REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 45, Número 2, 2004, Páginas 55–67

COMPLEX, SYMPLECTIC AND KÄHLER STRUCTURES ON FOUR DIMENSIONAL LIE GROUPS

GABRIELA OVANDO

ABSTRACT. In this paper we deal with left invariant complex and symplectic structures on simply connected four dimensional solvable real Lie groups. We parametrize such structures and we make use of this information to determine all left invariant Kähler structures. Finally, as an appendix we present explicitly the authomorphisms group of Lie algebras admitting complex structures.

1. INTRODUCTION

The study of complex and symplectic manifolds has attracted the attention of many authors interested in different fields in mathematics and physics since complex and symplectic structures have proved to be an important tool in the description and geometrization of several phenomena. A complex manifold posseses a complex structure on the underlying real manifold M, that is a differential tensor J defined on TM such that: (1) $J^2 = -\mathrm{Id}$ and (2) $N_J(X,Y) = 0$ for all $X, Y \in \Xi(M)$, the so-called Nijenhuis condition. It is also known that the existence of such a tensor J on a real manifold M, which satisfies condition (1) and (2) allows to construct complex coordinates on M.

If the manifold is homogeneous, the list of articles related with complex and/or symplectic questions is quite large. Existence conditions of symplectic forms on homogeneous spaces were found by Chu [Ch] in the beginning 1970s. On the other hand homogeneous complex manifolds for example were classified in different stages [O-R] [Wi].

From the point of view of Lie groups the existence problem of left invariant complex structures was treated by Samelson [Sm] and Wang [Wa] in the compact case and by Morimoto [Mo] in the non compact case. Detailed classifications of left invariant complex structures when the Lie group is $\mathrm{GL}(2,\mathbb{R})$, $U(2)$, $\mathrm{SL}(3,\mathbb{R})$ were given by Sasaki [Ss1], [Ss2], and when G is a simply connected solvable Lie algebra of dimension four by Snow [Sn] and the author [O1]. The last two works had the following goals: the classification of left invariant complex structures and the determination of the underlying complex manifold.

The addition of an extra assumption such as the compatibility condition between a complex structure and a symplectic structure introduces a new tool to be handled on the investigation of these objects.

(2000) Mathematics Subject Classification: 53C15, 53C55, 53D05, 22E25, 17B56
Acknowledgements. The author was partially supported by CONICET, FONCyT and SECYT-UNC (Argentina).

The purpose of this article is to parametrize complex, symplectic and Kähler structures in the four dimensional case. This results will be useful in the future to solve other problems such as the description of the moduli spaces of these structures, or the construction of manifolds with determined properties. In fact four dimensional solvable Lie groups provide an important source of applications in geometry. Invariant structures on the group, for instance special metrics [Al], [An] [B2], [D-S], [F], [J], complex and/or Kähler structures [A-C-F-M], [A-F-G-M], [F-G-G] [F-G], hypercomplex structures [B1], can be read off in $\mathbb{R}^4$, the universal covering group.

Our proofs are based in a case by case study, which make use of the classification of solvable real Lie algebras in dimension four. In sections III and IV by using explicit computations we describe the general form of complex and symplectic structures. As we said the classification of complex structures was given in [Sn] and [O1]. However in these papers neither parametrizations of complex subalgebras nor authomorphisms were given in all cases. Four dimensional symplectic Lie algebras were found in [M-R], however no symplectic structure was shown explicitely on any case. Thus our paper try to complete in some sense these works.

As a corollary we get exact symplectic structures in dimension four, which were also found by Campoamor [Ca] who related the existence of these kind of structures with the existence of invariants. In the last section we search for Kähler structures on four dimensional Lie algebras, that is compatible pairs (J, ω) where J is a complex structure and ω a symplectic structure.

Our study provide many examples of Lie algebras with different structures. In particular we get many Lie algebras with both complex and symplectic structures but no compatible pair (J, ω).

As an appendix we compute the authomorphisms group of Lie algebras admitting complex structures, which acts on the set of complex structures on each case.

The author thanks I. Dotti for useful comments and general supervision and specially M. Fernández for her valuable contributions and for suggesting the subject of this article.

If G is a Lie group then its Lie algebra will be denoted with a greek letter $\mathfrak{g}$ and identified as usual with the left invariant vector fields of G. Throughout this paper we assume that Lie groups are simply connected and real.

2. Four dimensional Lie algebras

Since our classifications follow a case by case study, we need the classification of four dimensional solvable Lie algebras. Hence in this section all four dimensional solvable Lie algebras are exposed (see for example [A-B-D-O] for a proof). Notations used in all of this paper are compatible with the exposed in the following paragraph.

Proposition 2.1. *Let $\mathfrak{g}$ be a solvable four dimensional real Lie algebra. Then if $\mathfrak{g} = span\{e_1,\, e_2, e_3, e_4\}$ is not abelian, it is equivalent to one and only one of the Lie algebras listed below:*

$\mathfrak{rh}_3$:	$[e_1, e_2] = e_3$
$\mathfrak{rr}_3$:	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3$
$\mathfrak{rr}_{3,\lambda}$:	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3 \qquad \lambda \in [-1, 1]$
$\mathfrak{rr}'_{3,\gamma}$:	$[e_1, e_2] = \gamma e_2 - e_3, [e_1, e_3] = e_2 + \gamma e_3 \qquad \gamma \geq 0$
$\mathfrak{r}_2\mathfrak{r}_2$:	$[e_1, e_2] = e_2,\ [e_3, e_4] = e_4$
$\mathfrak{r}'_2$:	$[e_1, e_3] = e_3,\ [e_1, e_4] = e_4,\ [e_2, e_3] = e_4,\ [e_2, e_4] = -e_3$
$\mathfrak{n}_4$:	$[e_4, e_1] = e_2,\ [e_4, e_2] = e_3$
$\mathfrak{r}_4$:	$[e_4, e_1] = e_1,\ [e_4, e_2] = e_1 + e_2, [e_4, e_3] = e_2 + e_3$
$\mathfrak{r}_{4,\mu}$:	$[e_4, e_1] = e_1,\ [e_4, e_2] = \mu e_2,\ [e_4, e_3] = e_2 + \mu e_3 \qquad \mu \in \mathbb{R}$
$\mathfrak{r}_{4,\alpha,\beta}$:	$[e_4, e_1] = e_1,\ [e_4, e_2] = \alpha e_2,\ [e_4, e_3] = \beta e_3,$
	with $\ -1 < \alpha \leq \beta \leq 1,\ \alpha\beta \neq 0$, or $\ -1 = \alpha \leq \beta \leq 0$
$\mathfrak{r}'_{4,\gamma,\delta}$:	$[e_4, e_1] = e_1,\ [e_4, e_2] = \gamma e_2 - \delta e_3,\ [e_4, e_3] = \delta e_2 + \gamma e_3 \quad \gamma \in \mathbb{R}, \delta > 0$
$\mathfrak{d}_4$:	$[e_1, e_2] = e_3,\ [e_4, e_1] = e_1,\ [e_4, e_2] = -e_2$
$\mathfrak{d}_{4,\lambda}$:	$[e_1, e_2] = e_3,\ [e_4, e_3] = e_3,\ [e_4, e_1] = \lambda e_1,\ [e_4, e_2] = (1 - \lambda)e_2 \quad \lambda \geq \frac{1}{2}$
$\mathfrak{d}'_{4,\delta}$:	$[e_1, e_2] = e_3,\ [e_4, e_1] = \frac{\delta}{2}e_1 - e_2, [e_4, e_3] = \delta e_3,\ [e_4, e_2] = e_1 + \frac{\delta}{2}e_2 \quad \delta \geq 0$
$\mathfrak{h}_4$	$[e_1, e_2] = e_3,\ [e_4, e_3] = e_3,\ [e_4, e_1] = \frac{1}{2}e_1,\ [e_4, e_2] = e_1 + \frac{1}{2}e_2$

Remark 2.2. Observe that $\mathfrak{r}_2\mathfrak{r}_2$ is the Lie algebra $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, where $\mathfrak{aff}(\mathbb{R})$ is the Lie algebra of the Lie group of affine motions of $\mathbb{R}$, $\mathfrak{r}'_2$ is the real Lie algebra underlying on the complex Lie algebra $\mathfrak{aff}(\mathbb{C})$, $\mathfrak{rr}_{3,-1}$ is the trivial extension of $\mathfrak{e}(1,1)$, the Lie algebra corresponding to the Lie group of rigid motions of the Minkowski 2-space; $\mathfrak{r}'_{3,0}$ is the trivial extension of $\mathfrak{e}(2)$, the Lie algebra of the Lie group of rigid motions of $\mathbb{R}^2$; $\mathfrak{rh}_3$ is the trivial extension of the three-dimensional Heisenberg Lie algebra denoted by $\mathfrak{h}_3$.

A Lie algebra is called *unimodular* if $\mathrm{tr}(\mathrm{ad}_x)$=0 for all $x \in \mathfrak{g}$, where tr denotes the trace of the map. The unimodular four-dimensional solvable Lie algebras algebras are: $\mathbb{R}^4$, $\mathfrak{rh}_3$, $\mathfrak{rr}_{3,-1}$, $\mathfrak{rr}'_{3,0}$, $\mathfrak{n}_4$, $\mathfrak{r}_{4,-1/2}$, $\mathfrak{r}_{4,\mu,-1-\mu}$ $(-1 < \mu \leq -1/2)$, $\mathfrak{r}'_{4,\mu,-\mu/2}$, $\mathfrak{d}_4$, $\mathfrak{d}'_{4,0}$.

Recall that a solvable Lie algebra is *completely solvable* when ad_x has real eigenvalues for all $x \in \mathfrak{g}$.

Remark 2.3. For an explanation concerning the Lie groups which admit a cocompact quotient, see for example the work of Oprea and Tralle [O-T]. In particular if G admits a discrete subgroup Γ with compact quotient, then the corresponding Lie algebra is unimodular [Mi].

3. Complex structures

In this section we recall basic definitions related to complex structures and the classification of these structures on four dimensional solvable real Lie algebras. Furthermore we parametrize any complex structure on a four dimensional solvable Lie algebra.

An *invariant complex structure* on a real Lie group G is a complex structure on the underlying manifold such that left multiplication ℓ_g, $g \in G$ (but not necessarily right multiplication) by elements of the group are holomorphic with respect to this structure. Because of the Newlander-Niremberg theorem [N-N] we have the following equivalent definition:

Definition 3.1. An invariant complex structure on a real Lie group G is an endomorphism J of $\mathfrak{g}$ the Lie algebra of G, such that:
(i) $J^2 = -Id$;
(ii) $0 = N_J(X,Y)$ where $N_J(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]$ $\forall X,Y \in \mathfrak{g}$

Condition ii) is called the *integrability condition* of J, that is $N_J(X,Y) = 0$ for all $X,Y \in \mathfrak{g}$.

By extending J in a natural way to $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, the complexification of the real Lie algebra $\mathfrak{g}$, it is possible to have an equivalent condition to that given in Definition 3.1. If we denote by σ the conjugation in $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$, that is, $\sigma(X+iY) = X - iY$, $X,Y \in \mathfrak{g}$, then a real Lie algebra $\mathfrak{g}$ has a left invariant complex structure if and only if $\mathfrak{g}^{\mathbb{C}}$ admits a decomposition as a direct sum of vector spaces:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{q} \oplus \sigma\mathfrak{q} \tag{1}$$

where $\mathfrak{q}$ is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$. In fact, condition (i) in Definition 3.1 gives a decomposition of $\mathfrak{g}^{\mathbb{C}}$ into a direct sum of subspaces (the eigenspaces of J). Condition (ii) is equivalent to the fact that these subspaces are subalgebras. Conversely, given a complex subalgebra $\mathfrak{q}$ satisfying (1), let J be the almost complex structure defined on $\mathfrak{g}^{\mathbb{C}}$ by $JX = -iX$, $J\sigma X = i\sigma X$, for $X \in \mathfrak{q}$. Since $J\sigma = \sigma J$, it is possible to define J on $\mathfrak{g}$. Using the fact that $\mathfrak{q}$ is a subalgebra, it is not hard to see that J is integrable.

Thus there exists a one to one correspondence between left invariant complex structures J and subalgebras $\mathfrak{q}$ satisfying (1), which are called *(invariant) complex subalgebras.*

Two invariant complex structures J_1 and J_2 on a real Lie group are said to be *equivalent* if there exists $x \in \mathrm{Aut}(\mathfrak{g})$ such that $xJ_1 = J_2x$. Thus $x : (\mathfrak{g}, J_1) \to (\mathfrak{g}, J_2)$ is a *a complex isomophism.* In terms of invariant complex subalgebras, a complex isomorphism is an authomorphism $y \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ such that $y\sigma = \sigma y$ and $y\mathfrak{q}_1 = \mathfrak{q}_2$, where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are complex subalgebras.

In a more general setting one defines the equivalence relation of Lie algebras admitting complex structure, say an isomorphism $\alpha : (\mathfrak{g}_1, J_1) \to (\mathfrak{g}_2, J_2)$ is complex if $\alpha \circ J_1 = J_2 \circ \alpha$.

Invariant complex structures were classified by J. Snow [Sn] and G. Ovando [O1] in the four dimensional solvable case. In the respective papers we find only particular examples in which all complex structure were parametrized. In this section we present a parametrization of all complex structures on any four dimensional solvable real Lie algebra, which could be used in the future for a study of the correponding moduli space. We also compute the respective authomorphisms that can be found in the Appendix.

Proposition 3.2. *Let G be a simply connected four dimensional solvable real Lie group. Then in the notation of Proposition 2.1, the following table shows the non-equivalent classes of complex subalgebras $\mathfrak{q}$ (*[Sn], [O1]*),and the general form of any such complex subalgebra, denoted by $\mathfrak{Q}$, only in those cases when they exist:*

Case	$\mathfrak{q}$	$\mathfrak{Q}$
$\mathfrak{rh}_3$:	$\langle e_1+ie_2, e_3+ie_4\rangle$	$\langle e_1+b_1e_2+d_1e_4, e_3+d_2e_4\rangle$, $\operatorname{Im} b_1 \operatorname{Im} d_2 \neq 0$
$\mathfrak{rr}_{3,0}$:	$\langle e_1+ie_2, e_3+ie_4\rangle$	$\langle e_1+b_1e_2+d_1e_4, e_3+d_2e_4\rangle$ $\operatorname{Im} b_1 \operatorname{Im} d_2 \neq 0$
$\mathfrak{rr}_{3,1}$:	$\langle e_1+ie_4, e_3+ie_2\rangle$	$\langle e_1+b_1e_2+d_1e_4, b_2e_2+e_3\rangle$ $\operatorname{Im} b_2 \operatorname{Im} d_1 \neq 0$
$\mathfrak{rr}'_{3,0}$: $\mathfrak{rr}'_{3,\gamma}$: $\gamma \neq 0$	$\langle e_1+ie_4, e_2+ie_3\rangle$ $\left\{\begin{array}{l}\langle e_1+ie_4, e_2-(2\gamma+i)e_3\rangle,\\ \langle e_1+ie_4, e_2-(2\gamma-i)e_3\rangle\end{array}\right.$	$\langle e_1+c_1e_3+d_1e_4, e_2-(2\gamma+\varepsilon i)e_3\rangle$ $\varepsilon=\pm1$, $\operatorname{Im} d_1 \neq 0$
$\mathfrak{r}_2\mathfrak{r}_2$:	$\langle e_1+ie_2, e_3+ie_4\rangle$	$\langle e_1+b_1e_2+d_1e_4, e_3+d_2e_4\rangle$ $\operatorname{Im} b_1 \operatorname{Im} d_2 \neq 0$
$\mathfrak{r}'_2$	$\langle e_1+ie_3, e_2+ie_4\rangle$ $\langle e_1+b_1e_2, e_3+ie_4\rangle$, $\operatorname{Im} b_1 \neq 0$	$\langle e_1+c_1e_3+d_1e_4, e_2-d_1e_3+c_1e_4\rangle$ $(\operatorname{Im} c_1)^2+(\operatorname{Im} d_1)^2 \neq 0$ $\langle e_1+\varepsilon ie_2+d_1e_4, e_3-\varepsilon ie_4\rangle$, $\varepsilon=\pm1$ $\langle e_1+b_1e_2+d_1e_4, e_3+\varepsilon ie_4\rangle$ $\varepsilon=\pm1$, $\operatorname{Im} b_1 \neq 0$
$\mathfrak{r}_{4,1}$:	$\langle e_4+ie_3, e_1+ie_2\rangle$	$\langle e_4+b_1e_2+c_1e_3, e_1+b_2e_2\rangle$ $\operatorname{Im} c_1 \operatorname{Im} b_2 \neq 0$
$\mathfrak{r}_{4,\alpha,1}$: $\alpha \neq 1$	$\langle e_4+ie_2, e_1+ie_3\rangle$	$\langle e_4+b_1e_2+c_1e_3, e_1+c_2e_3\rangle$ $\operatorname{Im} b_1 \operatorname{Im} c_2 \neq 0$
$\mathfrak{r}_{4,\alpha,\alpha}$: $\alpha \neq 0,1$	$\langle e_4+ie_1, e_2+ie_3\rangle$	$\langle e_4+a_1e_1+c_1e_3, a_2e_1+e_2+c_2e_3\rangle$ $\operatorname{Im} c_1 \operatorname{Im} a_2 - \operatorname{Im} a_1 \operatorname{Im} c_2 \neq 0$
$\mathfrak{r}'_{4,\gamma,\delta}$:	$\langle e_4+ie_1, e_2+ie_3\rangle$, $\langle e_4+ie_1, e_2-ie_3\rangle$	$\langle e_4+a_1e_1+c_1e_3, e_2+\varepsilon ie_3\rangle$ $\varepsilon=\pm1$, $\operatorname{Im} a_1 \neq 0$
$\mathfrak{d}_4$:	$\langle e_1-ie_3, e_4+ie_2\rangle$, $\langle e_4+i(e_2+e_3), e_1-ie_3\rangle$	$\langle e_4+b_1e_2+c_1e_3, e_1-b_1e_3\rangle$ $\operatorname{Im} b_1 \neq 0$ $\langle e_4+a_1e_1+c_1e_3, e_2-a_1e_3\rangle$ $\operatorname{Im} a_1 \neq 0$
$\mathfrak{d}_{4,1}$	$\langle e_4-ie_1, e_2+ie_3\rangle$	$\langle e_4-c_2e_1+c_1e_3, e_2+c_2e_3\rangle$ $\operatorname{Im} c_2 \neq 0$
$\mathfrak{d}_{4,1/2}$	$\langle e_4+ie_3, e_1+ie_2\rangle$ $\langle e_4+ie_3, e_1-ie_2\rangle$ $\langle e_4+ie_1, e_2-2ie_3\rangle$	$\langle e_4+b_1e_2+c_1e_3, e_1+b_2e_2+2b_1e_3\rangle$ $2\operatorname{Im}^2 b_1 - \operatorname{Im} c_1 \operatorname{Im} b_2 \neq 0$ $\langle e_4+a_1e_1+c_1e_3, a_2e_1+e_2-2a_1e_3\rangle$ $2\operatorname{Im}^2 a_1 + \operatorname{Im} c_1 \operatorname{Im} a_2 \neq 0$
$\mathfrak{d}_{4,\lambda}$ $\lambda \neq 1/2, 1$	$\langle e_4-\lambda ie_1, e_2+ie_3\rangle$ $\langle e_4+(1-\lambda)ie_2, e_1+ie_3\rangle$	$\langle e_4-\lambda c_2e_1+c_1e_3, e_2+c_2e_3\rangle$ $\langle e_4+(1-\lambda)c_2e_2+c_1e_3, e_1+c_2e_3\rangle$ $\operatorname{Im} c_2 \neq 0$
$\mathfrak{d}'_{4,0}$	$\langle e_4-ie_3, e_1+ie_2\rangle$ $\langle e_4+ie_3, e_1+ie_2\rangle$	$\langle e_4+c_1e_3-\varepsilon ic_2e_2, c_2e_3+e_1+\varepsilon ie_2\rangle$ $\langle e_4+c_1e_3-\varepsilon ic_2e_2, c_2e_3+e_1+\varepsilon ie_2\rangle$ $\varepsilon=\pm1$, $\operatorname{Re} c_2 \operatorname{Im} c_2 + \operatorname{Im} c_1 \neq 0$
$\mathfrak{d}'_{4,\delta}$ $\delta > 0$	$\langle e_4+ie_3, e_1-ie_2\rangle$, $\langle e_4-ie_3, e_1+ie_2\rangle$, $\langle e_4+ie_3, e_1+ie_2\rangle$, $\langle e_4-ie_3, e_1-ie_3\rangle$	$\langle e_4+c_1e_3+(\delta/2-\varepsilon i)c_2e_2,$ $e_1+\varepsilon ie_2+c_2e_3\rangle$, $\varepsilon=\pm1$, $\varepsilon \operatorname{Im} c_1 + \operatorname{Im} c_2(-\frac{\delta}{2}\operatorname{Im} c_2 - \varepsilon \operatorname{Re} c_2) \neq 0$
$\mathfrak{h}_4$	$\langle e_4+ie_2, e_1+2ie_3\rangle$	$\langle e_4+b_1e_2+c_1e_3, 2b_1e_3+e_1\rangle$ $\operatorname{Im} b_1 \neq 0$

Proof. The following reasoning must be applied to each case. If there exists $\mathfrak{q}$ an invariant complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$, then there are elements $U, V \in \mathfrak{g}^{\mathbb{C}}$ of the form:

$$U = e_4 + a_1e_1 + b_1e_2 + c_1e_3, \qquad V = a_2e_1 + b_2e_2 + c_2e_3$$

(by changing subindices if necessary) which are a basis of $\mathfrak{q}$ and

$$[U,V] = \beta V \qquad\qquad \beta \in \mathbb{C} \tag{2}$$

From (2) we get a system of equations, such that, the existence of solutions (that is, the existence of coefficients a_i, b_i, c_i and β) is equivalent to the existence of complex subalgebras,when we imposed an extra condition to make of the set $\{U, V, \sigma U, \sigma V\}$ a basis of $\mathfrak{g}^{\mathbb{C}}$. The equivalence classes can be obtained by the action of the authomorphisms group of these Lie algebras. These technical computations are done by following the theory of the introduction of this section. The authomorphisms group can be found in the Appendix.

Examples of special classes of complex structures are the abelian ones and those that determine a complex Lie bracket on $\mathfrak{g}$. The following propositions show examples of these structures in dimension four. The proofs are implicit in the table 3.2.

A complex structure J is said to be *abelian* if it satisfies $[JX, JY] = [X,Y]$ for all $X, Y \in \mathfrak{g}$, or in terms of complex subalgebras, if $\mathfrak{q}$ (or $\sigma\mathfrak{q}$) is abelian. The following proposition shows all Lie algebras admitting such structure, making use of notations of (2.1) and (3.2).

Proposition 3.3. *If $\mathfrak{g}$ is a four-dimensional Lie algebra admitting an abelian complex structure, then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras $\mathbb{R}^4$, $\mathbb{R}\times\mathfrak{h}_3$, $\mathbb{R}^2\times\mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R})\times\mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C})$, $\mathfrak{d}_{4,1}$.*

A complex structure J introduces on $\mathfrak{g}$ a structure of complex Lie algebra if $J \circ ad_X = ad \circ JX$ for all $X \in \mathfrak{g}$, and so $(\mathfrak{g}, J)$ is a *complex Lie algebra*, and that means that the corresponding simply connected Lie group is also complex, that is, left and right multiplication by elements of the Lie group are holomorphic maps.

Proposition 3.4. *Let $\mathfrak{g}$ be a four dimensional Lie algebra such that $(\mathfrak{g}, J)$ is a complex Lie algebra, then $\mathfrak{g}$ is either $\mathbb{R}^4$ or $\mathfrak{aff}(\mathbb{C}) = \mathfrak{r}_2'$.*

4. Symplectic structures on four dimensional Lie algebras

A *symplectic structure* on a 2n-dimensional Lie algebra $\mathfrak{g}$ is a closed 2-form $\omega \in \Lambda^2(\mathfrak{g}^*)$ such that ω has maximal rank, that is, ω^n is a volume form on the corresponding Lie group. Lie algebras (groups) admitting symplectic structures are called *symplectic* Lie algebras (resp. Lie groups).

Our aim now is to study the existence of left invariant symplectic structures on a four dimensional solvable real Lie group. It is known [Ch] that four dimensional symplectic Lie algebras are solvable.

Left invariant symplectic structures are those which are invariant by left translation of elements of the group, that is,

$$\ell_g^*\omega = \omega \circ \ell_{g*}$$

The left invariance property allows to work on the Lie algebra $\mathfrak{g}$.

Denoting by $\{e^i\}$ the dual basis on $\mathfrak{g}^*$ of the basis $\{e_i\}$ on $\mathfrak{g}$ (see(2.1)), the Proposition 4.1 describes symplectic structures in the four dimensional case.

Proposition 4.1. *Let $\mathfrak{g}$ be a symplectic real Lie algebra of dimension four. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras equipped with a symplectic form as follows:*

$\mathfrak{rh}_3$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4$
$a_{14}a_{23} - a_{13}a_{24} \neq 0$

$\mathfrak{rr}_{3,0}$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{12}a_{34} \neq 0$

$\mathfrak{rr}_{3,-1}$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3,$
$a_{14}a_{23} \neq 0$

$\mathfrak{rr}'_{3,0}$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3$
$a_{14}a_{23} \neq 0$

$\mathfrak{r}_2\mathfrak{r}_2$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{34}e^3 \wedge e^4,$
$a_{12}a_{34} \neq 0$

$\mathfrak{r}'_2$: $\omega = a_{12}e^1 \wedge e^2 + a_{13-24}(e^1 \wedge e^3 - e^2 \wedge e^4) + a_{14+23}(e^1 \wedge e^4 + e^2 \wedge e^3)$
$a^2_{14+23} + a^2_{13-24} \neq 0$

$\mathfrak{n}_4$: $\omega = a_{12}e^1 \wedge e^2 + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{12}a_{34} \neq 0$

$\mathfrak{r}_{4,0}$: $\omega = a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$

$\mathfrak{r}_{4,-1}$: $\omega = a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{13}a_{24} \neq 0$

$\mathfrak{r}_{4,-1,\beta}$: $\omega = a_{12}e^1 \wedge e^2 + a_{14}e^1 \wedge e^3 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{14}a_{23} \neq 0,\ \beta \neq -1, 0, 1$

$\mathfrak{r}_{4,-1,-1}$: $\omega = a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{12}a_{34} - a_{13}a_{24} \neq 0$

$\mathfrak{r}_{4,\alpha,-\alpha}$: $\omega = a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{14}a_{23} \neq 0,\ \alpha \neq -1, 0$

$\mathfrak{r}'_{4,0,\delta}$: $\omega = a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4 + a_{34}e^3 \wedge e^4,$
$a_{14}a_{23} \neq 0,\ \delta \neq 0$

$\mathfrak{d}_{4,1}$: $\omega = a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4,$
$a_{12-34} \neq 0$

$\mathfrak{d}_{4,2}$: $\omega = a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4,$
$-a^2_{12-34} + a_{14}a_{23} \neq 0$

$\mathfrak{d}_{4,\lambda}$: $\omega = a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4,$
$a_{12-34} \neq 0,\ \lambda \neq 1, 2$

$\mathfrak{d}'_{4,\delta}$: $\omega = a_{12-\delta 34}(e^1 \wedge e^2 - \delta e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4$
$a_{-12+\delta 34} \neq 0,\ \delta \neq 0$

$\mathfrak{h}_4$: $\omega = a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4$
$a_{12-34} \neq 0$

Proof. As all cases should be worked out in a similar way, we will give as example the proof of the case $\mathfrak{r}'_2$, that is the Lie algebra which corresponds to $\mathfrak{aff}(\mathbb{C})$.

From the definition of the Lie bracket it follows that $de^1 = 0 = de^2$ and simple computations following the properties mentioned at the beginning of the section allow to get $-de^3 = e^1 \wedge e^3 - e^2 \wedge e^4$, and $-de^4 = e^1 \wedge e^4 + e^2 \wedge e^3$. At the next level we have:

$$d(e^1 \wedge e^3) = -e^1 \wedge e^2 \wedge e^4,\ d(e^1 \wedge e^4) = e^1 \wedge e^2 \wedge e^3,\ d(e^2 \wedge e^3) = -e^1 \wedge e^2 \wedge e^3$$

$$d(e^2 \wedge e^4) = -e^1 \wedge e^2 \wedge e^4, \qquad d(e^3 \wedge e^4) = -2e^1 \wedge e^3 \wedge e^4$$

Thus any 2-form ω that is closed has the form $\omega = a_{12}e^1 \wedge e^2 + a_{13-24}(e^1 \wedge e^3 - e^2 \wedge e^4) + a_{14+23}(e^1 \wedge e^4 + e^2 \wedge e^3)$. Now ω is symplectic if it satisfies the conditions mentioned at the Table (4.1) for $\mathfrak{r}_2'$.

Exact symplectic structures in dimension four were found in [Ca]. From the previous table we also determined as corollary this kind of structures.

Corollary 4.2. *A four dimensional solvable Lie algebra admits an exact symplectic structure if and only if $\mathfrak{g}$ is one of the following attached with the respective symplectic structure*

Case	ω	Condition
$\mathfrak{r}_2\mathfrak{r}_2$	$a_{12}e^1 \wedge e^2 + a_{34}e^3 \wedge e^4$	$a_{12}a_{34} \neq 0$
$\mathfrak{r}_2'$	$a_{13-24}(e^1 \wedge e^3 - e^2 \wedge e^4) + a_{14+23}(e^1 \wedge e^4 + e^2 \wedge e^3)$	$a_{14+23}^2 + a_{13-24}^2 \neq 0$
$\mathfrak{d}_{4,1}$	$a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4$	$a_{12-34} \neq 0$
$\mathfrak{d}_{4,\lambda}\lambda \neq 1$	$a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4$	$a_{12-34} \neq 0$
$\mathfrak{d}_{4,\delta}'\delta \neq 0$	$a_{-12+\delta 34}(-e^1 \wedge e^2 + \delta e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4$	$a_{-12+\delta 34} \neq 0$
$\mathfrak{h}_4$	$a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + a_{14}e^1 \wedge e^4 + a_{24}e^2 \wedge e^4$	$a_{12-34} \neq 0$

5. Kähler structures

In this section we search for (left invariant) Kähler structures on a four dimensional Lie group G, that is for symplectic forms ω on $\mathfrak{g}$ such that there exists a complex structure J, which is compatible with ω:

$$\omega(JX, JY) = \omega(X, Y) \qquad \text{for all } X, Y \in \mathfrak{g} \qquad .$$

To this end it is necessary to compare the results in Tables (3.2) and (4.1).

Notice that the existence problem of compatible pairs (ω, J) is set up to complex isomorphism. In fact, assume that there is a complex structure J_1 such that there is a symplectic structure ω satisfying $\omega(J_1X, J_1Y) = \omega(X, Y)$ for all $X, Y \in \mathfrak{g}$ and assume that J_1 is equivalent to J_2. Hence there exists an automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g})$ such that $J_2 = \sigma_*^{-1} J_1 \sigma_*$. Then it holds

$$\omega(X, Y) = \sigma^{*^{-1}} \sigma^* \omega(X, Y) = \sigma^{*^{-1}} \omega(J_1\sigma^* X, J_1 \sigma^* Y) = \omega(J_2X, J_2Y).$$

In the following table for a given complex structure we describe the general form of a compatible symplectic structure when it exists.

Proposition 5.1. *Let G be a solvable real Lie group of dimension four. Then the following Table shows for a given left invariant complex structure the general form of a compatible left invariant symplectic structure ω on G, when it exists (the so defined complex structure must be extended by imposing $J^2 = -\,\mathrm{Id}$):*

Case	Complex structure	Kähler structure
$\mathfrak{rr}_{3,0}$	$Je_1 = e_2,\ Je_3 = e_4$	$a_{12}e^1 \wedge e^2 + a_{34}e^3 \wedge e^4$ $a_{12}a_{34} \neq 0$
$\mathfrak{rh}_3$	$Je_1 = e_2,\ Je_3 = e_4$	$a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4-$ $-a_{14}e^2 \wedge e^3 + a_{13}e^2 \wedge e^4$ $a_{13}^2 + a_{14}^2 \neq 0$
$\mathfrak{rr}_{3,0}$	$Je_1 = e_2,\ Je_3 = e_4$	$a_{12}e^1 \wedge e^2 + a_{34}e^3 \wedge e^4$ $a_{12}a_{34} \neq 0$
$\mathfrak{rr}'_{3,0}$	$Je_1 = e_4,\ Je_2 = e_3$	$a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3$ $a_{14}a_{23} \neq 0$
$\mathfrak{r}_2\mathfrak{r}_2$	$Je_1 = e_2,\ Je_3 = e_4$	$a_{12}e^1 \wedge e^2 + a_{34}e^3 \wedge e^4$ $a_{12}a_{34} \neq 0$
$\mathfrak{r}'_2$	$J_1e_1 = e_3,\ J_1e_2 = e_4$	$a_{13-24}(e^1 \wedge e^3 - e^2 \wedge e^4)+$ $+a_{14+23}(e^1 \wedge e^4 + e^2 \wedge e^3),\ a_{13-24}^2 + a_{14+23}^2 \neq 0$
	$J_2e_1 = -e_2,\ J_1e_3 = e_4$	$a_{12}(e^1 \wedge e^2) + a_{13-24}(e^1 \wedge e^3 - e^2 \wedge e^4)+$ $+a_{14+23}(e^1 \wedge e^4 + e^2 \wedge e^3),\ a_{13-24}^2 + a_{14+23}^2 \neq 0$
$\mathfrak{r}_{4,-1,-1}$	$Je_4 = e_1,\ Je_2 = e_3$	$a_{12}e^1 \wedge e^2 + a_{13}e^1 \wedge e^3 + a_{14}e^1 \wedge e^4+$ $-a_{13}e^2 \wedge e^4 + a_{12}e^3 \wedge e^4,\ a_{12}^2 + a_{13}^2 \neq 0$
$\mathfrak{r}'_{4,0,\delta}$	$J_1e_4 = e_1,\ J_1e_2 = e_3,$ $J_2e_4 = e_1,\ J_2e_2 = -e_3$	$a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3$ $a_{14}a_{23} \neq 0$
$\mathfrak{d}_{4,1}$	$J_1e_1 = e_4,\ J_1e_2 = e_3$	$a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4) + e_{14}(e^1 \wedge e^4)$ $a_{12-34} \neq 0$
$\mathfrak{d}_{4,2}$	$J_1e_4 = -e_2,\ J_1e_1 = e_3$	$a_{14}e^1 \wedge e^4 + a_{14}e^2 \wedge e^3 + a_{24}e^2 \wedge e^4$ $a_{14} \neq 0$
	$J_2e_4 = -2e_1,\ J_2e_2 = e_3$	$a_{14}e^1 \wedge e^4 + a_{23}e^2 \wedge e^3$ $a_{14}a_{23} \neq 0$
$\mathfrak{d}_{4,1/2}$	$J_1e_4 = e_3,\ J_1e_1 = e_2$	$a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4)$ $a_{12-34} \neq 0$
	$J_2e_4 = e_3,\ J_2e_1 = -e_2$	$a_{12-34}(e^1 \wedge e^2 - e^3 \wedge e^4)$ $a_{12-34} \neq 0$
$\mathfrak{d}'_{4,\delta}$ $\delta \neq 0$	$J_1e_4 = e_3,\ J_1e_1 = e_2$ $J_2e_4 = -e_3,\ J_2e_1 = e_2$ $J_3e_4 = -e_3,\ J_3e_1 = -e_2$ $J_4e_4 = e_3,\ J_4e_1 = e_2$	$a_{12-\delta 34}(e^1 \wedge e^2 - \delta e^3 \wedge e^4)$ $a_{12-34} \neq 0$

Table *5.1*

Proof. We will give the proof in the case $\mathfrak{r}_2'$ since all cases should be handled in a similar way. As we can see on the Table (3.2) the complex structures on $\mathfrak{r}_2'$ are given by: $J_1e_1 = e_3$, $J_1e_2 = e_4$; and for the other type of complex structures, denoting $a_1 \in \mathbb{C}$ by $a_1 = \mu + i\nu$, with $\nu \neq 0$; we have $J_{\mu,\nu}e_1 = \frac{\mu}{\nu}e_1 + (\frac{\nu^2+\mu^2}{\nu})e_2$, $J_{\mu,\nu}e_3 = e_4$. On the other hand any sympletic structure has the form: $\omega = a_{12}(e^1\wedge e^2)+a_{13-24}(e^1\wedge e^3 - e^2\wedge e^4) + a_{14+23}(e^1\wedge e^4 + e^2\wedge e^3)$, with $a_{14+23}^2 + a_{13-24}^2 \neq 0$. Assuming that there exists a Kähler structure it holds $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in \mathfrak{g}$ and this condition produces equations on the coefficients of ω which should be verified in each case.

Hence for J_1 we need to compute only the following:

$$\omega(e_1, e_2) = a_{12} = \omega(e_3, e_4) \qquad \text{and} \qquad \omega(e_1, e_4) = a_{14+23} = \omega(e_3, -e_2)$$

Thus these equalities impose the condition $a_{12} = 0$. And so any Kähler structure corresponding to J_1 has the form $\omega = a_{13-24}(e^1\wedge e^3 - e^2\wedge e^4)+a_{14+23}(e^1\wedge e^4+e^2\wedge e^3)$ with $a_{13-24}^2 + a_{14+23}^2 \neq 0$.

For the second case correspondig to $J_{\mu,\nu}$, by computing $\omega(e_2, e_4)$, $\omega(e_1, e_3)$, we get respectively:

$$\begin{array}{lrcl} \text{i)} & (1+\frac{1}{\nu})a_{24-13} & = & \frac{\mu}{\nu}a_{14+23} \\ \text{ii)} & (1+\frac{\mu^2+\nu^2}{\nu})a_{24-13} & = & -\frac{\mu}{\nu}a_{14+23} \end{array}$$

By comparing i) and ii) we get:

$$(1+\tfrac{1}{\nu})a_{24-13} = -(1+\tfrac{\mu^2+\nu^2}{\nu})a_{24-13}$$

and this equality implies iii)$a_{24-13} = 0$ or iv)$1+\frac{1}{\nu}+1+\frac{\mu^2+\nu^2}{\nu} = 0$. As $a_{24-13} \neq 0$ (since in this case we would also get $a_{14+23} = 0$ and this would be a contradiction) then it must hold iv), that is $1+\mu^2+\nu^2 = -2\nu$ and that implies $\mu^2 = -2\nu-1-\nu^2 = -(\nu+1)^2$ and so that is possible only if $\mu = 0$ and $\nu = -1$. For this complex structure J, given by $Je_1 = -e_2$ $Je_3 = e_4$, it is not difficult to prove that for any symplectic structure ω it always holds $\omega(JX, JY) = \omega(X, Y)$, that is, any symplectic structure on $\mathfrak{g}$ is compatible with J. In this way we have completed the proof of the assertion.

Remark 5.2. Compare the results of (3.2) (4.1) and (5.1) to get Lie algebras having complex and symplectic structures but no Kähler structures. This occurs in $\mathfrak{h}_4$, $\mathfrak{d}_{4,\lambda}$ $\lambda \neq 1, 2, 1/2$.

6. Appendix: Authomorphisms of four dimensional solvable Lie algebras

In this section we compute the authomorphisms of the four dimensional Lie algebras admitting a complex structure (3.2).

The authomorphisms group of many of these Lie algebras were computed in [O2], e.g. for all four dimensional solvable real Lie algebras with a three dimensional commutator.

We list the respective authomorphisms groups using matricial representations in the ordered basis $\{e_1, e_2, e_3, e_4\}$ of (2.1). In all cases the matrix must be non singular.

$$\mathfrak{rh}_3: \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{pmatrix} \quad a_{33} = a_{11}a_{22} - a_{12}a_{21} \qquad \mathfrak{rr}_{3,0}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix}$$

$$\mathfrak{r}_{3,1}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix} \qquad \mathfrak{r}'_{3,\gamma}: \gamma \neq 0 \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & -a_{23} & a_{22} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}$$

$$\mathfrak{rr}'_{3,0}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & -a_{23} & a_{22} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{23} & -a_{22} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}$$

$$\mathfrak{r}_2\mathfrak{r}_2: \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & a_{23} & a_{24} \\ 1 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$$

$$\mathfrak{r}'_2: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & -a_{34} & a_{33} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{34} & -a_{33} \end{pmatrix}$$

$$\mathfrak{r}_{4,1}: \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a_{32} = a_{44}a_{21},\ a_{33} = a_{44}^2 a_{11} \qquad \mathfrak{r}_{4,\varepsilon}: \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \varepsilon = 0, -1$$

$$\mathfrak{r}_{4,\alpha,1}: \alpha \neq 0,1 \quad \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{r}_{4,\alpha,\alpha}: \alpha \neq 0 \quad \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathfrak{r}'_{4,\gamma,\delta}: \gamma \in \mathbb{R}, \delta > 0 \quad \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & -a_{23} & a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{h}_4: \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ 2a_{11}a_{24} & a_{32} & a_{11}^2 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a_{32} = 2(a_{24}(2a_{11} + a_{12}) - a_{14}a_{11})$$

$$\mathfrak{d}_4 \quad \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ -a_{24}a_{11} & -a_{14}a_{22} & a_{11}a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & a_{12} & 0 & a_{14} \\ a_{21} & 0 & 0 & a_{24} \\ a_{14}a_{21} & a_{24}a_{12} & -a_{12}a_{21} & a_{34} \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\mathfrak{d}_{4,1/2}: \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{d}_{4,\lambda}: \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & a_{32} & a_{11}a_{22} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a_{31} = 2(a_{11}a_{24} - a_{14}a_{21}),\quad a_{32} = 2(a_{12}a_{24} - a_{22}a_{14}),\quad a_{33} = a_{12}a_{21} - a_{22}a_{11} \qquad \lambda > 1/2 \quad a_{31} = \tfrac{a_{11}a_{24}}{1-\lambda}, \text{ iff } \lambda \neq 1,\quad a_{32} = -\tfrac{a_{14}a_{22}}{\lambda}$$

$$\mathfrak{d}'_{4,\delta}: \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ -a_{12} & a_{11} & 0 & a_{24} \\ a_{31} & a_{32} & a_{11}^2 + a_{22}^2 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$\begin{pmatrix} a_{31} \\ a_{32} \end{pmatrix} = \frac{1}{(\delta/2)^2 + 1} \begin{pmatrix} \delta/2 & 1 \\ -1 & \delta/2 \end{pmatrix} \begin{pmatrix} a_{12} & a_{11} \\ -a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix}$$

References

[A-B-D-O] A. Andrada, L. Barberis, I. Dotti, G. Ovando, Product structures on four dimensional solvable Lie algebras, Homology, Homotopy and Applications, **7**, 9-13, (2005).

[Al] D. Alekseevsky, *Homogeneous Riemannian spaces of negative curvature*, Math. USSR Sbornik **25**(1) (1975), 87–109.

[An] A. Andrada, Hypersymplectic four dimensional Lie algebras, arXiv:math.DG/0310462.

[A-C-F-M] L. C. de Andrés, L. Cordero, M. Fernández, J. J. Mencía, *Examples of four-dimensional compact locally conformal Kähler solvmanifolds*, Geom. Dedicata **29**, No.2, 227–232 (1989).

[A-F-G-M] L. C. de Andrés, M. Fernández, A. Gray, J.Mencía, *Moduli spaces of complex structures on compact four dimensional nilmanifolds*, Boll. Unione Mat. Ital., VII Ser., A 5, No.3, 381–389 (1991).

[B1] M. L. Barberis, *Hypercomplex structures on four dimensional Lie groups*, Proc. Amer. Math. Soc. **125** (1997), 1043–1054.

[B2] M.L. Barberis, *Hyper-Kähler metrics conformal to left invariant metrics on four dimensional Lie groups*, Math. Phys., Analysis and Geometry **6 (1)**, (2003), 1–8.

[Bo] N. Bourbaki, *Éléments de mathématics - Groupes et algèbres de Lie*, Hermann, PARIS, (1972).

[Ca] R. Campoamor Stursberg, Towards a characerization of exact symplectic Lie algebras $\mathfrak{g}$ in terms of the invariants for the coadjoint representation, arXiv:math-ph/0301004 v1 6 Jan 2003.

[Ch] B. Y. Chu, Symplectic homogeneous spaces, Trans. Amer. Math. Soc., **197** (1974), 145 - 159.

[D-S] V. De Smedt, S. Salamon, *Anti-self dual metrics on Lie groups*, Contemp. Math. **308** (2002), 63–75.

[F] A. Fino, *Almost Kähler* 4*-dimensional Lie groups with* J*-invariant Ricci tensor*, preprint math.DG/0308014.

[F-G] M. Fernández and A. Gray, *Compact symplectic solvmanifolds not admitting complex structures*, Geom. Dedic. **34** (1990), 295–299.

[F-G-G] M. Fernández, M. Gotay, A. Gray, Compact parallelizable four dimensional symplectic and complex manifolds, Proc. of the A.M.S., **103** No. 4, (1988), 1209 - 1212.

[H] E. Heintze, On homogeneous manifolds of negative curvature, Math. Ann, **211** (1974), 23 - 34.

[J] G.R. Jensen, *Homogeneous Einstein metrics of dimension four*, J. Differential Geom. **3** (3) (1969), 309–349.

[K-N] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Interscience Publisher, **I - II**, (1969).

[H-L] A. Hucleberry, E. Livorni, *A classification of homogeneous surfaces*, Can. J. Math., **33**, (1981), 1097 - 1110.

[Mi] J. Milnor, *Curvatures of left invariant metrics on Lie groups* , Advances in Mathematics, **21** (1976), 293–329.

[Mo] A. Morimoto, Structures complexes sur les groupes de Lie semi-simples , C.R., **242** (1956), 1101 - 1103.

[M-R] A. Medina, P. Revoy: Groupes de Lie à structure symplectique invariante. "Symplectic Geometry, Grupoids and Integrable Systems, Séminaire Sud-Rhodanien de Géométrie" (P. Dazord et A. Weinstein Eds.), Mathematical Sciences Research Institute Publications, 247 - 266, New York-Berlin-Heidelberg, Springer Verlag 1991.

[N-N] A. Newlander, L. Niremberg, Complex analytic coordinates in almost complex manifolds, Ann of Math, **65** (1957), 391 - 404.

[O1] G. Ovando, Invariant complex structures on solvable real Lie groups, Man. Math., **103**, (2000), 19-30.

[O2] G. Ovando, Estructuras complejas y sistemas hamiltonianos en grupos de Lie solubles, Tesis doctoral, Fa.M.A.F., Univ. Nac. de Córdoba, (2002).

[O-R] K. Oeljeklaus, W. Richtofer, Homogeneous complex surfaces, Math. Ann., (1984), 273 - 292.

[O-T] J. Oprea, A. Tralle, *Symplectic manifolds with no Kähler structure*, Lecture Notes in Math. **1661**, Springer Verlag (1997).

[Sl] S.M. Salamon, Complex structures on nilpotent Lie algebras, Manuscripta Math, **66** (1990), 397 - 412.

[Sm] H. Samelson, A class of complex-analytic manifolds, Port. Math., (1953), 129 - 132.

[Ss1] T. Sasaki, Classification of left invariant complex structures on $GL(2, \mathbb{R})$ and $U(2)$, Kumamoto J. Sci. (Math.), **14** (1981), 115 - 123.

[Ss2] T. Sasaki, Classification of left invariant complex structures on $SL(3, \mathbb{R})$, Kumamoto J. Sci. (Math.), **15** (1982), 59 - 72.

[SD] D. Snow, Invariant Complex Structures On Reductive Lie Groups, J. Reine Angew. Math., **371** (1986), 191 - 215.

[Sn] J. E. Snow, Invariant Complex Structures on Four Dimensional Solvable Real Lie Groups, Manuscripta Math, **66** (1990), 397 - 412.

[Va] V. S. Varadarajan, *Lie groups, Lie algebras and their representations*, Springer, (1984).

[Wa] H.C. Wang, Closed manifolds with homogeneous complex structure, Amer. J. Math., **76** (1954), 1 - 37.

[Wi] J. Winkelman, Classification of 3-dimensional Homogeneous Complex Manifolds, Topological Methods in Algebraic Tranformations Groups, Proc. of a Conf. at Rutgers Univ., Birkhäuser (1989), 191 - 210.

Gabriela Ovando
CIEM-FaMAF,
Universidad Nacional de Córdoba,
Córdoba 5000, Argentina.
ovando@mate.uncor.edu

Recibido: 26 de agosto de 2004
Revisado: 15 de marzo de 2005