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# DIFFERENTIAL OPERATORS ON SMOOTH SCHEMES AND EMBEDDED SINGULARITIES.

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*En recuerdo de Angel Larrotonda*

Abstract. Differential operators on smooth schemes have played a central role in the study of embedded desingularization.

J. Giraud provides an alternative approach to the form of induction used by Hironaka in his Desingularization Theorem (over fields of characteristic zero). In doing so, Giraud introduces technics based on differential operators. This result was important for the development of algorithms of desingularization in the late 80's (i.e. for constructive proofs of Hironaka's theorem).

More recently, differential operators appear in the work of J. Wlodarczyk ([35]), and also on the notes of J. Kollár ([25]).

The form of induction used in Hironaka's Desingularization Theorem, which is a form of elimination of one variable, is called *maximal contact*. Unfortunately it can only be formulated over fields of characteristic zero.

In this paper we report on an alternative approach to elimination of one variable, which makes use of higher differential operators. These results open the way to new invariants for singularities over fields of positive characteristic ([34]).

## Contents



## Part 1. Introduction.

Let $V$ be a smooth scheme over a field $k$ of characteristic zero, and let $X \subset V$ be a singular subscheme. Hironaka proves embedded desingularization of $X$,

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considering as invariants the Hilbert-Samuel functions at the points of $X$. His proof is based on the *reduction* of Hilbert Samuel functions by monoidal transformations ([22]).

There is second theorem of Hironaka, used in his proof of *reduction* of Hilbert Samuel functions, which is called *Log-resolution* of ideals in smooth schemes. For this second theorem, which we discuss below, the invariant considered is the order of the ideal at the points of the smooth scheme.

In [17], both theorems are linked in a different way. In fact, if $X \subset V$ is defined by a sheaf of ideals $J\mathcal{O}_V$, then desingularization is proved by considering the order of the ideal $J$ at points in $W$, and hence avoiding the use of Hilbert Samuel functions.

Let $V$ be a smooth scheme over a field $k$, and let $J \subset \mathcal{O}_V$ be a non-zero sheaf of ideals. Define a function

$$ord_J : V \to \mathbb{Z}$$

where $ord_J(x)$ denotes the order of $J_x$ at the local regular ring $\mathcal{O}_{V,x}$. Let $b$ denote the biggest value achieved by this function (the biggest order of $J$). The pair $(J, b)$ is *the object of interest* in Log principalization of ideals. There is a closed set attached to this pair in $V$, namely the set of points where $J$ has order $b$; and there is also a notion of transformation of such pairs by blowing up suitable regular centers.

We will attach to $(J, b)$ a graded subring of $\mathcal{O}_V[W]$ (sheaf of polynomial rings), namely a graded algebra (Rees algebra) of the form

$$\oplus_{r\geq 0} I_r W^r,$$

defined uniquely in terms of $J$ and $b$.

Actually the Rees algebras that we will consider are closely related to Kollárs notion of *tuned ideals*.

We will show that there is a closed set in $V$ naturally attached to such Rees algebra, and also a notion of transformation. Of course the interest here is on the case of smooth schemes over fields of positive characteristic, where a weak form of *elimination* of one variable is discussed.

For any non-negative integer $s$ the sheaf of $k$-linear differential operators, say $Diff_k^s$, is coherent and locally free over $V$.

There is a natural identification, say $Diff_k^0 = \mathcal{O}_V$, and for each $s \geq 0$ there is a natural inclusions $Diff_k^s \subset Diff_k^{s+1}$.

If $U$ is an affine open set in $V$, each $D \in Diff_k^s(U)$ is a differential operator: $D : \mathcal{O}_V(U) \to \mathcal{O}_V(U)$. We define an extension of a sheaf of ideals $J \subset \mathcal{O}_V$, say $Diff_k^s(J)$, so that over the affine open set $U$, $Diff_k^s(J)(U)$ is the extension of $J(U)$ defined by adding all elements $D(f)$, for all $D \in Diff_k^s(U)$ and $f \in J(U)$.

So $Diff^0(J) = J$, and $Diff^s(J) \subset Diff^{s+1}(J)$ as sheaves of ideals in $\mathcal{O}_V$. Let $V(J) \subset V$ be the closed set defined by $J \in \mathcal{O}_V$. So

$$V(J) \supset V(Diff^1(J)) \supset \cdots \supset V(Diff^{s-1}(J)) \supset V(Diff^s(J)) \dots$$

It is simple to check that the order of the ideal at the local regular ring $\mathcal{O}_{V,x}$ is $\geq s$ if and only if $x \in V(Diff^{s-1}(J))$.

The previous observations say that $ord_J : V \to \mathbb{Z}$ is an upper-semi-continuous function, and that the highest order of $J$ (at points $x \in V$) is $b$, if $V(Diff^b(J)) = \emptyset$ and $V(Diff^{b-1}(J)) \neq \emptyset$. Let

$$\begin{array}{ccc} V & \stackrel{\pi}{\longleftarrow} & V_1 \\ \cup & & \cup \\ Y & & \pi^{-1}(Y) = H \end{array}$$

denote the blow up of $W$ at a smooth irreducible sub-scheme $Y$, and $H$ is the exceptional hypersurface. If $Y \subset V(Diff^{b-1}(J)))$ we say that $\pi$ is $b$-permissible. In such case

$$J\mathcal{O}_{V_1} = I(H)^b J_1,$$

where $I(H)$ is the sheaf of functions vanishing along the exceptional hypersurface $H$.

If $\pi$ is $b$-permissible, $J_1$ has at most order $b$ at points of $W_1$ (i.e. that $V(Diff^b(J_1))$ $= \emptyset$). If, in addition, $J_1$ has no point of order $b$, then we say that $\pi$ defines a $b$-*simplification* of $J$.

If $V(Diff^{b-1}(J_1)) \neq \emptyset$, let $V_1 \stackrel{\pi_1}{\longleftarrow} V_2$ denote the monoidal transformation with center $Y_1 \subset V(Diff^b(J_1))$. We say that $\pi_1$ is $b$-permissible, and set

$$J_1\mathcal{O}_{V_2} = I(H_1)^b J_2.$$

It turns out that $J_2$ has at most points of order $b$. If it does, define a $b$-permissible transformation at some smooth irreducible center $Y_2 \subset V(Diff^{b-1}(J_2)))$.

For $J$ and $b$ as before, we define, by iteration, a $b$-permissible sequence

$$V \stackrel{\pi}{\longleftarrow} V_1 \stackrel{\pi_1}{\longleftarrow} V_2 \stackrel{\pi_2}{\longleftarrow} \dots V_r \stackrel{\pi_r}{\longleftarrow} V_{r+1},$$

and a factorization $J_{n-1}\mathcal{O}_{V_n} = I(H_n)^b J_n$.

Let $H_i \subset V_n$ denote the strict transform of exceptional hypersurface $H_i \subset V_{i-1}$. Note that:

1) $\{H, H_1, \dots, H_{n-1}\}$ are the irreducible components of the exceptional locus of $V \leftarrow V_n$.

2) The total transform of $J$ relates to $J_n$ by an expression of the form

$$J\mathcal{O}_{V_n} = I(H)^{a_0} I(H_1)^{a_1} \cdots I(H_{n-1})^{a_0} J_n.$$

We say that this $b$-permissible sequence defines a $b$-*simplication* of $J \subset \mathcal{O}_V$ if $\cup H_i$ has normal crossings, and $V(Diff^{b-1}(J_n)) = \emptyset$ (i.e. $J_n$ has order at most $b-1$ at $W_n$).

When $k$ is a field of *characteristic zero*, and $b$ is the highest order of a sheaf of ideals $J \subset \mathcal{O}_V$, Hironaka proves that there is a $b$-simplification. Furthermore, taking this as starting point, he indicates how to achieve resolution of singularities.

Hironaka's theorem of resolution of singularities is existential, precisely because his proof of $b$-simplification is existential.

The achievement of constructive resolution of singularities was to provide an algorithm. So given $J \subset \mathcal{O}_V$ and $b$ as before, as input, the algorithm defines a $b$-simplification.

An advantage of a constructive proof of resolution of singularities, over the original existential proof, is that constructive resolutions are equivariant, they provide resolution en étale topology, they are compatible with change of base field etc. (see [32]).

Another advantage of the algorithm of $b$-simplification, already mentioned above, is that it simplifies the proof of desingularization ([17]).

The key point for $b$-simplification, already used in Hironaka's proof, is a form of induction. In fact, Hironaka proves $b$-simplification, by induction on the dimension of the ambient space $V$. To simplify matters, assume that $J$ is locally principal, and let $b$ denote the highest order of $J$ along points in $V$, which is now smooth over a field of characteristic zero. Let

$$\{ord_J \geq b\}$$

denote the closed set $\{x \in V/ord_J(x) \geq b\}$ (or say $= b$).

Fix a closed point $x \in \{ord_J \geq b\}$, and a regular system of parameters $\{x_1, x_2, \ldots, x_n\}$ at $\mathcal{O}_{V,x}$. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, set $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$\Delta^\alpha = (\frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_n!}) \frac{\partial^{\alpha_1}}{\partial^{\alpha_1} x_1} \cdots \frac{\partial^{\alpha_n}}{\partial^{\alpha_n} x_n}.$$

If $J_x$ is locally generated by $f \in \mathcal{O}_{V,x}$, then $f$ has order $b$ at $\mathcal{O}_{V,x}$, and

$$(Diff^{b-1}(J))_x = \langle f, \Delta^\alpha(f)/0 \leq |\alpha| < b \rangle.$$

The key point is that, the order of $(Diff^{b-1}(J))_x$ at $\mathcal{O}_{V,x}$ is one. This holds when $k$ is a field of characteristic zero.

Recall that $V(Diff^{b-1}(< f >)) = \{ord_J \geq b\}$ locally at $x$. One way to check that $(Diff^{b-1}(J))_x$ has order one at $\mathcal{O}_{V,x}$, is to check this at the completion $\hat{\mathcal{O}}_{V,x}$, say $R = k'[[x_1, .., x_n]]$. We may choose the system of parameters so that, for a suitable unit $u$:

$$u.f = f_1 = Z^b + a_1 Z^{b-1} + \cdots + a_b \in S[Z]$$

$S = k[[x_1, .., x_{n-1}]]$, and $Z = x_n$.

As $k$ is a field of characteristic zero, $S[Z] = S[Z_1]$, where $Z_1 = Z + \frac{1}{b} a_1$, and

$$f_1 = Z_1^b + a'_2 Z_1^{b-2} + \cdots + a'_b.$$

Then:

A) $Z_1 \in Diff^{b-1}(f)$ (in fact $\frac{\partial^{b-1} f}{\partial^{b-1} Z} \in Diff^{b-1}(f)$). In particular the ideal $Diff^{b-1}(f)$ has order one at $x$, and the closed set $\{ord_J \geq b\}$ is locally included in a smooth scheme of dimension $n-1$.

B)(*Elimination.*) $\{ord\ f \geq b\}(\subset V(Z_1))$ can be described as

$$\{ord\ f \geq b\} = \cap_{2 \leq i \leq b} \{ord\ a'_i \geq b - i\}.$$

C) (*Stability of elimination.*) Both A), and the description in B), are preserved by any $b$-permissible sequence of transformations.

We will not go into details of A), B) and C). But let us point out the elimination of one variable in (B). In fact the closed set $\{ord\ f \geq b\}$ defined in terms of $f$, is also described as $\cap_{2 \leq i \leq b} \{ord\ a'_i \geq b - i\}$, where now the $a'_i$ involve one variable less.

As indicated above, A),B), and C), together, conform the essential reason and argument in resolution of singularities in characteristic zero. They rely entirely on the hypothesis of characteristic zero. For instance A) does not hold over fields of positive characteristic; so there is no way to formulate this form of induction over arbitrary fields.

The objective of these notes is to report on an entirely different approach to induction, which can at least be formulated over arbitrary fields.

Suppose, for simplicity, that $V$ is affine, that $f$ is global in $\mathcal{O}_V$, and that $b$ is the highest order of $J = \langle f \rangle$. We reformulate the study $b$-sequences of transformations over $J$. In doing so we replace $J$ by a graded ring subring of $\mathcal{O}_V[W]$. In this case we consider the subring

$$\mathcal{O}_V[fW^b](\subset \mathcal{O}_V[W]).$$

In general, if $V$ is affine, we define a *Rees algebra* as a subring of $\mathcal{O}_V[W]$ generated by a finite set, say

$$\{f_1W^{n_1}, f_2W^{n_2}, \dots, f_sW^{n_s}\}.$$

These subrings can also be expressed as $\bigoplus_{k\geq 0} I_kW^k$, $I_0 = \mathcal{O}_V$, and each $I_k$ is an ideal. We say that $\bigoplus_{k\geq 0} I_kW^k$ has *differential structure*, say Diff-structure, if $D(I_N) \subset I_{N-r}$ for $0 \leq r \leq N$, and $D \in Diff^r_k$.

Diff-structures appear in [23] and [24](see 4.2), and they are closely related to the notion of *tuned ideals* introduced by J Kollár.

It is easy to show that any Rees algebra spans a smallest Diff-structure containing it. Diff-structures are known to have important geometric properties, which make them objects of particular interest. In this paper we report on a characteristic free form of *elimination* defined for Diff-structures (see (B) above).

We also study here a natural compatibility of monoidal transforms and Diff-structures. This is done via Taylor development in positive characteristic (see also [33]). So it makes sense to formulate *stability of elimination* (see (C) above) over arbitrary fields. Here results are stronger over fields of characteristic zero, where they provide an alternative approach to induction in desingularization theorems.

New invariants for singularities arise, in positive characteristic, when studying this form of *elimination* in the setting of Diff-structures.

## 1. Monoidal transformations and Hironaka's topology.

Fix a smooth scheme $V$ over a field $k$, an ideal $J \subset \mathcal{O}_V$, and a positive integer $b$. Hironaka attaches to these data, say $(J, b)$, a closed set, say

$$\{ord_J \geq b\} := \{x \in V / \nu_x(J_x) \geq b\}$$

where $\nu_x(J_x)$ denote the order of $J$ at the local regular ring $\mathcal{O}_{V,x}$.

Given $(J, b)$ and $(J', b')$, then

$$\{ord_J \geq b\} \cap \{ord_{J'} \geq b'\} = \{ord_K \geq c\}$$

where $K = J^{b'} + J'^{b}$, and $c = b \cdot b'$. Set formally $(J, b) \odot (J', b') = (K, c)$.

There is also a notion of *permissible transformation* on these data $(J, b)$. Let $Y$ be a smooth subscheme in $V$, included in the closed $\{ord_J \geq b\}$, and let

$$\begin{array}{ccc} V & \xleftarrow{\pi} & V_1 \\ \cup & & \cup \\ Y & & \pi^{-1}(Y) = H, \end{array} \tag{1.0.1}$$

be the blow up of $V$ at a smooth sub-scheme $Y$. Note that

$$J\mathcal{O}_{V_1} = I(H)^b J_1,$$

where $I(H)$ is the sheaf of functions vanishing along the exceptional hypersurface $H$.

We call $(J_1, b)$ the *transform* of $(J, b)$ by the permissible monoidal transformation.

If $\pi$ is permissible for both $(J, b)$ and $(J', b')$, then it is permissible for $(K, c)$. Moreover, if $(J_1, b)$, $(J'_1, b)$, and $(K_1, c)$ denote the transforms, then $(J_1, b) \odot (J'_1, b') = (K_1, c)$.

We now define a *Rees algebra* over $V$ to be a graded noetherian subring of $\mathcal{O}_V[W]$, say:

$$\mathcal{G} = \bigoplus_{k \geq 0} I_k W^k,$$

where $I_0 = \mathcal{O}_V$ and each $I_k$ is a sheaf of ideals. And we assume that at any affine open set $U \subset V$, there is a finite set

$$\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\},$$

$n_i \geq 1$ and $f_i \in \mathcal{O}_V(U)$, so that the restriction of $\mathcal{G}$ to $U$ is

$$\mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}] (\subset \mathcal{O}_V(U)[W]).$$

To a Rees algebra $\mathcal{G}$ we attach a closed set:

$$Sing(\mathcal{G}) := \{x \in V / \nu_x(I_k) \geq k, \text{ for any } k \geq 1\},$$

where $\nu_x(I_k)$ denotes the order of the ideal $I_k$ at the local regular ring $\mathcal{O}_{V,x}$.

**Remark 1.1.** Rees algebras are related to Rees rings. A Rees algebra is a Rees ring if, given any affine open set $U \subset V$, and $\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$ as above, all degrees $n_i$ are one.

In general Rees algebras are integral closures of Rees rings in a suitable sense. In fact, if $N$ is a positive integer divisible by all $n_i$, it is easy to check that

$$\mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}] = \oplus_{r \geq 0} I_r W^r (\subset \mathcal{O}_V(U)[W]),$$

is integral over the Rees sub-ring $\mathcal{O}_V(U)[I_N W^N] (\subset \mathcal{O}_V(U)[W^N])$.

**Proposition 1.2.** *Given an affine open $U \subset V$, and $\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$ as above,*

$$Sing(\mathcal{G}) \cap U = \cap_{1 \leq i \leq s} \{ord(f_i) \geq n_i\}.$$

*Proof.* It is clear that $\nu_x(f_i) \geq n_i$ for $x \in Sing(\mathcal{G})$, $0 \leq i \leq s$. So

$$Sing(\mathcal{G}) \cap U \subset \cap_{1 \leq i \leq s} \{ord(f_i) \geq n_i\}.$$

On the other hand, for any index $N \geq 1$, $I_N(U) W^N$ is generated by elements of the form $G_N(f_1 W^{n_1}, \ldots, f_s W^{n_s})$, where $G_N(Y_1, \ldots, Y_s) \in \mathcal{O}_U[Y_1, \ldots, Y_s]$ is weighted homogeneous of degree $N$, provided each $Y_j$ has weight $n_j$. The reverse inclusion is now clear. $\square$

A monoidal transformation (1.0.1) is said to be *permissible* for $\mathcal{G}$ if $Y \subset Sing(\mathcal{G})$. In such case, for each index $k \geq 1$, there is a sheaf of ideals, say $I_k^{(1)} \subset \mathcal{O}_{V_1}$, so that

$$I_k\mathcal{O}_{V_1} = I(H)^k I_k^{(1)}.$$

One can easily check that

$$\mathcal{G}_1 = \bigoplus_{k\geq 0} I_k^{(1)} W^k$$

is a Rees algebra over $V_1$, which we call the *transform* of $\mathcal{G}$.

Let $\mathcal{G} = \bigoplus_{k\geq 0} I_k W^k$ be a Rees algebra on $V$, $U \subset V$ an affine open set, and let $\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$ be such that the restriction of $\mathcal{G}$ to $U$ is

$$\mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}](\subset \mathcal{O}_V(U)[W]).$$

**Proposition 1.3.** *Let $V \leftarrow V_1$ be a permissible transformation of $\mathcal{G}$. There is an open covering of $\pi^{-1}(U)$ by affine sets $U^{(l)}$, so that:*

*1) $\langle f_i \rangle = I(H \cap U^{(l)})^{n_i} \langle f'_i \rangle$ for suitable $f'_i \in \mathcal{O}_{V_1}(U^{(l)})$.*

*2) The restriction of $\mathcal{G}_1$ to $U^{(l)}$ is*

$$\mathcal{O}_{V_1}(U^{(l)})[f'_1 W^{n_1}, \ldots, f'_s W^{n_s}](\subset \mathcal{O}_{V_1}(U^{(l)})[W]).$$

*Proof.* 1) follows from Prop 1.2. For 2) argue as in the proof of Prop 1.2, by using the fact that each ideal $I_N$ is generated by weighted homogeneous polynomials on the element of $\mathcal{F}$.

Given two Rees algebras over $V$, say $\mathcal{G} = \bigoplus_{k\geq 0} I_k W^k$ and $\mathcal{G}' = \bigoplus_{k\geq 0} J_k W^k$, set $K_k = I_k + J_k$ in $\mathcal{O}_V$, and define:

$$\mathcal{G} \odot \mathcal{G}' = \bigoplus_{k\geq 0} K'_k W^k,$$

as the subalgebra of $\mathcal{O}_V[W]$ generated by $\{K_k W^k, k \geq 0\}$.

One can check that:

1) $Sing(\mathcal{G} \odot \mathcal{G}') = Sing(\mathcal{G}) \cap Sing(\mathcal{G}')$. In particular, if $\pi$ in (1.0.1) is permissible for $\mathcal{G} \odot \mathcal{G}'$, it is also permissible for $\mathcal{G}$ and for $\mathcal{G}'$.

2) Set $\pi$ as in 1), and let $(\mathcal{G} \odot \mathcal{G}')_1$, $\mathcal{G}_1$, and $\mathcal{G}'_1$ denote the transforms at $V_1$. Then:

$$(\mathcal{G} \odot \mathcal{G}')_1 = \mathcal{G}_1 \odot \mathcal{G}'_1.$$

## 2. Integral closure of Rees algebras and a notion of equivalence.

We say that two Rees algebras over $V$, say $\mathcal{G} = \bigoplus_{k\geq 0} I_k W^k$ and $\mathcal{G}' = \bigoplus_{k\geq 0} J_k W^k$, are *equivalent*, if both have the same integral closure in $\mathcal{O}_V[W]$.

If $\mathcal{G}$ and $\mathcal{G}'$ are equivalent, then:

1) $Sing(\mathcal{G}) = Sing(\mathcal{G}')$. In particular, $\pi$ in (1.0.1) is permissible for $\mathcal{G}$ if and only if it is so for $\mathcal{G}'$.

2) Set $\pi$ as in 1), and let $\mathcal{G}_1$ and $\mathcal{G}'_1$ denote the transforms at $V_1$. Then $\mathcal{G}_1$ and $\mathcal{G}'_1$ are equivalent over $V_1$.

This shows that equivalent Rees algebras define the same closed sets, and the same holds after any sequence of permissible transformations.

Given a smooth scheme $V$, and $(J,b)$ as in 1, we consider the Rees algebra generated over $\mathcal{O}_V$ by $JW^b$ (as graded subring of $\mathcal{O}_V[W]$).

**Proposition 2.1.** *If $\mathcal{G}$ and $\mathcal{G}'$ are the Rees algebras corresponding to Hironaka's pairs $(J,b)$ and $(J',b')$, then $\mathcal{G}\odot\mathcal{G}'$ is equivalent to the Rees algebra assigned to $(J,b)\odot(J',b')$.*

*Proof.* Fix an affine open set $U$ in $V$, $\{f_1,\dots,f_s\}\in\mathcal{O}_V(U)$ generators of $J(U)$, and $\{g_1,\dots,g_r\}\in\mathcal{O}_V(U)$ generators of $J'(U)$. Then:

i) The restriction of $\mathcal{G}$ to $U$ is

$$\mathcal{O}_V(U)[f_1W^b,\dots,f_sW^b](\subset\mathcal{O}_V(U)[W]).$$

ii) The restriction of $\mathcal{G}'$ is

$$\mathcal{O}_V(U)[g_1W^{b'},\dots,g_rW^{b'}](\subset\mathcal{O}_V(U)[W]).$$

iii) The restriction of $\mathcal{G}\odot\mathcal{G}'$ to $U$ is

$$\mathcal{O}_V(U)[f_1W^b,\dots,f_sW^b,g_1W^{b'},\dots,g_rW^{b'}](\subset\mathcal{O}_V(U)[W]).$$

iv) The restriction of the Rees algebra assigned to $(J,b)\odot(J',b')$ is generated by

$$\{(f_1^{\alpha_1}\cdots f_s^{\alpha_s})\cdot W^{bb'};(g_1^{\beta_1}\cdots g_s^{\beta_s})\cdot W^{bb'}/\alpha_1+\cdots+\alpha_s=b';\beta_1+\cdots+\beta_r=b\}.$$

One can finally check that both algebras in (iii) and (iv) have the same integral closure in $\mathcal{O}_V(U)[W]$.

□

## 3. On differential structures and Kollár's tuned ideals.

Here $V$ is smooth over a field $k$, so for each non-negative integer $r$ there is a locally free sheaf of differential operators of order $r$, say $Diff_k^r$.

**Definition 3.1.** We say that a Rees algebra $\bigoplus I_nW^n$ is a Diff-structure relative to the field $k$, if:

i) $I_n\supset I_{n+1}$.

ii) There is open covering of $V$ by affine open sets $\{U_i\}$, and for any $D\in Diff^{(r)}(U_i)$, and any $h\in I_n(U_i)$, then $D(h)\in I_{n-r}(U_i)$ provided $n\geq r$.

Given a sheaf of ideals $I\in\mathcal{O}_V$ there is a natural definition of an extension, say $Diff^{(r)}(I)$ (see Introduction). Note that (ii) can be reformulated by

ii') $Diff^{(r)}(I_n)\subset I_{n-r}$ for each $n$, and $0\leq r\leq n$.

Fix a closed point $x\in V$, and a regular system of parameters $\{x_1,\dots,x_n\}$ at $\mathcal{O}_{V,x}$. The residue field, say $k'$ is a finite extension of $k$, and the completion $\hat{\mathcal{O}}_{V,x}=k'[[x_1,\dots,x_n]]$.

The Taylor development is the continuous $k'$-linear ring homomorphism:

$$Tay : k'[[x_1, \ldots, x_n]] \to k'[[x_1, \ldots, x_n, T_1, \ldots, T_n]]$$

that map $x_i$ to $x_i + T_i$, $1 \leq i \leq n$. So for $f \in k'[[x_1, \ldots, x_n]]$, $Tay(f(x)) = \sum_{\alpha \in \mathbb{N}^n} g_\alpha T^\alpha$, with $g_\alpha \in k'[[x_1, \ldots, x_n]]$.

Define, for each $\alpha \in \mathbb{N}^n$, $\Delta^\alpha(f) = g_\alpha$. It turns out that

$$\Delta^\alpha(\mathcal{O}_{V,x}) \subset \mathcal{O}_{V,x},$$

and that $\{\Delta^\alpha, \alpha \in (\mathbb{N})^n, 0 \leq |\alpha| \leq c\}$ generate the $\mathcal{O}_{Z,x}$-module $Diff_k^c(\mathcal{O}_{Z,x})$ (i.e. generate $Diff_k^c$ locally at $x$).

**Theorem 3.2.** *For any Rees algebra $\mathcal{G}$ over a smooth scheme $V$, there is a Diff-structure, say $G(\mathcal{G})$ such that:*

*i) $\mathcal{G} \subset G(\mathcal{G})$.*

*ii) If $\mathcal{G} \subset \mathcal{G}'$ and $\mathcal{G}'$ is a Diff-structure, then $G(\mathcal{G}) \subset \mathcal{G}'$.*

*Furthermore, if $x \in V$ is a closed point, and $\{x_1, \ldots, x_n\}$ is a regular system of parameters at $\mathcal{O}_{V,x}$, and $\mathcal{G}$ is locally generated by*

$$\mathcal{F} = \{g_{n_i} W^{n_i}, n_i > 0, 1 \leq i \leq m\},$$

*then*

$$\mathcal{F}' = \{\Delta^\alpha(g_{n_i}) W^{n'_i - \alpha} / g_{n_i} W^{n_i} \in \mathcal{F}, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N})^n, \text{ and } 0 \leq |\alpha| < n'_i \leq n_i\} \tag{3.2.1}$$

*generates $G(\mathcal{G})$ locally at $x$.*

**Remark 3.3.** The local description in the Theorem shows that $Sing(\mathcal{G}) = Sing(G(\mathcal{G}))$.

In fact, as $\mathcal{G} \subset G(\mathcal{G})$, it is clear that $Sing(\mathcal{G}) \supset Sing(G(\mathcal{G}))$. For the converse note that if $\nu_x(g_{n_i}) \geq n_i$, then $\Delta^\alpha(g_{n_i})$ has order at least $n_i - |\alpha|$ at the local ring $\mathcal{O}_{V,x}$.

**3.4.** In general $\mathcal{G} \subset G(\mathcal{G})$, and equality holds if $\mathcal{G}$ is already a Diff-structure.

Let $\mathcal{G} = \bigoplus_{k \geq 0} I_r W^r$ be a Diff-structure, in particular it is integral over a Rees subring, say $\mathcal{O}_V[I_N W^N]$ for suitable $N$ (see 1.1). These ideals $I_N$ are called *tuned ideals* in [25], page 45.

The previous Theorem defines an operator $G$ that extends Rees algebras into Diff-structures. Another natural operator we have considered on Rees algebras it that defined by taking normalization. The next Theorem relates both notions of extensions.

**Theorem 3.5.** *Let $\mathcal{G}$ and $\mathcal{G}'$ be equivalent Rees algebras on a smooth scheme $V$, then $G(\mathcal{G})$ and $G(\mathcal{G}')$ are also equivalent (in the sense of 2).*

(see Th 6.12 [33]).

**Definition 3.6.** Fix $\mathcal{G} = \bigoplus I_k \cdot W^k$, a Rees algebra on $V$, and let $V \longleftarrow V'$ be a morphism of smooth schemes. We define the *total transform* of $\mathcal{G}$ to be

$$\pi^{-1}(\mathcal{G}) = \bigoplus I_k \mathcal{O}_{V'} \cdot W^k.$$

Namely the Rees algebra defined by the total transforms of the ideals $I_n$, $n \geq 0$.

**Theorem 3.7.** *Let $V' \xrightarrow{\pi} V$ be a morphism of smooth schemes, then:*
*i) if $\mathcal{G}$ is a Diff-structure on $V$, the total transform $\pi^{-1}(\mathcal{G})$ is a Diff-structure on $V'$.*
*ii) $Sing(\pi^{-1}(\mathcal{G})) = \pi^{-1}(Sing(\mathcal{G}))$.*

(See Th 5.4 [33])

## 4. On differential structures and monoidal transformations.

Let us briefly recall some previous results, where now $J \subset \mathcal{O}_V$ be the sheaf of ideals defining a hypersurface $X$ in the smooth scheme $V$.

So $Diff^0(J) = J$, and for each positive integer $s$ there is an inclusion $Diff^s(J) \subset Diff^{s+1}(J)$ as sheaves of ideals in $\mathcal{O}_V$, and hence $V(Diff^s(J)) \supset V(Diff^{s+1}(J))$.

Recall that $b$ is the highest multiplicity at points of $X$, if and only if $V(Diff^b(J)) = \emptyset$ and $V(Diff^{b-1}(J)) \neq \emptyset$ (i.e. if and only if $Diff^b(J) = \mathcal{O}_V$ and $Diff^{b-1}(J)$ is a proper sheaf of ideals).

The closed set of interest is the set of $b$-fold points of $X$ (i.e. $V(Diff^{b-1}(J))$). Consider now a $b$-permissible transformation, say

$$\begin{array}{ccc} V & \overset{\pi}{\longleftarrow} & V_1 \\ \cup & & \cup \\ Y & & \pi^{-1}(Y) = H \end{array}$$

(i.e. the blow up of $V$ at a smooth sub-scheme $Y$). In such case

$$J\mathcal{O}_{W_1} = I(H)^b J_1,$$

where $I(H)$ is the sheaf of functions vanishing along the exceptional hypersurface $H$.

In this case $J_1$ is the sheaf of ideals defining a hypersurface $X_1 \subset V_1$, which is the strict transform of the hypersurface $X$.

It is not hard to check that $J_1$ has at most order $b$ at points of $V_1$ (i.e. that $V(Diff^b(J_1)) = \emptyset$). If, in addition, $J_1$ has no point of order $b$, then we say that $\pi$ defines a $b$-*simplification* of $J$. At any rate, the closed set of interest is the set of $b$-fold points $X_1$.

If $V(Diff^{b-1}(J_1)) \neq \emptyset$, let $V_1 \overset{\pi_1}{\longleftarrow} V_2$ denote the monoidal transformation with center $Y_1 \subset V(Diff^{b-1}(J_1))$. So $\pi_1$ is $b$- permissible, and set

$$J_1\mathcal{O}_{V_2} = I(H_1)^b J_2.$$

So again $J_2$ has at most points of order $b$, and if it does, define a $b$-permissible transformation at some smooth center $Y_2 \subset V(Diff^{b-1}(J_2)))$.

So for $J$ and $b$ as before, we define, by iteration, a $b$-permissible sequence

$$V \overset{\pi}{\longleftarrow} V_1 \overset{\pi_1}{\longleftarrow} V_2 \overset{\pi_2}{\longleftarrow} \dots V_r \overset{\pi_r}{\longleftarrow} V_{r+1},$$

and a factorization $J_{n-1}\mathcal{O}_{V_n} = I(H_n)^b J_n$. Where $J_n$ is the sheaf of ideals defining a hypersurface $X_i \subset V_i$, which is the strict transform of $X$.

From the point of view of resolution it is clear that our interest is to define a $b$-permissible sequence so that $X_{r+1}$ has no $b$-fold points.

We say that a $b$-permissible sequence defines a $b$-*simplication* of $J \subset \mathcal{O}_W$ if the jacobian of $V \leftarrow V_{r+1}$ has normal crossings, and $V(Diff^{b-1}(J_{r+1})) = \emptyset$ (i.e. if $X_{r+1}$ has at most points of multiplicity $b-1$).

Hironaka attaches to the original data $J$ and $b$ the pair $(J, b)$. The closed set assigned to this pair in $V$ is $\{ord_J \geq b\} = V(Diff^{b-1}(J))$. In our case, the $b$-fold points of the hypersurface $X$.

We attached to the original data a Rees algebra (up to integral closure), namely $\mathcal{G} = \mathcal{O}_V[JW^b]$. And to this Rees algebra a closed set in $V$, namely $Sing(\mathcal{G})$, which is again $V(Diff^{b-1}(J))$.

Moreover, we extended $\mathcal{G}$ to a Diff-structure $G(\mathcal{G})$, and $Sing(\mathcal{G}) = Sing(G(\mathcal{G}))$ (Th. 3.2).

Let us focus on the $b$-permissible transformation $\pi$. The transform of Hironaka's pair is the pair $(J_1, b)$. The transformation $\pi$ is also permissible for both $\mathcal{G}$ and $G(\mathcal{G})$, defining transforms of Rees algebras, say $\mathcal{G}_1$ and $G(\mathcal{G})_1$ on $V_1$.

Note that, in our setting, $J_1$ is the ideal defining defining $X_1$, which is the strict transform of $X$. The closed set assigned to $(J_1, b)$ is the set of $b$-fold points of $X_1$. On the other hand, $\mathcal{G}_1 = \mathcal{O}_{V_1}[J_1W^b]$, is such that $Sing(\mathcal{G}_1)$ is again the set of $b$-fold points $X_1$. A similar relation holds between pairs $(J_i, b)$ and the Rees algebras $\mathcal{G}_i$ (transform of $\mathcal{G}$), for any $b$-permissible sequence.

The natural question is on how do the successive transforms of $G(\mathcal{G})$ relate to the transforms of $\mathcal{G}$. The following theorem will address this question (see Th 7.6 [33]). It proves that the $G$-operator on Rees algebras is, in a natural way, compatible with transformation.

**Theorem 4.1.** *(J. Giraud) Let $\mathcal{G}$ be a Rees algebra on a smooth scheme $V$, and let $V \longleftarrow V_1$ be a permissible (moniodal) transformation for $\mathcal{G}$. Let $\mathcal{G}_1$ and $G(\mathcal{G})_1$ denote the transforms of $\mathcal{G}$ and $G(\mathcal{G})$. Then:*

*1) $\mathcal{G}_1 \subset G(\mathcal{G})_1$.*

*2) $G(\mathcal{G}_1) = G(G(\mathcal{G})_1)$.*

**4.2.** Hironaka considers the notion of Diff-structures in [23] and also in [24]. In this last paper he provides an interesting geometric interpretation of the elements of the integral closure of a Diff-structure, say $\overline{G(\mathcal{G})}$, which we briefly discuss below.

Recall that given an ideal $J$ in a smooth scheme $V$, and a positive integer $b$, Hironaka defines a pair $(J, b)$ (actually a closely related notion of *idealistic exponent*). As mentioned in Section 1, there is a closed set in $V$ attached to the pair, and also a notion of *permissible* transforms of pairs.

We have assign a Rees algebra to $(J, b)$, say $\mathcal{G} = \mathcal{O}_V[JW^b]$; and a closed set to $\mathcal{G}$, namely $Sing(\mathcal{G})$. We have also defined transformations of of Rees algebras, in accordance to transformations of pairs.

Here we have discussed integral closure of Rees algebras, and also a $G$-operator on Rees algebras, as two different manners to extend a Rees algebra.

These two forms of extension of Rees algebras have a very particular *geometric property*. In fact, both extended algebras define the same closed set, and hence both admit the same transformations. Furthermore, the closed set defined by the transform of $\mathcal{G}$ by a sequence of transformation, is the same closed set defined by the transform of the integral closure of $\mathcal{G}$. Theorem 4.1 asserts that the same holds for the transform of $G$-extension of $\mathcal{G}$.

So given $\mathcal{G}$, it is quite natural to iterate both operators, by taking successively integral closure and Diff-structures, to obtain larger and larger extensions of $\mathcal{G}$ with this geometric property.

The result of Hironaka in [24] says that the $\overline{G(\mathcal{G})}$ is the biggest extension of $\mathcal{G}$ with this property. Namely that $Sing(\mathcal{G}) = Sing(\overline{G(\mathcal{G})})$, and that the same equality of singular locus holds after any sequence of transformations. Theorem 4.1 can also be proved using this geometric characterization of $\overline{G(\mathcal{G})}$. The approach in [33] is different, and does not make use the concept of infinitely near singular point, but rather on technics that will also be useful for [34].

## 5. Idealistic exponents versus basic objects.

Recall that two ideals, say $I$ and $J$, in a normal domain $R$ have the same integral closure if they are equal for any extension to a valuation ring (i.e. if $IS = JS$ for any ring homomorphism $R \to S$ on a valuation ring $S$). The notion extends naturally to sheaves of ideals.

Hironaka considers the following equivalence on pairs $(J, b)$ and $(J', b')$ over a smooth scheme $V$.

**Definition 5.1.** The pairs $(J, b)$ and $(J', b')$ are *idealistic* equivalent on $V$ if $J^{b'}$ and $(J')^b$ have the same integral closure.

**Proposition 5.2.** *Let $(J, b)$ and $(J', b')$ be idealistic equivalent. Then:*

*1) $Sing(J, b) = Sing(J', b')$.*

*Note, in particular, that any monoidal transform $V \leftarrow V_1$ on a center $Y \subset Sing(J, b) = Sing(J', b')$ defines transforms, say $(J_1, b)$ and $((J')_1, b')$ on $V_1$.*

*2)The pairs $(J_1, b)$ and $((J')_1, b')$ are* idealistic *equivalent on $V_1$.*

If two pairs $(J, b)$ and $(J', b')$ be idealistic equivalent over $V$, the same holds for the restrictions to any open subset of $V$, and also for restrictions in the sense of etale topology, and even for smooth topology (i.e. pull-backs by smooth morphisms $W \to V$).

Note that if $(J, b)$ and $(J', b')$ are idealistic equivalent, the they define the same closed set on $V$ (i.e. $Sing(J, b) = Sing(J', b')$), and the same holds for monoidal transformations, pull-backs by smooth schemes, and hence by concatenation of both kinds of transformations. When this last condition holds on the singular locus of two pairs we say that they *define the same close sets*.

**Definition 5.3.** Two pairs $(J, b)$ and $(J', b')$ are *basically* equivalent on $V$, if the define the same close sets.

The proposition says that if two pairs are idealistic equivalent over $V$, then they are basically equivalent.

An *idealistic exponent*, as defined by Hironaka in [23], is an equivalence class of pairs in the sense of idealistic equivalence. Whereas the notion of equivalence among basic objects (see [31] or [32]) is the second one. In fact, the key point for constructive desingularization was to define an algorithm of resolutions of pairs $(J, b)$, so that two basically equivalent pairs undergo exactly the same resolution.

**5.4.** There are two notions of equivalence on the context of Rees algebras over $V$. The first, already formulated in Section 2:

**Definition 5.5.** Two Rees algebras over $V$, say $\mathcal{G} = \bigoplus_{k\geq 0} I_k W^k$ and $\mathcal{G}' = \bigoplus_{k\geq 0} J_k W^k$, are *integrally equivalent*, if both have the same integral closure.

**Proposition 5.6.** *Let $\mathcal{G}$ and $\mathcal{G}'$ be two integrally equivalent Rees algebras over $V$ Then:*

*1) $Sing(\mathcal{G}) = Sing(\mathcal{G}')$.*

*Note, in particular, that any monoidal transform $V \leftarrow V_1$ on a center $Y \subset Sing(\mathcal{G}) = Sing(\mathcal{G}')$ defines transforms, say $(\mathcal{G})_1$ and $(\mathcal{G}')_1$ on $V_1$.*

*2)$(\mathcal{G})_1$ and $(\mathcal{G}')_1$ are integrally equivalent on $V_1$.*

If $\mathcal{G}$ and $\mathcal{G}'$ are *integrally* equivalent on $V$, the same holds for any open restriction, and also for pull-backs by smooth morphisms $W \to V$.

On the other hand, as $(\mathcal{G})_1$ and $(\mathcal{G}')_1$ are integrally equivalent, the they define the same closed set on $V_1$ (the same singular locus), and the same holds for further monoidal transformations, pull-backs by smooth schemes, and concatenations of both kinds of transformations.

When this condition holds on the singular locus of two Rees algebras over $V$, we say that they *define the same close sets*.

**Definition 5.7.** Two Rees algebras over $V$, say $\mathcal{G} = \bigoplus_{k\geq 0} I_k W^k$ and $\mathcal{G}' = \bigoplus_{k\geq 0} J_k W^k$, are basically equivalent, if both define the same closed sets.

The previous Proposition asserts that if $\mathcal{G} = \bigoplus_{k\geq 0} I_k W^k$ and $\mathcal{G}' = \bigoplus_{k\geq 0} J_k W^k$ are integrally equivalent, then they are basically equivalent.

**5.8.** We assign to a pair $(J, b)$ over a smooth scheme $V$ the Rees algebra, say:

$$\mathcal{G}_{(J,b)} = \mathcal{O}_V[J^b W^b],$$

which is a graded subalgebra in $\mathcal{O}_V[W]$.

**Proposition 5.9.** *1) Two pairs $(J, b)$ and $(J', b')$ are idealistically equivalent over a smooth scheme $V$, if and only if the Rees algebras $\mathcal{G}_{(J,b)}$ and $\mathcal{G}_{(J',b')}$ are integrally equivalent.*

*2) Two pairs $(J, b)$ and $(J', b')$ are basically equivalent over $V$, if and only if the Rees algebras $\mathcal{G}_{(J,b)}$ and $\mathcal{G}_{(J',b')}$ are basically equivalent.*

## 6. Projection of differential structures and elimination of one variable.

**6.1.** The notion of Rees algebra $\mathcal{G} = \bigoplus_{k\geq 1} I_k W^k$ parallels that of idealistic exponents in [23], and the notion of singular locus $Sing(\mathcal{G})$, is the natural analog for that defined for idealistic exponents.

We finally introduce a function, again a natural analog to that defined for idealistic exponents. Fix $x \in \mathrm{Sing}\ (\mathcal{G})$. Given $f_n W^n \in I_n W^n$, set

$$ord_x(f_n) = \frac{\nu_x(f_n)}{n} \in \mathbb{Q};$$

called the order of $f_n$ (weighted by $n$), where $\nu_x$ denotes the order at the local regular ring $\mathcal{O}_{Z,x}$. As $x \in \mathrm{Sing}\ (\mathcal{G})$ it follows that $ord_x(f_n) \geq 1$. We also define

$$ord_x(\mathcal{G}) = inf\{ord_x(f_n); f_n W^n \in I_n W^n\}.$$

So, in general $ord_x(\mathcal{G}) \geq 1$ for any $x \in \mathrm{Sing}\ (\mathcal{G})$.

**Proposition 6.2.** *1) If $\mathcal{G}$ is a Rees algebra generated over $\mathcal{O}_Z$ by $\mathcal{F} = \{g_{n_i} W^{n_i}, n_i > 0, 1 \leq i \leq m\}$, then*

$$ord_x(\mathcal{G}) = inf\{ord_x(g_{n_i}); 1 \leq i \leq m\}.$$

*And if $N$ is any common multiple of all $n_i, 1 \leq i \leq m$, then $ord_x(\mathcal{G}) = \frac{\nu(I_N)}{N}$.*

*2) If $\mathcal{G}$ and $\mathcal{G}'$ are graded structures with the same integral closure (e.g. if $\mathcal{G} \subset \mathcal{G}'$ is a finite extension), then, for any $x \in Sing(\mathcal{G})(= Sing(\mathcal{G}'))$*

$$ord_x(\mathcal{G}) = ord_x(\mathcal{G}').$$

*3) Set $G(\mathcal{G}) = \mathcal{G}'' = \bigoplus I''_n \cdot W^n$ (the extension of $\mathcal{G}$ to a differential structure), then for any $x \in Sing(\mathcal{G})(= Sing(\mathcal{G}''))$.*

$$ord_x(\mathcal{G}) = ord_x(\mathcal{G}'').$$

**6.3.** Let $\mathcal{G}$ be a Rees algebra, and fix a closed point $x \in Sing(\mathcal{G})$. We assume that at a affine open neighborhood of the point, say $U \subset V$, there is a finite set $\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$, $n_i \geq 1$ and $f_i \in \mathcal{O}_V(U)$, so that the restriction of $\mathcal{G}$ to $U$ is

$$\mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}](\subset \mathcal{O}_V(U)[W]).$$

Let

$$\mathcal{G}_x = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_{V,x}[W])$$

be the *localization* of $\mathcal{G}$ at $x$. As $x \in Sing(\mathcal{G})$, the order of $I_k$ at $\mathcal{O}_{V,x}$ is at least $k$. We say that $\mathcal{G}$ is *simple* at the singular point $x$, if for some positive index $k$, $I_k$ has order $k$. This amounts to saying that $ord_x(\mathcal{G}) = 1$; or equivalently, that for some $f_c W^{n_c} \in \mathcal{F}$, the element $f_c$ has order $n_c$ at $\mathcal{O}_{V,x}$.

Recall that $V(Diff^{n_c - 1}(< f_c >)) \subset Sing(\mathcal{G})$ locally at $x$.

We may choose the system of parameters $\{x_1, \ldots, x_n\}$ at $x$, so that at the completion $\hat{\mathcal{O}}_{V,x}$, say $R = k'[[x_1, .., x_n]]$:

$$u.f_c = Z^{n_c} + a_1 Z^{n_c - 1} + \cdots + a_{n_c} \in S[Z]$$

$S = k[[x_1, .., x_{n-1}]]$, and $Z = x_n$; where $u$ is a unit of $R$.

A similar result holds at a suitable étale neighborhood of $x$. We may assume that $f_{c_1}$ is a monic polynomial of degree $c_1$ in $S[Z]$, and of order $c_1$ in $S[Z]_{<M_S, Z>} \subset R$, where $S$ is regular.

Let $\pi : V \to V'$ be a smooth morphism defined at an étale neighborhood of $x$, where $V'$ is smooth, dim $V'$=dim $V$-1. We say that $\pi$ is *transversal* to $\mathcal{G}$ at $x$, if the previous setting holds for $R = \hat{\mathcal{O}}_{V,x}$, $S = \hat{\mathcal{O}}_{V',\pi(x)}$; and for some $f_c W^{n_c} \in \mathcal{F}$, where $f_c$ has order $n_c$ at $\mathcal{O}_{V,x}$.

In these conditions, a transversal morphism $\pi$, induces a finite morphism

$$\overline{\pi} : Spec(S[Z]/\langle f_{c_1}(Z)\rangle) \to Spec(S)\rangle.$$

Here we view $H = Spec(S[Z]/\langle f_{c_1}(Z)\rangle)$ as a hypersurface in $V$, and locally at $x$, $Sing(\mathcal{G})$ is included in the $c_1$-fold points of this hypersurface. So

$$\{ord\ f_{c_1} \geq n_{c_1}\} := V(Diff^{c_i - 1}(\langle f_{c_1}(Z)\rangle)) \subset H.$$

In this setting the finite morphism $\pi$ is one to one over a closed subset of $Y$, namely on the image of the $c_1$-fold points. Set

$$\begin{array}{ccc} Spec(S[Z]/ < f_{c_1}(Z) >) & \stackrel{\overline{\pi}}{\longrightarrow} & Spec(S) \\ \cup & & \cup \\ \{ord\ f_{c_1} \geq n_{c_1}\} & \stackrel{1\ to\ 1}{\longrightarrow} & \pi(\{ord\ f_{c_1} \geq n_{c_1}\}) \end{array} \tag{6.3.1}$$

Since $Sing(\mathcal{G}) \subset \{ord\ f_{c_1} \geq n_{c_1}\}$, $\pi$ induces a one to one map, say

$$Sing(\mathcal{G}) \stackrel{1\ to\ 1}{\longrightarrow} \pi(Sing(\mathcal{G})),$$

for any transversal morphism $\pi : Spec(S[Z]) \to Spec(S)$.

**Theorem 6.4.** *Let $\mathcal{G}$ be a Diff-structure over a smooth scheme $V$, and $x \in Sing(\mathcal{G})$ a closed point which we assume to be simple. Let $\pi : V \to V'$ be a smooth morphism defined at an étale neighborhood of $x$, where $V'$ is smooth, dim $V'$=dim $V$-1. Assume that $\pi$ is transversal at $x$. Then:*

*1) At a suitable neighborhood of $\pi(x)$, there is a Rees algebra $\mathcal{R}_{\mathcal{G}}$ over the smooth scheme $V'$, so that $\pi(Sing(\mathcal{G})) = Sing(\mathcal{R}_{\mathcal{G}})$.*

*2) The morphism $\pi$ induces a one-to-one map from $Sing(\mathcal{G})$ to $Sing(\mathcal{R}_{\mathcal{G}})$. Furthermore, setting $S = \mathcal{O}_{V',\pi(x)}$, and $S[Z]$ as before, then the one-to-one map is that described above.*

The formulation of the theorem is independent of the choice of $f_{c_1}$ of order $n_{c_1}$ at $\mathcal{O}_{V,x}$. However given a finite morphisms as that in (6.3.1), and a smooth center $Y_1 \subset Sing(\mathcal{R}_{\mathcal{G}})$, there is a unique and smooth center $Y \subset Sing(\mathcal{G})$ mapping isomorphically to $Y_1$ via $\overline{\pi}$ (and hence via $\pi$). Set $Y_1 = \pi(Y)$.

So both $Y$ in $V$, and $\pi(Y)$ in $Spec(S)$, are regular centers.

Let now $V \leftarrow V_1$, and $Spec(S) \leftarrow U_1$, denote the monoidal transformations at $Y$ and $\pi(Y)$ respectively; and let $H'$ denote the strict transform of $H$. The hypersurface $H'$ has at most points of multiplicity $n_{c_1}$. Let $F (\subset H')$ denotes the closed set of points of multiplicity $n_{c_1}$. After replacing $V_1$ by a suitable neighborhood of $F$, we may assume that there is a finite morphism, say $H' \to U_1$, compatible with $\pi$.

As the regular center $Y$ was chosen in $Sing(\mathcal{G})$, then a weighted transform, say

$$\mathcal{G}_1 = \bigoplus I_n^{(1)} \cdot W^k (\subset \mathcal{O}_{V_1}[W])$$

is defined, and $Sing(\mathcal{G}_1) \subset F$. So locally at a point $y \in Sing(\mathcal{G}_1)$ there is a finite morphism

$$\pi' : Spec(S'[Z]/\langle f'_{c_1}(Z)\rangle) \to U_1,$$

where $f'_{c_1}$ is a strict transform of $f_{c_1}$. Let $\mathcal{G}'_1$ be the Diff-structure generated by $\mathcal{G}_1$. According to the previous Theorem, locally at $\pi'(y)$ there is an elimination algebra, say

$$\mathcal{R}_{\mathcal{G}'_1} \subset \mathcal{O}_{U_1,\pi'(y)}[W].$$

On the other hand, $Y_1 = \pi(Y) \subset Sing(\mathcal{R}_{\mathcal{G}})$, so there is also a weighted transform

$$(\mathcal{R}_{\mathcal{G}})_1 \subset \mathcal{O}_{U_1}[W].$$

The question now is to relate the Rees algebra $(\mathcal{R}_{\mathcal{G}})_1$ with $\mathcal{R}_{\mathcal{G}'_1}$, locally at the point $\pi(y)$.

**Proposition 6.5.** *With the setting as above:*

*1) There is a natural inclusion* $(\mathcal{R}_{\mathcal{G}})_1 \subset \mathcal{R}_{\mathcal{G}'_1}$.

*2) Over fields of characteristic zero both* $(\mathcal{R}_{\mathcal{G}})_1$ *and* $\mathcal{R}_{\mathcal{G}'_1}$ *define the same Diff-structure, up to integral closure.*

Here $(\mathcal{R}_{\mathcal{G}})_1$ is the transform of $\mathcal{R}_{\mathcal{G}}$ by one monoidal transformation. If we could guarantee that $Sing(\mathcal{R}_{\mathcal{G}})_1 = \pi'(Sing(\mathcal{G}_1)$, we could identify the singular locus of $\mathcal{G}_1$ (i.e. of $\mathcal{G}'_1$) with the singular locus of the transform of $\mathcal{R}_{\mathcal{G}}$. If furthermore, this link between $\mathcal{G}$ and $\mathcal{R}_{\mathcal{G}}$ is preserved by any sequence of monoidal transformations, then we have achieved a way of representing the singular locus of $\mathcal{G}$ which is stable by monoidal transformations.

Part 2) in the previous Proposition ensures that this is the case over fields of characteristic zero, providing an alternative form of stability of elimination (see (C) in Introduction). This is not the case over fields of positive characteristic, but it is the starting point for new invariants in that context.

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