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DOLBEAULT COHOMOLOGY AND DEFORMATIONS OF NILMANIFOLDS

SERGIO CONSOLE

Abstract. In these notes I review some classes of invariant complex structures on nilmanifolds for which the Dolbeault cohomology can be computed by means of invariant forms, in the spirit of Nomizu's theorem for de Rham cohomology.
Moreover, deformations of complex structures are discussed. Small deformations remain in some cases invariant, so that, by Kodaira-Spencer theory, Dolbeault cohomology can be still computed using invariant forms.

1. Introduction

A nilmanifold is a compact quotient $M = G/\Gamma$, where G is a real simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G. The homogeneous space M is complex if it is endowed with an invariant complex structure J.

Nilmanifolds provide examples of symplectic manifolds with no Kähler structure (see for instance [OT]). Indeed Benson and Gordon [BG] proved that a complex nilmanifold is Kähler if and only if it is a torus (see also [H]).

This is one reason for the interest in computing Dolbeault cohomology. Since nilmanifolds are homogeneous spaces, it is natural to expect this to be possible using invariant differential forms. However, whether this is true in general is still an open problem.

In these notes I shall review some classes of complex structures whose Dolbeault cohomology is computed by means of the complex of forms of type (p, q) on the (complexified) Lie algebra, i.e., $H^{*,*}_{\overline{\partial}}(M) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$, where $\mathfrak{g}$ is the Lie algebra of G. Special attention will be devoted to the class of abelian complex structures [BDM, CF, CFP, CFGU, MPPS]. On one hand, this is because proving that $H^{*,*}_{\overline{\partial}}(M) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ is simpler when dealing with abelian complex structures, and suggests how to tackle other situations. On the other hand, and more importantly, abelian complex structures have a very strong property related to deformations: they admit a locally complete family of deformations consisting (solely) of invariant complex structures. Using Kodaira-Spencer theory, one can

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show that $H^{*,*}_{\overline{\partial}}(M,J) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ still holds for any small deformation J of an abelian complex structure (cf. Theorem 4.2).

2. Real nilmanifolds

Recall that G is (k-step) *nilpotent* if and only if there exists a (minimal) integer k such that $0 \neq \mathfrak{g}^{k-1} \supset \mathfrak{g}^k = \{0\}$ where

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]\,. \quad \text{(lower central series)}$$

If G is nilpotent with *rational structure constants* or, equivalently, $\mathfrak{g}$ has a *rational structure* (i.e., there exists a rational $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$) then a result of Mal'čev [M] asserts that there exists a discrete subgroup Γ such that $M = G/\Gamma$ is compact.

The computation of de Rham cohomology can be carried out by means of invariant forms. This is a result of Nomizu going back to the fifties [N]. We will outline the main ideas, since some of them can be used to understand Dolbeault cohomology.

Let us begin remarking that the complex $\bigwedge^* M$ of differential forms on M can be regarded as that of Γ-invariant forms on G. On the other hand, the complex of G-invariant forms on G can be identified with the exterior algebra $\bigwedge^* \mathfrak{g}^*$. By this, the differential of $\alpha \in \bigwedge^k \mathfrak{g}^*$ is given by

$$d\alpha(x_1, \ldots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} \alpha([x_i, x_j], x_1, \ldots \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1})$$

The differential complex

$$\cdots \to \bigwedge^{k-1} \mathfrak{g}^* \xrightarrow{d} \bigwedge^{k} \mathfrak{g}^* \xrightarrow{d} \bigwedge^{k+1} \mathfrak{g}^* \to \ldots$$

is usually called the Chevalley-Eilenberg complex.

Theorem 2.1 (Nomizu [N])**.** *$H^k_{dR}(M) \cong H^k(\mathfrak{g})$, for any k, where $H^k(\mathfrak{g})$ is the cohomology of the Chevalley-Eilenberg complex.*

Sketch of the proof: Set $\mathfrak{h} := \mathfrak{g}^{k-1}$ where $\mathfrak{g}^{k-1}$ is the last non-zero term in the lower central series. Observe that $\mathfrak{h}$ is central, in particular abelian. Let H be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Since G is simply-connected and H is connected, H is simply-connected as well. Hence H is diffeomorphic to $\mathbb{R}^n$.

By standard results on discrete subgroups [R] $H\Gamma/H$ and $H \cap \Gamma$ are discrete cocompact subgroups. Thus, $\overline{M} := G/H\Gamma$ is a compact nilmanifold with dimension smaller than M and $\mathbb{T}^n := H/H \cap \Gamma$ is a torus. Moreover, $\mathbb{T}^n \hookrightarrow M \xrightarrow{\pi} \overline{M}$ is a fibration.

Next, consider the Leray-Serre spectral sequence $E^{p,q}_*$ associated with the above fibration. One has

$$\begin{aligned} &E^{p,q}_2 = H^p_{dR}(\overline{M}, H^q_{dR}(\mathbb{T}^n)) \cong H^p_{dR}(\overline{M}) \otimes \textstyle\bigwedge^q \mathbb{R}^n\,, \\ &E^{p,q}_\infty \Rightarrow H^{p+q}_{dR}(M)\,. \end{aligned}$$

The idea is to construct a second spectral sequence $\tilde{E}^{p,q}_*$ which is the Leray-Serre spectral sequence relative to the complex of G-invariant forms (i.e., the

Chevalley-Eilenberg complex). Since $\bigwedge^* \mathfrak{g}^*$ is subcomplex of $\bigwedge^* M$, $\tilde{E}_*^{p,q} \subseteq E_*^{p,q}$ and

$$\begin{aligned} \tilde{E}_2^{p,q} &= H^p(\mathfrak{g}/\mathfrak{h}) \otimes \textstyle\bigwedge^q \mathbb{R}^n\,, \\ \tilde{E}_\infty^{p,q} &\Rightarrow H^{p+q}(\mathfrak{g})\,. \end{aligned}$$

Now, since $\overline{M}$ is a nilmanifold of lower dimension than M, an inductive argument on the dimension shows that $H^p_{dR}(\overline{M}) \cong H^p(\mathfrak{g}/\mathfrak{h})$ for any p. Thus $E_2 = \tilde{E}_2$ and one has the equality $E_\infty = \tilde{E}_\infty$. Therefore, $H^k_{dR}(M) \cong H^k(\mathfrak{g})$ for any k. □

3. Complex structures

We now assume that M has an *invariant complex structure* J, so that J comes from a (left-invariant) complex structure J on $\mathfrak{g}$. Recall that J determines a decomposition of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ and of its dual

$$\left\{ \begin{array}{lcl} \mathfrak{g}^{\mathbb{C}} &=& \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1} \\ (\mathfrak{g}^{\mathbb{C}})^* &=& \bigwedge^{1,0}(\mathfrak{g}^{\mathbb{C}})^* \oplus \bigwedge^{0,1}(\mathfrak{g}^{\mathbb{C}})^* = \mathfrak{g}^{*(1,0)} \oplus \mathfrak{g}^{*(0,1)} \end{array} \right.$$

Moreover, J is integrable. This means that the Nijenhuis tensor

$$N_J(Z,W) = [Z,W] + J[JZ,W] + J[Z,JW] - [JZ,JW], \quad Z,W \in \mathfrak{g}$$

vanishes or, equivalently,

$$d\mathfrak{g}^{*(1,0)} \subset \mathfrak{g}^{*(1,1)} \oplus \mathfrak{g}^{*(2,0)}\,.$$

Here $\mathfrak{g}^{*(p,q)} = \bigwedge^{p,q}(\mathfrak{g}^{\mathbb{C}})^*$ is the space of (p,q)-forms on $\mathfrak{g}$ and the differential d is extended to $\mathfrak{g}^{\mathbb{C}}$ by $\mathbb{C}$-linearity.

3.1. **Dolbeault cohomology.** Observe that, by the above relation, J is complex (integrable) if and only if $\overline{\partial}^2 = 0$. This allows to define a Dolbeault complex for $\bigwedge^{p,q}(\mathfrak{g}^{\mathbb{C}})^* = \mathfrak{g}^{*(p,q)}$, i.e., for the $G-$invariant forms of type (p,q) on M. We denote by $H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ the cohomology of this complex. On the other hand, $H^{*,*}_{\overline{\partial}}(M)$ is the cohomology of the Dolbeault complex $(\bigwedge^{p,q}(M), \overline{\partial})$ of Γ-invariant forms on M.

In the next Sections, we will discuss the following

Problem. *For which complex structure J does the isomorphism*

$$(*) \qquad H^{*,*}_{\overline{\partial}}(M) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$$

hold?

In general, we have the following

Lemma 3.1 ([CF])**.** *The inclusion $\bigwedge^{p,q}(\mathfrak{g}^{\mathbb{C}}) \hookrightarrow \bigwedge^{p,q}(M)$ induces an injective morphism*

$$H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}}) \overset{j}{\hookrightarrow} H^{*,*}_{\overline{\partial}}(M)\,.$$

Proof. If $M = G/\Gamma$ is endowed with an invariant Hermitian metric, $\overline{\partial}$ and its adjoint $\overline{\partial}^*$ preserve G-invariant forms but also the forms orthogonal to the G-invariant ones. Thus, if $j[\omega] = 0$, for some $[\omega] \in H^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$, we may assume

$j(\omega) = \overline{\partial}\varphi$ with φ lying in the orthogonal complement of the G-invariant forms. Therefore, $\overline{\partial}^*\overline{\partial}\varphi$ is a G-invariant form. Since

$$0 = \langle \varphi, \overline{\partial}^*\overline{\partial}\varphi\rangle = \langle \overline{\partial}\varphi, \overline{\partial}\varphi\rangle\,,$$

$\overline{\partial}\varphi$ must be zero, which implies $[\omega] = 0$ in $H^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$. □

We let $H^{\perp *,*}_{\overline{\partial}}(M)$ denote the cohomology of the Dolbeault complex of Γ-invariant forms orthogonal to the G-invariant ones [CF].

The isomorphism $H^{*,*}_{\overline{\partial}}(M) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ can be rephrased as

$$(**) \qquad \dim H^{\perp *,*}_{\overline{\partial}}(M) = 0\,.$$

We begin examining some important subclasses of complex structures for which the isomorphism $(*)$ (or equivalently $(**)$) holds.

3.2. **Complex parallelizable nilmanifolds.** A nilmanifold $M = G/\Gamma$ is called *complex parallelizable* if G is a complex Lie group, or equivalently

$$d\mathfrak{g}^{*(1,0)} \subset \mathfrak{g}^{*(2,0)}$$

It is known that for complex parallelizable nilmanifolds the isomorphism (*) holds. Actually one can say more.

Theorem 3.2 (Sakane [S])**.** *If M is a complex parallelizable nilmanifold*

$$H^{p,q}_{\overline{\partial}}(M) \cong \bigwedge^p \mathfrak{g}^{*(1,0)} \otimes H^q(\mathfrak{g}^{*(0,1)})\,,$$

where $H^q(\mathfrak{g}^{*(0,1)})$ is the cohomology of the complex

$$\cdots \to \bigwedge^q \mathfrak{g}^{*(0,1)} = \mathfrak{g}^{*(0,q)} \stackrel{\overline{\partial}}{\to} \bigwedge^{q+1} \mathfrak{g}^{*(0,1)} = \mathfrak{g}^{*(0,q+1)} \stackrel{\overline{\partial}}{\to} \cdots$$

In particular,

$$H^{p,0}_{\overline{\partial}}(M) \cong \bigwedge^p \mathfrak{g}^{*(1,0)} = \mathfrak{g}^{*(p,0)}\,.$$

We shall not prove this, focusing instead on another class of complex structures (the so-called abelian ones). We briefly explain why the Dolbeault cohomology is not simply equal to the cohomology of $G-$invariant forms but the stronger result of Theorem 3.2 holds. Since $d\mathfrak{g}^{*(1,0)} \subset \mathfrak{g}^{*(2,0)}$, $\overline{\partial}\mathfrak{g}^{*(p,0)} = 0$, so the Dolbeault complex $\bigwedge^{p,q}(\mathfrak{g}^{\mathbb{C}})^* = \mathfrak{g}^{*(p,q)}$ splits as

$$\begin{array}{ccccc}
\mathfrak{g}^{*(p,q)} = & \mathfrak{g}^{*(p,0)} & \otimes & \mathfrak{g}^{*(0,q)} \\
 & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \overline{\partial}} \\
\mathfrak{g}^{*(p,q+1)} = & \mathfrak{g}^{*(p,0)} & \otimes & \mathfrak{g}^{*(0,q+1)}
\end{array}$$

In other words, $\overline{\partial}'(\omega \otimes \bar{\omega}') = \omega \otimes \overline{\partial}\bar{\omega}'$, with $\omega \in \mathfrak{g}^{*(p,0)}$ and $\bar{\omega}' \in \mathfrak{g}^{*(0,q)}$. Hence, the cohomology of the Dolbeault complex $(\bigwedge^{p,q}(\mathfrak{g}^{\mathbb{C}})^*, \overline{\partial})$ is the same as the cohomology of $(\mathfrak{g}^{*(p,0)} \otimes \mathfrak{g}^{*(0,q)}, \overline{\partial}')$, which equals $\bigwedge^p \mathfrak{g}^{*(1,0)} \otimes H^q(\mathfrak{g}^{*(0,1)})$.

3.3. **Abelian complex structures.** A complex structure J is called *abelian* [BDM] if $\mathfrak{g}^{1,0}$ is abelian. This means

$$[JX, JY] = [X, Y], \quad \forall X, Y \in \mathfrak{g}\,,$$

or equivalently

$$d\mathfrak{g}^{*(1,0)} \subset \mathfrak{g}^{*(1,1)}\,.$$

Nilmanifolds with abelian complex structures are therefore dual to complex parallelizable ones, in some sense. The following result should thus be read as the counterpart to Theorem 3.2.

Theorem 3.3 ([CF, CFP, CFGU]). *If M is a nilmanifold endowed with an abelian complex structure*

$$H^{p,q}_{\overline{\partial}}(M) \cong H^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})\,.$$

Moreover,

$$H^{0,q}_{\overline{\partial}}(M) = H^q(M, \mathcal{O}_M) \cong \bigwedge{}^q \mathfrak{g}^{*(0,1)} = \mathfrak{g}^{*(0,q)}\,,$$

where $\mathcal{O}_M$ is the structure sheaf of M.

Proof. We use a method very similar to the proof of Nomizu's Theorem. We consider the upper central series $\{\mathfrak{g}_\ell\}$ given by $\mathfrak{g}_0 = \{0\}$, $\mathfrak{g}_1 = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$, $\mathfrak{g}_\ell = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{g}_{\ell-1}\}$, $\mathfrak{g}_k = \mathfrak{g}$.

If J is abelian $J\mathfrak{g}_\ell \subseteq \mathfrak{g}_\ell$ (observe that this is not true for the lower central series).

Moreover, $\mathfrak{g}_1$ is central, in particular abelian, and we have the following exact sequence of Lie algebras

$$0 \to \underset{\text{abelian}}{\mathfrak{g}_1} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}_1 \to 0\,.$$

Let G_1 be the connected Lie subgroup of G with Lie algebra $\mathfrak{g}_1$ (G_1 is actually diffeomorphic to $\mathbb{R}^n$, since it is abelian and simply-connected), and let G^1 be the simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}/\mathfrak{g}_1$.

Given the uniform discrete subgroup $\Gamma \subset G$, since $\mathfrak{g}_1$ is a rational subalgebra of $\mathfrak{g}$, $p_0(\Gamma)$ is uniform in G^1 (cf. [R]). So we have the holomorphic fibration

$$\pi_0 : M = G/\Gamma \to M^1 = G^1/p_0(\Gamma)$$

whose fibre is the torus $\mathbb{T}^n = G_1/\Gamma \cap G_1$.

The main tool to get cohomological information about the total space of this bundle is Borel's spectral sequence [Hi, Appendix II by A. Borel, Theorem 2.1]: *Let $p : P \to B$ be a holomorphic fibre bundle, with compact connected fibre F, and P and B connected. If F is Kähler there is a spectral sequence (E_r, d_r), $(r \geq 0)$ with the following properties:*

1. *E_r is 4-graded by fibre-degree, base-degree and type;*
2. *If $p + q = u + v$, then ${}^{p,q}E_2^{u,v} \cong \sum_k H^{k,u-k}_{\overline{\partial}}(B) \otimes H^{p-k,q-u+k}_{\overline{\partial}}(F)$;*
3. *${}^{p,q}E_\infty^{*,*} \Rightarrow H^{*,*}_{\overline{\partial}}(P)$.*

As for the proof of Nomizu's Theorem, one can use a first Borel spectral sequence E for the complex of forms invariant by the lattice and a second one, denoted by $\tilde{E}$, for the forms invariant by the group action (the forms on the Lie algebra).

We proceed by induction on the dimension. We have

$$\begin{aligned} {}^{p,q}E_2^{u,v} &\cong \textstyle\sum_k H_{\overline{\partial}}^{k,u-k}(M^1) \otimes \bigwedge^{p-k,q-u+k}(\mathbb{C}^n)\,, \\ {}^{p,q}\tilde{E}_2^{u,v} &\cong \textstyle\sum_k H_{\overline{\partial}}^{k,u-k}((\mathfrak{g}/\mathfrak{g}_1)^{\mathbb{C}}) \otimes \bigwedge^{p-k,q-u+k}(\mathbb{C}^n) \end{aligned}$$

Now, M^1 is a nilmanifold with an abelian complex structure of dimension lower than M, so $H_{\overline{\partial}}^{k,u-k}(M^1) \cong H_{\overline{\partial}}^{k,u-k}((\mathfrak{g}/\mathfrak{g}_1)^{\mathbb{C}})$. Thus $E_2 = \tilde{E}_2$ and $E_\infty = \tilde{E}_\infty$. This implies that $H_{\overline{\partial}}^{p,q}(M) \cong H_{\overline{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}})$.

We want to show that $H_{\overline{\partial}}^{0,q}(M) = H^q(M, \mathcal{O}_M) \cong \bigwedge^q(\mathfrak{g}^{*(0,1)})$. Similarly to the complex parallelizable case, we have that $\overline{\partial}\mathfrak{g}^{*(0,q)} = 0$, for any q, since $d\mathfrak{g}^{*(1,0)} \subset \mathfrak{g}^{*(1,1)}$, and the Dolbeault complex $\bigwedge^{0,*}(\mathfrak{g}^{\mathbb{C}})^* = \mathfrak{g}^{*(0,*)}$ is the zero complex. □

One can also compute the Dolbeault cohomology with coefficients in the tangent sheaf of M. This was done in [MPPS] for 2-step nilmanifolds with an abelian complex structure and in [CFP] in the general case.

Theorem 3.4 ([CFP, MPPS]). *Let M be nilmanifold with an abelian complex structure and denote by Θ_M the holomorphic tangent sheaf of M. Then $H^*(M, \Theta_M)$ is canonically isomorphic with the cohomology of a complex constructed using invariant forms and invariant vectors.*

3.4. **Nilpotent complex structures.** An important property of abelian complex structures (used in 3.3) is that the complex structure J preserves the elements of the upper central series $\{\mathfrak{g}_\ell\}$, i.e., $J\mathfrak{g}_\ell \subseteq \mathfrak{g}_\ell$.

Given a complex structure J on a $2n$-dimensional nilpotent Lie algebra $\mathfrak{g}$, one may modify the upper central series so that it is preserved by J. For this purpose, consider the series $\{\mathfrak{g}_\ell^J\}$ defined inductively by $\mathfrak{g}_0^J = \{0\}$ and

$$\mathfrak{g}_\ell^J = \{X \in \mathfrak{g} \,:\, [X, \mathfrak{g}] \subseteq \mathfrak{g}_{\ell-1}^J,\, [JX, \mathfrak{g}] \subseteq \mathfrak{g}_{\ell-1}^J\}, \quad \ell \geq 1.$$

A complex structure J is said to be *nilpotent* [CFGU] if the series satisfies $\mathfrak{g}_k^J = \mathfrak{g}$ for some positive integer k.

Note that an abelian complex structure is always nilpotent, with $\mathfrak{g}_\ell^J = \mathfrak{g}_\ell$, for any $\ell \geq 0$.

Again, for nilpotent complex structures the isomorphism (*) holds.

Theorem 3.5 ([CFGU]). *If M is a nilmanifold endowed with a nilpotent complex structure*

$$H_{\overline{\partial}}^{p,q}(M) \cong H_{\overline{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}})\,.$$

The proof relies on a similar technique to that of abelian complex structures.

3.5. **Rational complex structures.** A complex structure J is called *rational* if it is compatible with the rational structure of the lattice Γ, meaning $J(\mathfrak{g}_{\mathbb{Q}}) \subset \mathfrak{g}_{\mathbb{Q}}$. For this class as well we have

Theorem 3.6 ([CF])**.** *If M is a nilmanifold endowed with a rational complex structure*

$$H^{*,*}_{\overline{\partial}}(M) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})\,.$$

Sketch of the proof: One key point in the proof is to modify the lower central series so that it is preserved by J. For each i, one considers the smallest subalgebra $\mathfrak{g}^i_J$ containing $\mathfrak{g}^i$ which is J invariant. More precisely, $\mathfrak{g}^i_J = \mathfrak{g}^i + J\mathfrak{g}^i$. One obtains a descending series in which each term fits into an exact sequence of Lie algebras

$$\begin{array}{cccccc}
0 \to & \mathfrak{g}^1_J \longrightarrow & \mathfrak{g} \xrightarrow{p_0} & \underset{\text{abelian}}{\mathfrak{g}/\mathfrak{g}^1_J} \neq 0 \to 0 \\
 & & \cdots & \\
0 \to & \mathfrak{g}^i_J \longrightarrow & \mathfrak{g}^{i-1}_J \overset{p_{i-1}}{\to} & \underset{\text{abelian}}{\mathfrak{g}^{i-1}_J/\mathfrak{g}^i_J} \to 0 \\
 & & \cdots & \\
0 \to & \underset{\text{abelian}}{\mathfrak{g}^s_J} \longrightarrow & \mathfrak{g}^{s-1}_J \overset{p_{s-1}}{\to} & \underset{\text{abelian}}{\mathfrak{g}^{s-1}_J/\mathfrak{g}^s_J} \to 0
\end{array}$$

Here, s is the smallest integer such that $\mathfrak{g}^{s+1}_J = \{0\}$.

In general, the subalgebra $\mathfrak{g}^i_J$ is not a rational subalgebra of $\mathfrak{g}^{i-1}_J$ (in which case one would have trouble with discrete subgroups). But if J is rational, any $\mathfrak{g}^i_J$ is rational in $\mathfrak{g}^{i-1}_J$.

The passage to Lie groups and then to the quotient by uniform subgroups yields a set of holomorphic fibrations of nilmanifolds.

For these one considers a generalization of Borel's spectral sequence (E_r, d_r) to the case of non-Kähler fibres [FW, L]. Once again, one considers two spectral sequences: one for the complex of forms invariant by the lattice and another for the forms invariant by the group action. An inductive argument (starting from $\ell = s$ and the "last fibration" whose fibres and base are tori) yields the proof. $\square$

4. Deformations

Abelian complex structures behave well with respect to deformations. Indeed

Theorem 4.1 ([CFP, MPPS])**.** *Any abelian invariant complex structure on a nilmanifold $M = G/\Gamma$ has a locally complete family of deformations (parametrized by a subset* $\mathrm{Kur} \subset \mathbb{C}^N$*) consisting entirely of invariant complex structures.*

Thus, any deformation of an abelian invariant complex structure is necessarily equivalent to an invariant one, at least if the deformation is sufficiently small.

The main point for the proof is that the holomorphic tangent sheaf Θ_M can be computed using invariant forms and invariant vectors (Theorem 3.4). This implies that there are harmonic representatives of $H^k(M, \Theta_M)$, which are the main ingredient for Kuranishi deformations [K]. Moreover, this harmonic theory can be exploited on finite dimensional vector spaces of invariant forms and invariant vectors. Thus, Kuranishi's inductive process remains within invariant tensors.

4.1. **Stability results.** If we deform a complex structure J of some type (abelian, nilpotent), the deformed structure J_t can still be invariant but will not be of the same type in general. One calls a complex structure *stable* if it remains of the same type when deformed.

The deformation of abelian complex structures are not stable in this sense, beginning from real dimension six [MPPS].

However, on a nilmanifold M with an abelian complex structure, a vector in the virtual parameter space $H^1(M, \Theta_M)$ is integrable to a 1-parameter family of abelian complex structures if and only if it lies in a linear subspace that defines the abelian condition infinitesimally (see [MPPS] for 2-step nilmanifolds and [CFP] in the general case).

In real dimension six, the deformation of abelian complex structures is stable within the class of nilpotent complex structures. There are examples showing that this phenomenon does not occur in higher dimension [CFP]. Consequently, neither nilpotent complex structures are stable.

Finally, one can see that rational complex structures are not stable under deformations, too [CF, Section 7, page 122].

4.2. **Deformations and Dolbeault cohomology.** Suppose we have a deformation J_t $(t \in B)$ of a complex structure J_0 consisting (at least locally) of invariant complex structures.

Then we have (cf. [CF] in the case of rational complex structures)

Theorem 4.2. *Suppose that $H^{*,*}_{\overline{\partial}}(M, J_0) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ holds that for a complex structure J_0. Then this isomorphism still holds for any small deformation J_t $(t \in B)$ of J_0 consisting of invariant complex structures.*

Sketch of the proof: The idea goes as follows.

For J_0 we know that $\dim H^{\perp *,*}_{\overline{\partial}}(M, J_0) = 0$. Now, by Kodaira-Spencer theory [KS], $\dim H^{\perp *,*}_{\overline{\partial}}(M, J_t)$ is an upper semicontinuous function, so there exists a sufficiently small neighbourhood U_0 of J_0 such that

$$\forall J_t \in U_0 \qquad \dim H^{\perp *,*}_{\overline{\partial}}(M, J_t) \leq \dim H^{\perp *,*}_{\overline{\partial}}(M, J_0) = 0 \,.$$

Thus $H^{*,*}_{\overline{\partial}}(M, J_t) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ in $U_0 \subset B$. □

We conclude with an application of this result. By Theorem 4.1, in fact, given an abelian complex structure J, for any small deformation J_t $(t \in U_0 \subseteq \mathrm{Kur})$, we have $H^{*,*}_{\overline{\partial}}(M, J_t) \cong H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$.

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Sergio Console
Dipartimento di Matematica
Università di Torino
via Carlo Alberto 10
10123 Torino, Italy
`sergio.console@unito.it`

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