics of the theory of singular integrals.
3. By interpolation, it follows that the operators $\tilde{T}_N f(x)$ are uniformly bounded in $L^p(dx)$, $1 < p < \infty$.
4. Finally (we do not go into the details), they study the maximal operator $\tilde{T}^* f(x) = \sup_N |\tilde{T}_N f(x)|$ controlling it by the Hardy–Littlewood maximal operator
$$Mf(x) = \sup_{\eta,\varepsilon>0} \frac{1}{\eta+\varepsilon} \int_{-\eta}^{\varepsilon} |f(x+t)|\, dt.$$
In this way, they prove that $\tilde{T}^* f$ is of weak type $(1,1)$ and of strong type (p,p), $1 < p < \infty$.

3. Our approach: the weighted case

In order to study the problem in weighted spaces, we have to change the approach. We notice that the modified operators $\tilde{T}_N$ and the partial sums

$$S_N f(x) = \sum_{k=N_1}^{N_2} v_k(E_k f(x) - E_{k-1} f(x))$$

are two-sided operators in the sense that for fixed x the value of $\tilde{T}_N f(x)$ depend on the values of $f(y)$ for $y < x$ and $y > x$. Consequently, the maximal operator that controls $\tilde{T}_N$ and S_N is the two-sided Hardy-Littlewood maximal operator M. In this way, if we follow the approach in [6] to study the good weights for T_N, we are led to consider Muckenhoupt A_p weights, i.e.,the good weights for the two-sided Hardy-Littlewood maximal operator M. However, if we look at the original operator T_N we realize that they are one-sided operators in the sense that for fixed x the values of the series depend only on the values of $f(y)$ for $y > x$. Therefore, the one-sided Hardy-Littlewood maximal operator

$$M^+ f(x) = C \sup_{h>0} \frac{1}{h} \int_0^h |f(x+t)|\, dt$$

is the natural candidate to control the operators T_N and the natural weights for our problem are the good weights for M^+ which constitute a class of weights wider than the Muckenhoupt A_p weights. Following these remarks, our idea in [3] is to study directly the convergence of $T_N f$ without going through the conditional expectations. In what follows, we give an idea about the approach in [3].

As we indicated above, instead of studying the modified operators $\tilde{T}_N$, we go directly to the operators $T_N = K_N * f$. We start obtaining the uniform boundedness of the operators T_N in the unweighted case, giving a different proof which is independent of the results in [6]. An easy computation shows that

$$\sup_N |\hat{K}_N(x)| \le C,$$

where $\hat{K}_N$ stands for the Fourier transform of the kernel. It follows that the operators T_N are uniformly bounded in $L^2(dx)$, i.e.,

$$\sup_N \|T_N f\|_{L^2(dx)} \le C \|f\|_{L^2(dx)}.$$

Another property, less evident, is the condition D_r, $1 \le r < \infty$: If $1 \le r < \infty$ an $\varepsilon_{i-1} < x \le \varepsilon_i$ then

$$\sum_{j=i+\alpha}^{\infty} \varepsilon_j^{1/r'} \left(\int_{\varepsilon_j}^{\varepsilon_{j+1}} |K_N(x-y) - K_N(-y)|^r \, dy \right)^{1/r} < \infty,$$

where α is a natural number depending on ρ. It is easy to see that if $\varepsilon_{i-1} < |x| < \varepsilon_i$ then there exists C_ρ such that

$$\{y : |y| > C_\rho |x|\} \subset \cup_{j=i+\alpha}^{\infty} (\varepsilon_j, \varepsilon_{j+1}).$$

That relation and condition D_1 imply Hörmander's condition:

$$\int_{\{y:|y|>C_\rho|x|\}} |K_N(x-y) - K_N(-y)|\, dy \le C.$$

Hörmander's condition and the uniform boundednes of the Fourier transform give (see [5])

$$\sup_N |\{x : |T_N f(x)| > \lambda\}| \le \frac{C}{\lambda} \int |f|$$

and

$$\sup_N \int |T_N f|^p \le C_p \int |f|^p, \quad 1 < p < \infty.$$

Our next step is to study the uniform boundedness of T_N in weighted spaces. In order to do that, we use the one-sided sharp maximal function defined by

$$f^{+,\#}(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ dy,$$

where $z^+ = 0$ if $z < 0$ and $z^+ = z$ if $z \ge 0$. Clearly, $f^{+,\#}$ is dominated by the one-sided Hardy-Littlewood maximal function. More precisely:

$$f^{+,\#}(x) \le CM^+f(x) = C \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|.$$

Notice that the opposite inequality is not true since for f increasing we have that $f^{+,\#}(x) = 0$. The sharp function $f^{+,\#}$ is related to the good weights for M^+. Let us recall the characterizations of the good weights for M^+ [13].

- The operator M^+ satisfies the weak type $(1,1)$ inequality

$$\int_{\{x: M^+f(x) > \lambda\}} w \le \frac{C}{\lambda} \int |f| w$$

if and only if $w \in A_1^+$, i.e., there exists C such that $M^- w(x) \le C w(x)$ a.e.

- The operator M^+ satisfies the strong type (p,p) inequality, $1 < p < \infty$,

$$\int |M^+ f|^p w \le C \int |f|^p w$$

if and only if $w \in A_p^+$, i.e., there exists C such that

$$A_p^+ : \quad \left(\int_a^b w \right)^{\frac{1}{p}} \left(\int_b^c w^{1-p'} \right)^{\frac{1}{p'}} \le C(c-a), \quad (p + p' = pp')$$

for all $a < b < c$.

As in the case of the usual sharp maximal function, it can be shown [10] that if $w \in \cup_p A_p^+$ then

$$\int |M^+ f|^r w \le C \int |f^{+,\#}|^r w, \quad 0 < r < \infty,$$

under the assumption that the left hand side is finite.

The uniform boundednes of T_N and condition D_r give the following lemma.

Lemma 3.1. *Let $s > 1$. There exists C such that*

$$|T_N f|^{+,\#}(x) \le C \left(M^+ |f|^s \right)^{1/s}(x).$$

This lemma and the above relation between $M^+ f$ and $f^{+,\#}$ allows to get the uniform boundedness of T_N in weighted spaces.

Theorem 3.2. *Let $1 < p < \infty$ and $w \in A_p^+$. There exists C such that*

$$\int |T_N f|^p w \leq C \int |f|^p w$$

Proof. We sketch the proof of the lemma. Since $w \in A_p^+$, there exists $s > 1$ such that $w \in A_{p/s}^+$ [13]. Therefore

$$\int |T_N f|^p w \leq \int |M^+(T_N f)|^p w \leq C \int (|T_N f|^{+,\#})^p w.$$

By the lemma, the last term is dominated by

$$C \int (M^+|f|^s)^{p/s} w.$$

The proof finishes using that $w \in A_{p/s}^+$ and the characterization of the strong type inequality for M^+. □

Remark 3.3. *We notice that if we use the classical sharp maximal function then we obtain the last result only for weights in the Muckenhoupt classes which are a subclass of A_p^+ (we remind that the increasing weights belong to the A_p^+ classes).*

In order to get the boundednes of the maximal operator, the key result is the following inequality: If

$$T_M^* f(x) = \sup_{|N_1|,|N_2| \leq M} |T_{N_1,N_2} f(x)|$$

then for every $s \in (1,\infty)$ there exists a constant C such that

$$T_M^* f(x) \leq C \left[M^+(|T_{-M,M} f|)(x) + \left(M^+|f|^s\right)^{1/s}(x) \right].$$

Combining this inequality and the uniform boundedness of the operators T_N in weighted spaces, we easily obtain the following theorem.

Theorem 3.4. *If $1 < p < \infty$ and $w \in A_p^+$ then there exists C such that*

$$\int_{\mathbb{R}} |T^* f|^p w \leq C \int_{\mathbb{R}} |f|^p w, \quad f \in L^p(w),$$

and $T_N f$ converges a.e. and in $L^p(w)$ for all $f \in L^p(w)$.

The result about the convergence follows from the strong type inequality for for T^* and the convergence for the functions in the Schwartz class.

If $p = 1$ we obtain the following result.

Theorem 3.5. *If $w \in A_1^+$ then*

$$\int_{\{x: T^* f(x) > \lambda\}} \leq \frac{C}{\lambda} \int_{\mathbb{R}} |f| w$$

and $T_N f$ converges a.e and in measure for all $f \in L^1(w)$.

The proof of this theorem is unified in the sense that we do not need to do first the case of the Lebesgue measure ($w = 1$).

In order to finish this section we point out that we are able to obtain the behavior in BMO. More precisely, if $f \in BMO$ then there exists C such that

$$\sup_N \|T_N f\|_{BMO} \leq C\|f\|_{BMO}$$

and $T_N f$ converges in the weak $*$ topology of BMO.

4. Final remark

Using transference arguments as in [9], the weighted results in the real line can be used to obtain convergence results in Ergodic Theory. In what follows, we state one of them.

Assume that $\{T_t : t \in \mathbb{R}\}$ is a strongly continuous group of positive linear operators in $L^p(d\mu)$, $1 < p < \infty$. For $\varepsilon > 0$ the average $A_\varepsilon f$ is defined by

$$A_\varepsilon f(x) = \frac{1}{\varepsilon}\int_0^\varepsilon T_t f(x)\,dt.$$

Let $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence and let us consider the operators

$$R_N f(x) = \sum_{k=N_1}^{N_2} v_k(A_{\varepsilon_k} f(x) - A_{\varepsilon_{k-1}} f(x))$$

where v_k is a bounded sequence. Under these assumptions we can prove the following theorem.

Theorem 4.1. *Assume that the averages are uniformly bounded in $L^p(d\mu)$, i.e.,*

$$\sup_{\varepsilon>0} \|A_\varepsilon f\|_p \leq C\|f\|_p.$$

Then $R_N f$ converges a.e. and in $L^p(d\mu)$ for all $f \in L^p(d\mu)$.

The details will appear in a forthcoming paper.

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