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THREE-DIMENSIONAL ALMOST CO-KÄHLER MANIFOLDS WITH HARMONIC REEB VECTOR FIELDS

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ABSTRACT. Let M^3 be a three-dimensional almost co-Kähler manifold whose Reeb vector field is harmonic. We obtain some local classification results of M^3 under some additional conditions related to the Ricci tensor.

1. Introduction

Let (M, g) be an m-dimensional Riemannian manifold and (T^1M, g_S) its unit tangent sphere bundle furnished with the standard Sasakian metric g_S. A unit vector field V is a map V from (M, g) into (T^1M, g_S). V is said to be *harmonic* if it is a critical point of the *energy function* defined on the set $\mathfrak{X}^1(M)$ of all unit vector fields. The map $V : (M, g) \to (T^1M, g_S)$ defines a *harmonic map* if and only if V is harmonic and trace$\{Y \to R(\nabla_Y V, V)X\}$ vanishes for any vector field X on M, where R denotes the curvature tensor. V is said to be *minimal* if it is a critical point of the *volume functional* defined on $\mathfrak{X}^1(M)$.

In this paper, we aim to investigate curvature properties of a three-dimensional almost co-Kähler manifold with harmonic Reeb vector field. Almost co-Kähler manifolds were first introduced by Blair [1] and studied by Goldberg and Yano [7] and Olszak et al. [6, 10]. Such manifolds are actually the almost cosymplectic manifolds studied in the above literature. We adopt this new terminology since the co-Kähler manifolds are really odd dimensional analogues of Kähler manifolds (see [3, 8]). Recently, Perrone in [11] obtained a complete classification of three-dimensional homogeneous almost co-Kähler manifolds, and in particular gave also a local characterization of such manifolds under a condition of local symmetry. Applying this result, Y. Wang in [15, 16] and the first author of the present paper in [14] obtained some local classifications of three-dimensional almost co-Kähler manifolds. Note that Perrone in [12] characterized the minimality of the Reeb vector field of three-dimensional almost co-Kähler manifolds.

In this paper, after giving some fundamental formulas of almost co-Kähler manifolds in Section 2, we present some necessary lemmas in Section 3. In Section 4, we

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investigate three-dimensional non-co-Kähler almost co-Kähler manifolds M^3 with harmonic Reeb vector fields. Generalizing Perrone's results shown in [11, 12], we obtain some local classification theorems of M^3 under some additional conditions like cyclic-parallel Ricci tensor, η-Einstein condition, and strong η-parallel Ricci tensor.

2. Almost co-Kähler manifolds

If there exist a $(1,1)$-type tensor field ϕ, a global vector field ξ, and a 1-form η on a $(2n+1)$-dimensional smooth manifold M^{2n+1} such that

$$\phi^2 = -\operatorname{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity endomorphism, then the triplet (ϕ, ξ, η) is said to be an *almost contact structure* and M^{2n+1} is said to be an *almost contact manifold*, denoted by $(M^{2n+1}, \phi, \xi, \eta)$. Moreover, ξ is called the *characteristic* or the *Reeb vector field*. From (2.1) one obtains $\phi(\xi) = 0$, $\eta \circ \phi = 0$, and $\operatorname{rank}(\phi) = 2n$. We define an almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right), \tag{2.2}$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a smooth function defined on the product. An almost contact structure is said to be *normal* if the above almost complex structure J is integrable, i.e., J is a complex structure. According to Blair [2], the normality of an almost contact structure is expressed by $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

for any vector fields X, Y on M^{2n+1}.

If on an almost contact manifold there exists a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any vector fields X, Y, then g is said to be *compatible* with respect to the almost contact structure. An almost contact manifold equipped with a compatible Riemannian metric is said to be an *almost contact metric manifold*, denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. The *fundamental* 2-*form* Φ on an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y.

In this paper, by an *almost co-Kähler manifold* we mean an almost contact metric manifold such that both the 1-form η and 2-form Φ are closed (see [3]). An almost co-Kähler manifold is said to be a *co-Kähler manifold* (see [8]) if the associated almost contact structure is normal, which is also equivalent to $\nabla\phi = 0$, or equivalently, $\nabla\Phi = 0$. Note that the (almost) co-Kähler manifolds in fact are the (almost) cosymplectic manifolds studied in [1, 2, 6, 7, 10].

On an almost co-Kähler manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, we set $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $h' = h \circ \phi$ (note that both h and h' are symmetric operators with respect to the metric g). The following formulas can be found in Olszak [10] and Perrone [11]:

$$h\xi = 0, \quad h\phi + \phi h = 0, \quad \mathrm{tr}(h) = \mathrm{tr}(h') = 0, \tag{2.4}$$

$$\nabla_\xi \phi = 0, \quad \nabla \xi = h', \quad \mathrm{div}\, \xi = 0, \tag{2.5}$$

$$\nabla_\xi h = -h^2\phi - \phi l, \tag{2.6}$$

$$\phi l \phi - l = 2h^2, \tag{2.7}$$

where $l := R(\cdot\,,\xi)\xi$ is the Jacobi operator along the Reeb vector field, and tr and div denote the trace and divergence operators with respect to the metric g, respectively. The well-known Ricci tensor S is defined by

$$S(X,Y) = g(QX,Y) = \mathrm{tr}\{Z \to R(Z,X)Y\},$$

where Q denotes the associated Ricci operator with respect to the metric g.

3. Three-dimensional almost co-Kähler manifolds with harmonic Reeb vector fields

In this paper, let $(M^3, \phi, \xi, \eta, g)$ be a three-dimensional almost co-Kähler manifold. According to the second term of relation (2.5) we obtain

$$(\mathcal{L}_\xi g)(X,Y) = 2g(h'X,Y).$$

Here we remark that a three-dimensional almost co-Kähler manifold is co-Kähler if and only if the Reeb vector field ξ is a Killing vector field, or equivalently, $h = 0$ (see also [7, Proposition 3]).

Note that the minimality and harmonicity of the Reeb vector field of a three-dimensional almost co-Kähler manifold were characterized by Perrone [11, 12, 13].

Lemma 3.1 ([12, Theorems 3.1, 4.2])**.** *On a three-dimensional almost co-Kähler manifold, the Reeb vector field is minimal if and only if it is harmonic, and this is also equivalent to requiring that it is an eigenvector field of the Ricci operator.*

Lemma 3.2 ([11, Theorem 4.2])**.** *On a three-dimensional almost co-Kähler manifold the Reeb vector field defines a harmonic map if and only if it is harmonic and* $\xi(\mathrm{tr}\, h^2) = 0$ *on every dense and open subset of the manifold.*

Applying the above results, Perrone in [11, p. 53] points out some examples of three-dimensional almost co-Kähler manifolds for which ξ defines a harmonic map. For examples of three-dimensional almost co-Kähler manifolds for which ξ is minimal we refer the reader to [12].

A Lie group G is said to be unimodular if its left invariant Haar measure is also right invariant. It is known that a Lie group G is unimodular if and only if the endomorphism $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_X(Y) = [X,Y]$ has trace equal to zero for any $X \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra associated to G. In what follows, let G be a three-dimensional unimodular Lie group with a left invariant metric g. From Milnor [9], we know that there exists a left invariant local orthonormal frame field $\{u_1, u_2, u_3\}$ satisfying

$$[u_1, u_2] = c_3 u_3, \quad [u_2, u_3] = c_1 u_1, \quad [u_3, u_1] = c_2 u_2, \tag{3.1}$$

where c_i, $i \in \{1, 2, 3\}$, are all constants.

We set $\xi := u_1$ and define a 1-form η by $\eta = g(\xi, \cdot)$. We also define a $(1,1)$-type tensor field ϕ by $\phi u_2 = u_3$, $\phi u_3 = -u_2$, and $\phi\xi = 0$. One can check that (ϕ, ξ, η, g) defines a left invariant almost contact metric structure on G. Perrone [11, Proposition 2.1] proved that an almost contact metric three-manifold is an almost co-Kähler manifold if and only if $\nabla\xi$ is symmetric and $\operatorname{div}\xi = 0$. By (3.1) we know that $\nabla\xi$ is symmetric if and only if $c_1 = 0$. Moreover, $\operatorname{div}\xi = 0$ holds already (see also Cho [4]). Thus, we state that there exists a left invariant almost co-Kähler structure on any three-dimensional Lie group G whose Lie algebra is given by

$$[u_1, u_2] = c_3 u_3, \quad [u_2, u_3] = 0, \quad [u_3, u_1] = c_2 u_2, \quad \forall\, c_2, c_3 \in \mathbb{R}. \tag{3.2}$$

Using the Koszul formula from (3.2) we obtain (see Perrone [11, p. 54])

$$(\nabla_{u_i} u_j) = \begin{pmatrix} 0 & \frac{c_2+c_3}{2} u_3 & -\frac{c_2+c_3}{2} u_2 \\ \frac{c_2-c_3}{2} u_3 & 0 & \frac{c_3-c_2}{2} u_1 \\ \frac{c_2-c_3}{2} u_2 & \frac{c_3-c_2}{2} u_1 & 0 \end{pmatrix} \tag{3.3}$$

for any $i, j \in \{1, 2, 3\}$. It follows from (3.2)–(3.3) that

$$Q\xi = -\frac{(c_2 - c_3)^2}{2}\xi, \quad Qu_2 = \frac{c_2^2 - c_3^2}{2} u_2, \quad Qu_3 = \frac{c_3^2 - c_2^2}{2} u_3. \tag{3.4}$$

On a three-dimensional unimodular Lie group there exists a left invariant non-co-Kähler almost co-Kähler structure whose Reeb vector field is harmonic.

Let M^3 be a three-dimensional almost co-Kähler manifold. Following Perrone [12], let $\mathcal{U}_1$ be the open subset of M^3 on which $h \neq 0$ and $\mathcal{U}_2$ the open subset defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3. For any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$, we find a local orthonormal basis $\{\xi, e_1, e_2 = \phi e_1\}$ of three distinct unit eigenvector fields of h in a certain neighborhood of p. On $\mathcal{U}_1$ we assume that $he_1 = \lambda e_1$ and hence $he_2 = -\lambda e_2$, where λ is assumed to be a positive function. Note that λ is continuous on M^3 and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 3.3 ([12, Lemma 2.1])**.** *On $\mathcal{U}_1$ we have*

$$\nabla_\xi e_1 = fe_2, \quad \nabla_\xi e_2 = -fe_1, \quad \nabla_{e_1}\xi = -\lambda e_2, \quad \nabla_{e_2}\xi = -\lambda e_1,$$
$$\nabla_{e_1} e_1 = \frac{1}{2\lambda}\Big(e_2(\lambda) + \sigma(e_1)\Big)e_2, \quad \nabla_{e_2} e_2 = \frac{1}{2\lambda}\Big(e_1(\lambda) + \sigma(e_2)\Big)e_1,$$
$$\nabla_{e_2} e_1 = \lambda\xi - \frac{1}{2\lambda}\Big(e_1(\lambda) + \sigma(e_2)\Big)e_2, \quad \nabla_{e_1} e_2 = \lambda\xi - \frac{1}{2\lambda}\Big(e_2(\lambda) + \sigma(e_1)\Big)e_1,$$
$$\nabla_\xi h = \left(\frac{1}{\lambda}\xi(\lambda)\,\mathrm{id} + 2f\phi\right)h,$$

where f is a smooth function and σ is the 1-form defined by $\sigma(\cdot) = S(\cdot, \xi)$.

Applying Lemma 3.3, we obtain the Ricci operator Q given by (see [12, Proposition 4.1])

$$Q = \alpha\,\mathrm{id} + \beta\eta \otimes \xi + \phi\nabla_\xi h - \sigma(\phi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_1 + \sigma(e_2)\eta \otimes e_2, \tag{3.5}$$

where $\alpha = \frac{1}{2}(r + \operatorname{tr}(h^2))$, $\beta = -\frac{1}{2}(r + 3\operatorname{tr}(h^2))$, and r denotes the scalar curvature. Moreover, using Lemma 3.3 we have the following Poisson brackets:

$$\begin{aligned}
&[\xi, e_1] = (f+\lambda)e_2, \quad [e_2, \xi] = (f-\lambda)e_1,\\
&[e_1, e_2] = \frac{1}{2\lambda}\Big(e_1(\lambda) + \sigma(e_2)\Big)e_2 - \frac{1}{2\lambda}\Big(e_2(\lambda) + \sigma(e_1)\Big)e_1.
\end{aligned} \tag{3.6}$$

4. Some local classification results

If the Reeb vector field of a three-dimensional almost co-Kähler manifold M^3 is harmonic ($\Leftrightarrow$ minimal), then we write $Q\xi = S(\xi,\xi)\xi$ and hence $\sigma(e_1) = \sigma(e_2) = 0$. It follows from (3.5) and Lemma 3.3 that

$$\begin{cases}
Q\xi = -2\lambda^2\xi,\\
Qe_1 = (\frac{1}{2}r + \lambda^2 - 2\lambda f)e_1 + \xi(\lambda)e_2,\\
Qe_2 = (\frac{1}{2}r + \lambda^2 + 2\lambda f)e_2 + \xi(\lambda)e_1
\end{cases} \tag{4.1}$$

holds on $\mathcal{U}_1$, where r denotes the scalar curvature. Moreover, on $\mathcal{U}_1$, by applying Lemma 3.3 again we obtain from (4.1) the following nine relations:

$$(\nabla_\xi Q)\xi = -4\lambda\xi(\lambda)\xi. \tag{4.2}$$

$$(\nabla_\xi Q)e_1 = \left[\xi\left(\frac{r}{2} + \lambda^2 - 2\lambda f\right) - 2f\xi(\lambda)\right]e_1 + [\xi(\xi(\lambda)) - 4\lambda f^2]e_2. \tag{4.3}$$

$$(\nabla_\xi Q)e_2 = \left[\xi\left(\frac{r}{2} + \lambda^2 + 2\lambda f\right) + 2f\xi(\lambda)\right]e_2 + [\xi(\xi(\lambda)) - 4\lambda f^2]e_1. \tag{4.4}$$

$$(\nabla_{e_1} Q)\xi = -4\lambda e_1(\lambda)\xi + \lambda\xi(\lambda)e_1 + \lambda\left(\frac{r}{2} + 3\lambda^2 + 2\lambda f\right)e_2. \tag{4.5}$$

$$\begin{aligned}
(\nabla_{e_1} Q)e_1 &= \lambda\xi(\lambda)\xi + [e_1(\xi(\lambda)) - 2fe_2(\lambda)]e_2\\
&\quad + \left[e_1\left(\frac{r}{2} + \lambda^2 - 2\lambda f\right) - \frac{\xi(\lambda)}{\lambda}e_2(\lambda)\right]e_1.
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
(\nabla_{e_1} Q)e_2 &= \lambda\left(\frac{r}{2} + 3\lambda^2 + 2\lambda f\right)\xi + [e_1(\xi(\lambda)) - 2fe_2(\lambda)]e_1\\
&\quad + \left[e_1\left(\frac{r}{2} + \lambda^2 + 2\lambda f\right) + \frac{\xi(\lambda)}{\lambda}e_2(\lambda)\right]e_2.
\end{aligned} \tag{4.7}$$

$$(\nabla_{e_2} Q)\xi = -4\lambda e_2(\lambda)\xi + \lambda\xi(\lambda)e_2 + \lambda\left(\frac{r}{2} + 3\lambda^2 - 2\lambda f\right)e_1. \tag{4.8}$$

$$\begin{aligned}
(\nabla_{e_2} Q)e_1 &= \lambda\left(\frac{r}{2} + 3\lambda^2 - 2\lambda f\right)\xi + [e_2(\xi(\lambda)) + 2fe_1(\lambda)]e_2\\
&\quad + \left[e_2\left(\frac{r}{2} + \lambda^2 - 2\lambda f\right) + \frac{\xi(\lambda)}{\lambda}e_1(\lambda)\right]e_1.
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
(\nabla_{e_2} Q)e_2 &= \lambda\xi(\lambda)\xi + [e_2(\xi(\lambda)) + 2fe_1(\lambda)]e_1\\
&\quad + \left[e_2\left(\frac{r}{2} + \lambda^2 + 2\lambda f\right) - \frac{\xi(\lambda)}{\lambda}e_1(\lambda)\right]e_2.
\end{aligned} \tag{4.10}$$

Applying (4.2), (4.6) and (4.10) in the well-known formula $\operatorname{div} Q = \frac{1}{2}\operatorname{grad} r$ we see that the following relation holds on $\mathcal{U}_1$:

$$\begin{aligned}\frac{1}{2}\operatorname{grad}(r) &= -2\lambda\xi(\lambda)\xi \\ &\quad + \left[e_1\left(\frac{r}{2}+\lambda^2-2\lambda f\right)+e_2(\xi(\lambda))+2fe_1(\lambda)-\frac{\xi(\lambda)}{\lambda}e_2(\lambda)\right]e_1 \\ &\quad + \left[e_2\left(\frac{r}{2}+\lambda^2+2\lambda f\right)+e_1(\xi(\lambda))-2fe_2(\lambda)-\frac{\xi(\lambda)}{\lambda}e_1(\lambda)\right]e_2.\end{aligned} \tag{4.11}$$

Lemma 4.1. *On a three-dimensional non-co-Kähler almost co-Kähler manifold whose Reeb vector field is harmonic, the scalar curvature is invariant along the Reeb vector field if and only if $\xi(\lambda)=0$. If $\xi(\lambda)=0$, we have*

$$e_1(\lambda-f)=e_2(\lambda+f)=0. \tag{4.12}$$

Proof. On any three-dimensional non-co-Kähler almost co-Kähler manifold, $\mathcal{U}_1$ is non-empty and λ is assumed to be non-zero. Taking the inner product of (4.11) with ξ we obtain that $\xi(r)=0$ if and only $\xi(\lambda)=0$. If $\xi(\lambda)=0$, using this in (4.11) we obtain

$$\begin{aligned}\frac{1}{2}\operatorname{grad}(r) &= \left[e_1\left(\frac{r}{2}+\lambda^2-2\lambda f\right)+2fe_1(\lambda)\right]e_1 \\ &\quad + \left[e_2\left(\frac{r}{2}+\lambda^2+2\lambda f\right)-2fe_2(\lambda)\right]e_2.\end{aligned}$$

In view of $\lambda\neq 0$, taking the inner product of the above relation with e_1 and e_2, respectively, we complete the proof. □

Before giving our main results we need the following lemma (see also [5]).

Lemma 4.2 ([14, Corollary 4.1])**.** *A three-dimensional co-Kähler manifold with constant scalar curvature is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the Riemannian product $\mathbb{R}\times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c\neq 0$.*

Definition 4.1. An almost contact metric manifold is said to be *η-Einstein* if the Ricci operator satisfies

$$Q=\alpha\,\mathrm{id}+\beta\eta\otimes\xi \tag{4.13}$$

for some functions α,β.

Obviously, an η-Einstein metric becomes an Einstein one if β vanishes. Recall that on any three-dimensional Riemannian manifold the curvature tensor can be written as

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X \\ &\quad - S(X,Z)Y-\frac{r}{2}\{g(Y,Z)X-g(X,Z)Y\}\end{aligned} \tag{4.14}$$

for any vector fields X,Y,Z. Applying $h=0$ in the second term of relation (2.5) we have $Q\xi=0$. Using this and putting $Y=Z=\xi$ in (4.14) we obtain the Ricci operator

$$Q = \frac{r}{2}\,\mathrm{id} - \frac{r}{2}\eta \otimes \xi. \tag{4.15}$$

Thus, a three-dimensional co-Kähler manifold is always η-Einstein. Also, it follows directly from (4.15) that

$$(\nabla_X Q)Y = \frac{1}{2}X(r)Y - \frac{1}{2}X(r)\eta(Y)\xi \tag{4.16}$$

for any vector fields X, Y.

Theorem 4.1. *Let M^3 be a three-dimensional non-co-Kähler almost co-Kähler manifold whose Reeb vector field is harmonic. Then M^3 is η-Einstein if and only if it is locally isometric to a unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space equipped with a left invariant almost co-Kähler structure.*

Proof. Let M^3 be an η-Einstein non-co-Kähler almost co-Kähler manifold of dimension three. Since $\mathcal{U}_1$ is non-empty, replacing X by e_1 and e_2 in $QX = \alpha X + \beta\eta(X)\xi$ respectively and comparing these with the last two terms of relation (4.1) we have

$$\xi(\lambda) = f = 0,$$

where we have used $\lambda \neq 0$. In view of $\xi(\lambda) = 0$, applying Lemma 4.1 we have that λ is a positive constant. In this case, (3.6) becomes

$$[\xi, e_1] = \lambda e_2, \quad [e_2, \xi] = -\lambda e_1, \quad [e_1, e_2] = 0.$$

According to Milnor [9] or Perrone [11, 12], we state that M^3 is locally isometric to the unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space equipped with a left invariant almost co-Kähler structure.

Conversely, from relations (3.4) and (4.13) we observe that the non-co-Kähler almost co-Kähler structure defined on the Lie group $E(1,1)$ is η-Einstein provided $c_3 = -c_2 \neq 0$. This completes the proof. □

Definition 4.2. The Ricci tensor of a Riemannian manifold is said to be *cyclic-parallel* if it satisfies

$$g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)Z, X) + g((\nabla_Z Q)X, Y) = 0 \tag{4.17}$$

for any vector fields X, Y, Z.

A Riemannian metric satisfying (4.17) can be called an Einstein-like metric since an Einstein metric satisfies (4.17), but the converse is not necessarily true.

Theorem 4.2. *Let M^3 be a three-dimensional non-co-Kähler almost co-Kähler manifold whose Reeb vector field is harmonic. Then, the Ricci tensor of M^3 is cyclic-parallel if and only if it is locally isometric to either the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space or the Heisenberg group H^3 or the group $E(1,1)$ of rigid motions of the Minkowski 2-space equipped with a left invariant almost co-Kähler structure.*

Proof. We assume that M^3 is non-co-Kähler with cyclic-parallel Ricci tensor. From (4.2) and $g((\nabla_\xi Q)\xi,\xi)=0$ we have $\xi(\lambda)=0$. Moreover, in view of $\lambda\neq 0$, from $g((\nabla_{e_i}Q)\xi,\xi)+2g((\nabla_\xi Q)\xi,e_i)=0$ for $i=1,2$, (4.5) and (4.8) we observe that λ is a non-zero constant. Thus, from $g((\nabla_\xi Q)e_1,e_1)+2g((\nabla_{e_1}Q)e_1,\xi)=0$, (4.3) and (4.6) we see that $\xi(f)=0$. Taking into account Lemma 4.1, we know that f is also a constant. In this case, (3.6) becomes

$$[\xi,e_1]=(f+\lambda)e_2,\quad [e_2,\xi]=(f-\lambda)e_1,\quad [e_1,e_2]=0.$$

According to Milnor [9] or Perrone [11, 12], we state that M^3 is locally isometric to the universal covering $\tilde{E}(2)$ of the group of rigid motions of the Euclidean 2-space if $f>\lambda$, $f>0$ or $f<-\lambda$, $f<0$; the Heisenberg group H^3 if $f=\lambda$ or $f=-\lambda$; the group $E(1,1)$ of rigid motions of the Minkowski 2-space if $0\leq f<\lambda$ or $-\lambda<f\leq 0$.

Conversely, on non-co-Kähler almost co-Kähler structures defined on the above three Lie groups, from Perrone [11, p. 55] we have (3.3) and (3.4). By using these relations, one can check easily that (4.17) is true. This completes the proof. □

Definition 4.3. The Ricci tensor of an almost contact metric manifold is said to be *strong η-parallel* if it satisfies

$$g((\nabla_X Q)Y,Z)=0 \tag{4.18}$$

for any vector field X and any vector fields Y,Z orthogonal to the Reeb vector field ξ.

Obviously, the strong η-parallelism of the Ricci tensor on an almost contact metric manifold is weaker than usual parallelism ($\nabla Q=0$) but stronger than usual *η-parallelism* (that is, (4.18) holds for any X,Y,Z orthogonal to ξ).

Lemma 4.3. *Let M^3 be a three-dimensional non-co-Kähler almost co-Kähler manifold whose Reeb vector field is harmonic. If the Ricci tensor of M^3 is strong η-parallel, then the scalar curvature r is invariant along the distribution $\{\xi\}^\perp$ and we have*

$$\xi(\xi(\lambda))=4\lambda f^2. \tag{4.19}$$

Proof. We assume that M^3 is non-co-Kähler. Thus, $\mathcal{U}_1$ is non-empty and Lemma 3.3 is applicable. The Ricci tensor of M^3 is strong η-parallel if and only if

$$g((\nabla_\xi Q)e_i,e_k)=0,\quad g((\nabla_{e_i}Q)e_j,e_k)=0,\qquad i,j,k=1,2.$$

Using relations (4.3)–(4.10) in the above relation and (4.11), we obtain

$$\mathrm{grad}(r)=-4\lambda\xi(\lambda)\xi. \tag{4.20}$$

It follows from (4.20) that $e_1(r)=e_2(r)=0$. Moreover, from $g((\nabla_\xi Q)e_1,e_2)=0$ and (4.3) we obtain (4.19). This completes the proof. □

Theorem 4.3. *Let M^3 be a three-dimensional almost co-Kähler manifold whose Reeb vector field defines a harmonic map. Then the Ricci tensor of M^3 is strong η-parallel if and only if one of the following statements occurs.*

(1) *M^3 is co-Kähler and in this case it is locally isometric to either the flat Euclidean space $\mathbb{R}^3$ or the product $\mathbb{R}\times N^2(c)$, where $N^2(c)$ denotes a Kähler surface of constant curvature $c \neq 0$.*
(2) *M^3 is non-co-Kähler and in this case it is locally isometric to a unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space equipped with a left invariant almost co-Kähler structure.*

Proof. We first consider the co-Kähler case. On any three-dimensional co-Kähler manifold, using $h = 0$ and $Q\xi = 0$ (deduced from the second term of (2.5)) in relation (4.14) we obtain (4.16). If the Ricci tensor of M^3 is strong η-parallel, it follows from (4.16) that the scalar curvature r is a constant. Then, the proof follows from Lemma 4.2.

Next let M^3 be a non-co-Kähler almost co-Kähler 3-manifold. As ξ defines a harmonic map, we obtain from Lemma 3.2 that $\xi(\lambda) = 0$. Therefore, using this in (4.19) we obtain $f = 0$, where we have used the fact $\lambda \neq 0$. In this context, applying Lemma 4.1 we see that λ is a non-zero constant. Now, (3.6) becomes

$$[\xi, e_1] = \lambda e_2, \quad [e_2, \xi] = -\lambda e_1, \quad [e_1, e_2] = 0.$$

Therefore, M^3 is locally isometric to the unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space equipped with a left invariant almost co-Kähler structure. Conversely, the proof follows from (3.4) and (3.3). □

Remark 4.1. On any three-dimensional co-Kähler manifold the Ricci tensor is parallel if and only if it is strong η-parallel. But there exists a strictly strong η-parallel Ricci tensor on a non-co-Kähler almost co-Kähler 3-manifold.

5. Almost co-Kähler structures on non-unimodular Lie groups

From now on we show that the Ricci tensor of a left invariant almost co-Kähler structure defined on a three-dimensional non-unimodular Lie group is cyclic-parallel but non-η-parallel, non-η-Einstein. In what follows, let G be a three-dimensional non-unimodular Lie group (see [9]) with a left invariant metric g whose Lie algebra is given by

$$[e_1, e_2] = \alpha e_2, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = \beta e_2, \tag{5.1}$$

where $\{e_1, e_2, e_3\}$ is an orthonomal basis with respect to g and $\alpha \neq 0$, $\beta \in \mathbb{R}$.

Now we set $\xi := e_3$ and define a 1-form η by $\eta = g(\xi, \cdot)$. We define a $(1,1)$-type tensor field ϕ by $\phi\xi = 0$, $\phi e_1 = e_2$ and $\phi e_2 = -e_1$. One can check that (G, ϕ, ξ, η, g) defines a three-dimensional left invariant non-co-Kähler almost co-Kähler manifold when $\beta \neq 0$ or a three-dimensional co-Kähler manifold when β vanishes (for more details see Perrone [11, Example 4.2]). Moreover, by using the Koszul formula and (5.1) we have

$$(\nabla_{e_i} e_j) = \begin{pmatrix} 0 & -\frac{\beta}{2}\xi & \frac{\beta}{2}e_2 \\ -\alpha e_2 - \frac{\beta}{2}\xi & \alpha e_1 & \frac{\beta}{2}e_1 \\ -\frac{\beta}{2}e_2 & \frac{\beta}{2}e_1 & 0 \end{pmatrix} \tag{5.2}$$

for any $i, j \in \{1,2,3\}$. Using (5.2), the Ricci operator of G is given by

$$\begin{aligned} Q\xi &= -\frac{\beta^2}{2}\xi - \alpha\beta e_2, \\ Qe_1 &= -\left(\alpha^2 + \frac{\beta^2}{2}\right)e_1, \quad Qe_2 = \left(-\alpha^2 + \frac{\beta^2}{2}\right)e_2 - \alpha\beta\xi. \end{aligned} \tag{5.3}$$

Remark 5.1. From (5.3) we see that the non-co-Kähler almost co-Kähler structure defined on the non-unimodular Lie group G can not be η-Einstein and the Reeb vector field ξ is not harmonic.

Using (5.2) and (5.3) we obtain

$$(\nabla_\xi Q)e_1 = \frac{1}{2}\beta^3 e_2 - \frac{1}{2}\alpha\beta^2\xi, \tag{5.4}$$

$$(\nabla_{e_1} Q)\xi = \frac{\beta}{2}(\alpha^2 - \beta^2)e_2 + \alpha\beta^2\xi, \tag{5.5}$$

$$(\nabla_{e_1} Q)e_2 = \frac{\beta}{2}(\alpha^2 - \beta^2)\xi - \alpha\beta^2 e_2. \tag{5.6}$$

Remark 5.2. From relation (5.6), we observe that the Ricci tensor of the non-co-Kähler almost co-Kähler structure defined on the non-unimodular Lie group G is not η-parallel.

Using (5.2) and (5.3) we obtain

$$\begin{aligned} &(\nabla_\xi Q)\xi = -\frac{1}{2}\alpha\beta^2 e_1, \quad (\nabla_\xi Q)e_2 = \frac{1}{2}\beta^3 e_1, \quad (\nabla_{e_2} Q)\xi = -\frac{1}{2}\alpha^2\beta e_1, \\ &(\nabla_{e_1} Q)e_1 = 0, \quad (\nabla_{e_2} Q)e_1 = -\frac{1}{2}\alpha^2\beta e\xi + \frac{1}{2}\alpha\beta^2 e_2, \quad (\nabla_{e_2} Q)e_2 = \frac{1}{2}\alpha\beta^2 e_1. \end{aligned} \tag{5.7}$$

Remark 5.3. From relations (5.4)–(5.6) and (5.7), we observe that the Ricci tensor of the non-co-Kähler almost co-Kähler structure defined on the non-unimodular Lie group G is cyclic-parallel.

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References

[1] D. E. Blair, *The theory of quasi-Sasakian structures*, J. Differential Geom. **1** (1967), 331–345. MR 0226538.

[2] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, 2nd edition, Progress in Mathematics, vol. 203, Birkhäuser, 2010. MR 2682326.

[3] B. Cappelletti-Montano, A. De Nicola, I. Yudin, *A survey on cosymplectic geometry*, Rev. Math. Phys. **25** (2013), 1343002, 55 pp. MR 3149323.

[4] J. T. Cho, *Reeb flow symmetry on almost cosymplectic three-manifolds*, Bull. Korean Math. Soc. **53** (2016), 1249–1257. MR 3534316.

[5] J. T. Cho, *Conformally flat normal almost contact* 3*-manifolds*, Honam Math. J. **38** (2016), 59–69 MR 3495943.

[6] P. Dacko, Z. Olszak, *On almost cosymplectic (k, μ, ν)-spaces*, PDEs, submanifolds and affine differential geometry, 211–220, Banach Center Publ., 69, Polish Acad. Sci. Inst. Math., Warsaw, 2005. MR 2189569.

[7] S. I. Goldberg, K. Yano, *Integrability of almost cosymplectic structures*, Pacific J. Math. **31** (1969), 373–382. MR 0251678.

[8] H. Li, *Topology of co-symplectic/co-Kähler manifolds*, Asian J. Math. **12** (2008), 527–543. MR 2481690.

[9] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 293–329. MR 0425012.

[10] Z. Olszak, *On almost cosymplectic manifolds*, Kodai Math. J. **4** (1981), 239–250. MR 0630244.

[11] D. Perrone, *Classification of homogeneous almost cosymplectic three-manifolds*, Differ. Geom. Appl. **30** (2012), 49–58. MR 2871704.

[12] D. Perrone, *Minimal Reeb vector fields on almost cosymplectic manifolds*, Kodai Math. J. **36** (2013), 258–274. MR 3081246.

[13] D. Perrone, *Almost contact metric manifolds whose Reeb vector field is a harmonic section*, Acta Math. Hungar. **138** (2013), 102–126. MR 3015965.

[14] W. Wang, *A class of three dimensional almost coKähler manifolds*, Palestine J. Math. **6** (2017), 111–118. MR 3562748.

[15] Y. Wang, *Almost co-Kähler manifolds satisfying some symmetry conditions*, Turkish J. Math. **40** (2016), 740–752. MR 3518243.

[16] Y. Wang, *Ricci tensors on three-dimensional almost coKähler manifolds*, Kodai Math. J. **39** (2016), 469–483. MR 3567225.

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