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T^*-EXTENSIONS AND ABELIAN EXTENSIONS OF HOM-LIE COLOR ALGEBRAS

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Abstract. We study hom-Nijenhuis operators, T^*-extensions and abelian extensions of hom-Lie color algebras. We show that the infinitesimal deformation generated by a hom-Nijenhuis operator is trivial. Many properties of a hom-Lie color algebra can be lifted to its T^*-extensions such as nilpotency, solvability and decomposition. It is proved that every finite-dimensional nilpotent quadratic hom-Lie color algebra over an algebraically closed field of characteristic not 2 is isometric to a T^*-extension of a nilpotent Lie color algebra. Moreover, we introduce abelian extensions of hom-Lie color algebras and show that there is a representation and a 2-cocycle, associated to any abelian extension.

1. Introduction

In 2012, Yuan [9] introduced the notion of a hom-Lie color algebra which can be viewed as an extension of Hom-Lie superalgebras to G-graded algebras, where G is any abelian group. In 2015, Abdaoui, Ammarto and Makhlouf defined representations and a cohomology of the Hom-Lie color algebra ([1, Section 3]). We recover a Lie color algebra when we have $\alpha = \mathrm{Id}_L$ in Definition 2.2.

In 1979, Scheunert investigated the Lie color algebras from a purely mathematical point of view and obtained generalizations of the PBW and Ado theorems ([7, Sections 4 and 7]). Scheunert and Zhang introduced the cohomology theory of Lie color algebras in [8]. Feldvoss described representations of Lie color algebras in [5]. Ma, Chen and Lin investigated T^*-extensions of the Lie color algebras by virtue of a cohomology and the representations in [6].

The first purpose of this paper is to define hom-Nijenhuis operators of the hom-Lie color algebra, showing that the infinitesimal deformation generated by a hom-Nijenhuis operator is trivial. Secondly, we study T^*-extensions of the hom-Lie color algebra by virtue of the cohomology and the representation, show that every finite-dimensional nilpotent quadratic Lie color algebra L over an algebraically closed field of characteristic not 2 is isometric to a T^*-extension of a nilpotent Lie color algebra B, and the nilpotent length of B is at most half of that of L. We also give

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the equivalence of T^*-extensions from the cohomological point of view. Finally, we introduce abelian extensions of hom-Lie color algebras, show that there is a representation and a 2-cocycle associated to any abelian extension.

The paper proceeds as follows. In Section 2, we summarize basic concepts and the cohomology theory of hom-Lie color algebras. We show that the direct sum of two hom-Lie color algebras is still a hom-Lie color algebra. An even homomorphism between hom-Lie superalgebras is a morphism if and only if its graph is a hom-subalgebra. In particular, any α-derivation gives rise to a derivation extension of the multiplicative hom-Lie color algebra $(L, [\cdot,\cdot]_L, \alpha)$ (see Theorem 2.9). In Section 3, we define hom-Nijenhuis operators of regular hom-Lie color algebras and show that the infinitesimal deformation generated by a hom-Nijenhuis operator is trivial. In Section 4, we show that T^*-extension preserves many properties such as nilpotency, solvability and decomposition in some sense. Moreover, we discuss the equivalence of T^*-extensions using cohomology. In Section 5, we introduce abelian extensions of hom-Lie color algebras and show that there is a representation and a 2-cocycle associated to any abelian extension.

2. HOM-LIE COLOR ALGEBRAS

Definition 2.1 ([7]). Let G be an abelian group and $\mathbb{K}$ be an arbitrary field. The map $\varepsilon : G \times G \to \mathbb{K}\backslash\{0\}$ is called a skew-symmetric bicharacter (or commutation factor) of G if $\forall f, g, h \in G$,

$$\varepsilon(f, g+h) = \varepsilon(f, g)\varepsilon(f, h),$$
$$\varepsilon(g+h, f) = \varepsilon(g, f)\varepsilon(h, f),$$
$$\varepsilon(g, h)\varepsilon(h, g) = 1.$$

The definition above implies, in particular, the following relations:

$$\varepsilon(a, 0) = \varepsilon(0, a) = 1, \quad \varepsilon(a, a) = \pm 1, \qquad \forall a \in G.$$

Throughout this paper, if x, y, z are homogeneous elements of a G-graded vector space and $|x|, |y|, |z| \in G$ denote their degrees respectively, then for convenience we write $\varepsilon(x, y)$ instead of $\varepsilon(|x|, |y|)$, $\varepsilon(x, y+z)$ instead of $\varepsilon(|x|, |y| + |z|)$, and so on. Moreover, when the notation $\varepsilon(x, y)$ appears, it means that x, y are homogeneous elements.

Definition 2.2 ([9]). A hom-Lie color algebra is a quadruple $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ consisting of a G-graded vector space L, a bicharacter ε, an even bilinear map $[\cdot,\cdot]_L : \wedge^2 L \to L$ (i.e., $[L_a, L_b]_L \subset L_{a+b}$) and an even homomorphism $\alpha : L \to L$ such that for homogeneous elements $x, y, z \in L$ we have

$$[x, y]_L = -\varepsilon(x, y)[y, x]_L \quad (\varepsilon\text{-skew symmetry}),$$
$$\circlearrowleft_{x,y,z} \varepsilon(z, x)[\alpha(x), [y, z]_L]_L = 0 \quad (\varepsilon\text{-hom-Jacobi identity}),$$

where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z. In particular, if α is a morphism of Lie algebras (i.e., $\alpha \circ [\cdot,\cdot]_L = [\cdot,\cdot]_L \circ \alpha^{\otimes 2}$), then we call $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ a multiplicative hom-Lie color algebra. A hom-Lie color algebra is called regular hom-Lie color algebra if α is an algebraic automorphism.

Remark 2.3. Lie color algebra is a generalization of Lie algebra and Lie superalgebra (if $G = \{0\}$, we have that $L = L_0$ is a Lie algebra and if $G = \mathbb{Z}_2 = \bar{0}, \bar{1}$ and $\varepsilon(\bar{1}, \bar{1}) = -1$, then L is a Lie superalgebra). hom-Lie color algebras also can be regarded as the extension of hom-Lie algebras and hom-Lie superalgebras.

Definition 2.4. Let L be a hom-Lie color algebra and I be a G-graded subspace of L. I is called a hom-subalgebra (resp. hom-ideal) of L if $[I, I]_L \subseteq I$ (resp. $[I, L]_L \subseteq I$) and $\alpha(I) \subseteq I$. Moreover, I is called a hom abelian ideal of L if $[I, I]_L = 0$.

Proposition 2.5. *Given two hom-Lie color algebras* $(L, [\cdot, \cdot]_L, \alpha, \varepsilon)$ *and* $(\Gamma, [\cdot, \cdot]_\Gamma, \beta, \varepsilon)$*, there is a hom-Lie color algebra* $(L \oplus \Gamma, [\cdot, \cdot]_{L\oplus\Gamma}, \alpha + \beta, \varepsilon)$*, where the bilinear map* $[\cdot, \cdot]_{L\oplus\Gamma} : \wedge^2(L \oplus \Gamma) \to L \oplus \Gamma$ *is given by*

$$[u_1 + v_1, u_2 + v_2]_{L\oplus\Gamma} = [u_1, u_2]_L + [v_1, v_2]_\Gamma, \quad \forall u_1, u_2 \in L,\ v_1, v_2 \in \Gamma,$$

and the linear map $(\alpha + \beta) : L \oplus \Gamma \to L \oplus \Gamma$ *is given by*

$$(\alpha + \beta)(u + v) = \alpha(u) + \beta(v), \quad \forall u \in L,\ v \in \Gamma.$$

Definition 2.6 ([2])**.** Let $(L, [\cdot, \cdot]_L, \alpha, \varepsilon)$ and $(\Gamma, [\cdot, \cdot]_\Gamma, \beta, \varepsilon)$ be two hom-Lie color algebras. An even homomorphism $\phi : L \to \Gamma$ is said to be a morphism of hom-Lie color algebras if

$$\phi[u, v]_L = [\phi(u), \phi(v)]_\Gamma, \quad \forall u, v \in L, \tag{2.1}$$

$$\phi \circ \alpha = \beta \circ \phi. \tag{2.2}$$

Denote by $\mathfrak{G}_\phi = \{(x, \phi(x)) \mid x \in L\} \subseteq L\oplus\Gamma$ the graph of a linear map $\phi : L \to \Gamma$.

Proposition 2.7. *An even homomorphism* $\phi : (L, [\cdot, \cdot]_L, \alpha, \varepsilon) \to (\Gamma, [\cdot, \cdot]_\Gamma, \beta, \varepsilon)$ *is a morphism of hom-Lie color algebras if and only if the graph* $\mathfrak{G}_\phi \subseteq L \oplus \Gamma$ *is a hom-subalgebra of* $(L \oplus \Gamma, [\cdot, \cdot]_{L\oplus\Gamma}, \alpha + \beta, \varepsilon)$.

Proof. Let $\phi : (L, [\cdot, \cdot]_L, \alpha, \varepsilon) \to (\Gamma, [\cdot, \cdot]_\Gamma, \beta, \varepsilon)$ be a morphism of hom-Lie color algebras. We have

$$[u + \phi(u), v + \phi(v)]_{L\oplus\Gamma} = [u, v]_L + [\phi(u), \phi(v)]_\Gamma = [u, v]_L + \phi[u, v]_L.$$

Then the graph $\mathfrak{G}_\phi$ is closed under the bracket operation $[\cdot, \cdot]_{L\oplus\Gamma}$. Furthermore, by (2.2), we have

$$(\alpha + \beta)(u + \phi(u)) = \alpha(u) + \beta \circ \phi(u) = \alpha(u) + \phi \circ \alpha(u),$$

which implies that $(\alpha + \beta)(\mathfrak{G}_\phi) \subseteq \mathfrak{G}_\phi$. Thus, $\mathfrak{G}_\phi$ is a hom-subalgebra of $(L \oplus \Gamma, [\cdot, \cdot]_{L\oplus\Gamma}, \alpha + \beta, \varepsilon)$.

Conversely, if the graph $\mathfrak{G}_\phi \subseteq L \oplus \Gamma$ is a hom-subalgebra of $(L \oplus \Gamma, [\cdot, \cdot]_{L\oplus\Gamma}, \alpha + \beta, \varepsilon)$, then we have

$$[u + \phi(u), v + \phi(v)]_{L\oplus\Gamma} = [u, v]_L + [\phi(u), \phi(v)]_\Gamma \in \mathfrak{G}_\phi,$$

which implies that

$$[\phi(u), \phi(v)]_\Gamma = \phi[u, v]_L.$$

Furthermore, $(\alpha + \beta)(\mathfrak{G}_\phi) \subseteq \mathfrak{G}_\phi$ yields that

$$(\alpha + \beta)(u + \phi(u)) = \alpha(u) + \beta \circ \phi(u) \in \mathfrak{G}_\phi,$$

which is equivalent to the condition $\beta\circ\phi(u) = \phi\circ\alpha(u)$, i.e., $\beta\circ\phi = \phi\circ\alpha$. Therefore, ϕ is a morphism of hom-Lie color algebras. □

Let $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ be a multiplicative hom-Lie color algebra. For any nonnegative integer k, denote by α^k the k-times composition of α, i.e.,

$$\alpha^k = \alpha\circ\cdots\circ\alpha \quad (k \text{ times}).$$

In particular, $\alpha^0 = \mathrm{Id}$ and $\alpha^1 = \alpha$. If $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ is a regular hom-Lie color algebra, we denote by α^{-k} the k-times composition of α^{-1}, the inverse of α.

Definition 2.8 ([2])**.** Let $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ be a multiplicative hom-Lie color algebra. A homogeneous bilinear map $D : L \to L$ of degree θ is said to be an α^k-derivation, where $k \in \mathbb{N}$, if it satisfies

$$D\circ\alpha = \alpha\circ D, \tag{2.3}$$

and

$$D[u,v]_L = [D(u), \alpha^k(v)]_L + \varepsilon(\theta,u)[\alpha^k(u), D(v)]_L, \quad \forall u,v \in L. \tag{2.4}$$

For a regular hom-Lie color algebra, α^{-k}-derivations can be defined similarly.

$\mathrm{Der}_{\alpha^s}(L)$ is the set of α^s-derivations of the multiplicative hom-Lie color algebra $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$. For any $u \in L$ satisfying $\alpha(u) = u$, define $\mathrm{ad}_k(u) : L \to L$ by

$$\mathrm{ad}_k(u)(v) = [u, \alpha^k(v)]_L, \quad \forall v \in L.$$

Then $\mathrm{ad}_k(u)$ is an α^k-derivation, which we call an inner α^{k+1}-derivation. In fact, we have

$$\mathrm{ad}_k(u)(\alpha(v)) = [u, \alpha^{k+1}(v)]_L = \alpha([u, \alpha^k(v)]_L) = \alpha\circ\mathrm{ad}_k(u)(v),$$

which implies that (2.3) in Definition 2.8 is satisfied. On the other hand, we have

$$\begin{aligned}
\mathrm{ad}_k u([v,w]_L) &= [u, \alpha^k([v,w]_L)]_L \\
&= [\alpha(u), [\alpha^k(v), \alpha^k(w)]_L]_L \\
&= -\varepsilon(u,w)(\varepsilon(u,v)[\alpha^{k+1}(v), [\alpha^k(w), u]_L]_L \\
&\quad + \varepsilon(v,w)[\alpha^{k+1}(w), [u, \alpha^k(v)]_L]_L \\
&= -\varepsilon(u,w)\varepsilon(u,v)[\alpha^{k+1}(v), [\alpha^k(w), u]_L]_L \\
&\quad - \varepsilon(u,w)\varepsilon(v,w)[\alpha^{k+1}(w), [u, \alpha^k(v)]_L]_L) \\
&= \varepsilon(u,v)[\alpha^{k+1}(v), [u, \alpha^k(w)]_L]_L + [[u, \alpha^s(v)]_L), \alpha^{k+1}(w)]_L \\
&= [\mathrm{ad}_k u(v), \alpha^{k+1}(w)]_L] + \varepsilon(u,v)[\alpha^{k+1}(v), \mathrm{ad}_k u(w)]_L].
\end{aligned}$$

Therefore, $\mathrm{ad}_k(u)$ is an α^{k+1}-derivation. Denote by $\mathrm{Inn}_{\alpha^k}(L)$ the set of inner α^k-derivations, i.e.,

$$\mathrm{Inner}_{\alpha^k}(L) = \{[u, \alpha^{k-1}(\cdot)]_L \mid u \in L, \alpha(u) = (u)\}.$$

At the end of this section, we consider the derivation extension of the multiplicative hom-Lie color algebra $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ and give an application of the α-derivation $\mathrm{Der}_\alpha(L)$.

For any even linear map $D : L \to L$, consider the vector space $L \oplus RD$. Define a bilinear bracket operation $[\cdot,\cdot]_D$ on $L \oplus RD$ by

$$[u + mD, v + nD]_D = [u, v]_L + mD(v) - \varepsilon(u, v)nD(u),$$

$$[u, v]_D = [u, v]_L, \quad [D, u]_D = -[u, D]_D = D(u), \qquad \forall u, v \in L.$$

Define a linear map $\alpha' : L \oplus RD \to L \oplus RD$ by $\alpha'(u + D) = \alpha(u) + D$.

Theorem 2.9. *With the above notations, $(L \oplus \mathbb{R}D, [\cdot,\cdot]_D, \alpha', \varepsilon)$ is a multiplicative hom-Lie color algebra if and only if D is an α-derivation of the multiplicative hom-Lie color algebra $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$.*

Proof. First, $[\cdot,\cdot]_D$ satisfies the ε-skew symmetry, for any $u, v, w \in L$, we have

$$[u + mD, v + nD]_D = [u, v]_L + mD(v) - \varepsilon(u, v)nD(u),$$

and

$$\begin{aligned}-\varepsilon(u, v)[v + nD, u + mD]_D &= -\varepsilon(u, v)([v, u]_L + nD(u) - \varepsilon(u, v)mD(v))\\ &= [u, v]_L + mD(v) - \varepsilon(u, v)nD(u)\\ &= [u + mD, v + nD]_D.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\alpha'([u + mD, v + nD,]_D) &= \alpha'([u, v]_L + mD(v) - \varepsilon(u, v)nD(u))\\ &= \alpha[u, v]_L + m\alpha \circ D(v) - \varepsilon(u, v)n\alpha \circ D(u),\end{aligned}$$

and

$$\begin{aligned}[\alpha'(u + mD), \alpha'(v + nD)]_D &= [\alpha(u) + mD, \alpha(v) + nD]_D\\ &= [\alpha(u), \alpha(v)]_L + mD \circ \alpha(v) - \varepsilon(u, v)nD \circ \alpha(u).\end{aligned}$$

Since α is an algebra morphism, α' is an algebra morphism if and only if

$$D \circ \alpha = \alpha \circ D.$$

By a direct calculation, we have

$$\begin{aligned}&[\alpha'(D), [u, v]_D]_D + [\alpha'(u), [v, D]_D]_D + \varepsilon(u, v)[\alpha'(v), [D, u]_D]_D\\ &\quad= D[u, v]_D - [\alpha(u), D(v)]_D + \varepsilon(u, v)[\alpha(v), D(u)]_D\\ &\quad= D[u, v]_D - [\alpha(u), D(v)]_D - [D(u), \alpha(v)]_D.\end{aligned}$$

Therefore, it is obvious that the ε-hom-Jacobi identity is satisfied if and only if the following condition holds:

$$D[u, v]_D - [\alpha(u), D(v)]_D - [D(u), \alpha(v)]_D = 0.$$

Thus, $(L \oplus \mathbb{R}D, [\cdot,\cdot]_D, \alpha', \varepsilon)$ is a multiplicative hom-Lie color algebra if and only if D is an α-derivation of $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$. ☐

Definition 2.10 ([1])**.** Let $(L, [\cdot,\cdot], \varepsilon, \alpha)$ be a Hom-Lie color algebra. Let (V, β) be a pair of G-graded vector space V and an even homomorphism of vectors spaces $\beta : V \to V$, and

$$L_g \cdot V_h \subseteq V_{g+h}, \quad \forall g, h \in G.$$

(V,β) is said to be L-module if

$$\beta(x\cdot m)=\alpha(x)\cdot\beta(m),$$
$$[x,y]\cdot\beta(m)=\alpha(x)\cdot(y\cdot m)-\varepsilon(x,y)\alpha(y)\cdot(x\cdot m),\quad \forall x,y\in L.$$

Now we introduce the cohomology theory of hom-Lie color algebras, which can be found in [1].

Let $V=\oplus_{g\in G}V_g$ be a graded L-module. Denote by $C^n(L,V)$ ($n\geq 0$, $C^0(L,V)=V$) the G-graded vector space spanned by all n-linear homogeneous maps $f: L\times\cdots\times L\to V$ such that

$$f(x_1,\dots,x_i,x_{i+1},\dots,x_n)=-\varepsilon(x_i,x_{i+1})f(x_1,\dots,x_{i+1},x_i,\dots,x_n).$$

The map f is called even (resp. of degree γ) when $f(x_1,\dots,x_i,\dots,x_n)\in V_0$ for all elements $(x_1,\dots,x_n)\in L^{\otimes n}$ (resp. $f(x_1,\dots,x_i,\dots,x_n)\in V_\gamma$ for all elements $(x_1,\dots,x_n)\in L^{\otimes n}$ of degree γ). An n-cochain on L with values in V is defined to be an n-cochain $f\in C^n(L,V)$ such that it is compatible with α and β in the sense that $f\circ\alpha=\beta\circ f$. Denote by $C^n_{\alpha,\beta}(L,V)$ the set of n-cochains:

$$C^n_{\alpha,\beta}(L,V)=\{f\in C^n(L,V)\mid f\circ\alpha=\beta\circ f\}.$$

Next, for a given integer r, we define the cobounday operator $\delta^n_r: C^n_{\alpha,\beta}(L,V)\to C^{n+1}_{\alpha,\beta}(L,V)$ by

$$\begin{aligned}
&\delta^n_r(f)(x_0,\dots,x_n)\\
&=\sum_{i=0}^{n}(-1)^i\varepsilon(f+x_0+\cdots+x_{i-1},x_i)\alpha^{n+r-1}(x_i)\cdot f(x_0,\cdots,\hat{x}_i,\dots,x_n)\\
&\quad+\sum_{0\leq i<j\leq n}(-1)^j\varepsilon(u_{i+1}+\cdots+u_{j-1},u_j)\\
&\qquad\cdot f(\alpha(x_0),\cdots,[x_i,x_j],\cdots,\widehat{\alpha(x_j)},\cdots,\alpha(x_n)).
\end{aligned}\tag{2.5}$$

Lemma 2.11 ([1]). *Let $(L,[\cdot,\cdot],\varepsilon,\alpha)$ be a hom-Lie color algebra and (V,β) be an L-module. Then the pair $(\oplus_{n\geq 0}C^n_{\alpha,\beta},\delta^n_r)$ is a cohomology complex. That is, the maps δ^n_r satisfy $\delta^n_r\circ\delta^{n-1}_r=0$, $\forall n\geq 2$, $\forall r\geq 1$.*

Let $Z^n_r(L,V)$ (resp. $B^n_r(L,V)$) denote the kernel of δ^n_r (resp. the image of δ^{n-1}_r). The spaces $Z^n_r(L,V)$ and $B^n_r(L,V)$ are graded submodules of $C^n_{\alpha,\beta}(L,V)$, and we have

$$B^n_r(L,V)\subseteq Z^n_r(L,V).$$

The elements of $Z^n_r(L,V)$ are called n-cocycles, and the elements of $B^n_r(L,V)$ are called the n-coboundaries. Thus, we define a so-called cohomology group

$$H^n_r(L,V)=\frac{Z^n_r(L,V)}{B^n_r(L,V)}.$$

3. Hom-Nijenhuis operator of hom-Lie color algebras

Definition 3.1 ([2]). Let $(L, [\cdot,\cdot], \varepsilon, \alpha)$ be a hom-Lie color algebra. A representation of L is a triple (V, ρ, β), where V is a G-graded vector space, $\beta \in \mathrm{End}(V)_0$ and $\rho : L \to \mathrm{End}(V)$ is an even linear map satisfying

$$\rho([x,y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \varepsilon(x,y)\rho(\alpha(y)) \circ \rho(x), \quad \forall x, y \in L. \tag{3.1}$$

Definition 3.2 ([2]). A representation ρ of a multiplicative hom-Lie color algebra $(L, [\cdot,\cdot], \varepsilon, \alpha)$ on a G-graded vector space (V, β) is a representation of a hom-Lie color algebra such that

$$\beta(\rho(x)(v)) = \rho(\alpha(x))(\beta(v)), \quad \forall x \in L,\ v \in V. \tag{3.2}$$

Now, we introduce the adjoint representations of a hom-Lie color algebra.

Lemma 3.3 ([1]). *Let $(L, [\cdot,\cdot], \varepsilon, \alpha)$ be a hom-Lie color algebra and $\mathrm{ad} : L \to \mathrm{End}(L)$ an operator defined for $x \in L$ by $\mathrm{ad}(x)(y) = [x, y]$. Then (L, ad, α) is a representation of L.*

Definition 3.4 ([1]). For any integer s, the α^s-adjoint representation of the regular hom-Lie color algebra $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$, which we denote by ad_s, is defined by

$$\mathrm{ad}_s(u)(v) = [\alpha^s(u), v]_L, \quad \forall u, v \in L.$$

In particular, we use ad to represent ad_0.

Lemma 3.5 ([1]). *With the above notations, we have*

$$\mathrm{ad}_s(\alpha(u)) \circ \alpha = \alpha \circ \mathrm{ad}_s(u);$$
$$\mathrm{ad}_s([u,v]_L) \circ \alpha = \mathrm{ad}_s(\alpha(u)) \circ \mathrm{ad}_s(v) - \varepsilon(u,v)\mathrm{ad}_s(\alpha(v)) \circ \mathrm{ad}_s(u).$$

Thus, the α^s-adjoint representation is well defined.

For the α^s-adjoint representation ad_s, we obtain the α^s-adjoint complex $(C^{\bullet}_{\alpha}(L; L), d_s)$ and the corresponding cohomology

$$H^k(L; \mathrm{ad}_s) = Z^k(L; \mathrm{ad}_s)/B^k(L; \mathrm{ad}_s).$$

Let $\psi \in C^2_{\alpha}(L; L)$ be a bilinear operator commuting with α. Consider a t-parametrized family of bilinear operations

$$[u, v]_t = [u, v]_L + t\psi(u, v).$$

Since ψ commutes with α, α is a morphism with respect to the bracket $[\cdot,\cdot]_t$ for every t. If all the brackets $[\cdot,\cdot]_t$ endow $(L, [\cdot,\cdot]_t, \alpha, \varepsilon)$ with regular hom-Lie color algebra structures, we say that ψ generates an infinitesimal deformation of the regular hom-Lie color algebra $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$. By computing the ε-hom-Jacobi identity of $[\cdot,\cdot]_t$, this is equivalent to the conditions

$$\varepsilon(w,u)\psi(\alpha(u), \psi[v,w]) + c.p.(u,v,w) = 0;$$
$$\varepsilon(w,u)(\psi(\alpha(u), [v,w]_L) + [\alpha(u), \psi[v,w]_L]_L) + c.p.(u,v,w) = 0.$$

An infinitesimal deformation is said to be trivial if there is a linear operator $N \in C^1_\alpha(L;L)$ satisfying $T_t = \mathrm{Id} + tN$ and

$$T_t[u,v]_t = [T_t(u), T_t(v)]_L.$$

Definition 3.6. A linear operator $N \in C^1_\alpha(L,L)$ is called a hom-Nijenhuis operator if we have

$$[Nu, Nv]_L = N[u,v]_N, \tag{3.3}$$

where the bracket $[\cdot,\cdot]_N$ is defined by

$$[u,v]_N = [Nu,v]_L + [u,Nv]_L - N[u,v]_L. \tag{3.4}$$

Theorem 3.7. *Let $N \in C^1_\alpha(L,L)$ be a hom-Nijenhuis operator. Then an infinitesimal deformation of the regular hom-Lie color algebra $(L,[\cdot,\cdot]_L,\alpha)$ can be obtained by putting*

$$\psi(u,v) = \delta_0 N(u,v) = [u,v]_N. \tag{3.5}$$

Furthermore, this infinitesimal deformation is trivial.

Proof. Since $\psi = \delta_0 N$, $\delta_0\psi = 0$ is valid. To see that ψ generates an infinitesimal deformation, we need to check the ε-hom-Jacobi identity for ψ. By (3.3), (3.4) and (3.5), we have

$$\begin{aligned}
&\varepsilon(w,u)\psi(\alpha(u),\psi(v,w)) + c.p.(u,v,w)\\
&\quad= \varepsilon(w,u)[\alpha(u),[v,w]_N]_N + c.p.(u,v,w)\\
&\quad= \varepsilon(w,u)([N\alpha(u),[Nv,w]_L]_L + [N\alpha(u),[v,Nw]_L]_L - [N\alpha(u),N[v,w]_L]_L\\
&\qquad - N[\alpha(u),[Nv,w]_L]_L - N[\alpha(u),[v,Nw]_L]_L + N[\alpha(u),N[v,w]_L]_L\\
&\qquad + [\alpha(u),N[v,w]_N]_L) + c.p.(u,v,w)\\
&\quad= \varepsilon(w,u)\big([N\alpha(u),[Nv,w]_L]_L + [N\alpha(u),[v,Nw]_L]_L - [N(u),N[v,w]_L]_L\\
&\qquad - N[\alpha(u),[Nv,w]_L]_L - N[\alpha(u),[v,Nw]_L]_L + N[\alpha(u),N[v,w]_L]_L\\
&\qquad + [\alpha(u),[Nv,Nw]_L]_L\big) + c.p.(u,v,w)\\
&\quad= \varepsilon(w,u)\big(\underbrace{[N\alpha(u),[Nv,w]_L]_L}_{(1)} + \underbrace{[N\alpha(u),[v,Nw]_L]_L}_{(2)} - [N\alpha(u),N[v,w]_L]_L\\
&\qquad \underbrace{- N[\alpha(u),[Nv,w]_L]_L}_{(3)} \underbrace{- N[\alpha(u),[v,Nw]_L]_L}_{(4)} + N[\alpha(u),N[v,w]_L]_L\\
&\qquad \underbrace{+ [\alpha(u),[Nv,Nw]_L]_L}_{(5)}\big)\\
&\qquad + \varepsilon(u,v)\big(\underbrace{[N\alpha(v),[Nw,u]_L]_L}_{(5')} + \underbrace{[N\alpha(v),[w,Nu]_L]_L}_{(1')} - [N\alpha(v),N[w,u]_L]_L\\
&\qquad \underbrace{- N[\alpha(v),[Nw,u]_L]_L}_{(4')} - \underbrace{N[\alpha(v),[w,Nu]_L]_L}_{(6')} + N[\alpha(v),N[w,u]_L]_L
\end{aligned}$$

$$\underbrace{+[\alpha(v),[Nw,Nu]_L]_L}_{(2')}\Big)$$
$$+\varepsilon(v,w)\Big(\underbrace{[N\alpha(w),[Nu,v]_L]_L}_{(2'')}+\underbrace{[N\alpha(w),[u,Nv]_L]_L}_{(5'')}-[N\alpha(w),N[u,v]_L]_L$$
$$\underbrace{-N[\alpha(w),[Nu,v]_L]_L}_{(6'')}\underbrace{-N[\alpha(w),[u,Nv]_L]_L}_{(3'')}+N[\alpha(w),N[u,v]_L]_L$$
$$\underbrace{+[\alpha(w),[Nu,Nv]_L]_L}_{(1'')}\Big),$$

where $c.p.(u,v,w)$ denotes summation over the cyclic permutation on u,v,w. Since N is a Hom-Nijenhuis operator, we get

$$-[N\alpha(u),N[v,w]_L]_L+N[\alpha(u),N[v,w]_L]_L=\underbrace{N^2[\alpha(u),[v,w]_L]_L}_{(7)}-\underbrace{N[N\alpha(u),[v,w]_L]_L}_{(6)},$$
$$-[N\alpha(v),N[w,u]_L]_L+N[\alpha(v),N[w,u]_L]_L=\underbrace{N^2[\alpha(v),[w,u]_L]_L}_{(7')}-\underbrace{N[N\alpha(v),[w,u]_L]_L}_{(3')}$$

and

$$-[N\alpha(w),N[u,v]_L]_L+N[\alpha(w),N[u,v]_L]_L=\underbrace{N^2[\alpha(w),[u,v]_L]_L}_{(7'')}-\underbrace{N[N\alpha(w),[u,v]_L]_L}_{(4'')}.$$

Thus $(i)+(i)'+(i)''=0$, for $i=1,\dots,7$ by the ε-hom-Jacobi identity. This proves that ψ generates a deformation of the regular Hom-Lie conformal algebra $(L,[\cdot,\cdot]_L,\alpha)$.

Let $T_t=\mathrm{Id}+tN$. Then

$$\begin{aligned}T_t[u,v]_t&=(\mathrm{Id}+tN)([u,v]_L+t\psi(u,v))\\&=(\mathrm{Id}+tN)([u,v]_L+t[u,v]_N)\\&=[u,v]_L+t([u,v]_N+N[u,v]_L)+t^2N[u,v]_N.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[T_t(u),T_t(v)]_L&=[u+tNu,v+tNv]_L\\&=[u,v]_L+t([Nu,v]_L+[u,Nv]_L)+t^2[Nu,Nv]_L.\end{aligned}$$

By (3.4) and (3.5), we have

$$T_t[u,v]_t=[T_t(u),T_t(v)]_L,$$

which implies that the infinitesimal deformation is trivial. □

4. T^*-extensions of hom-Lie color algebras

The method of T^*-extension was introduced in [4] and the T^*-extension of an algebra is quadratic. For the theory of quadratic (color) hom-Lie algebras the reader is referred to [2, 3].

Definition 4.1 ([2])**.** Let $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ be a hom-Lie color algebra. A bilinear form f on L is said to be invariant if

$$f([x,y],z) = f(x,[y,z]), \quad \forall x,y,z \in L,$$

and ε-symmetric if

$$f(x,y) = \varepsilon(x,y)f(y,x).$$

A subspace I of L is called isotropic if $I \subseteq I^{\perp}$.

Definition 4.2. A bilinear form f on a hom-Lie color algebra $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ is said to be nondegenerate if

$$L^{\perp} = \{x \in L \mid f(x,y) = 0,\ \forall y \in L\} = 0,$$

and colorconsistent if f satisfies

$$f(x,y) = 0, \quad \forall x \in L_{|x|},\ y \in L_{|y|},\ |x| + |y| \neq 0.$$

Throughout this section, we only consider colorconsistent bilinear forms.

Definition 4.3 ([2])**.** A hom-Lie color algebra $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ is called quadratic hom-Lie color algebra if there exists a nondegenerate, ε-symmetric and invariant bilinear form f on L such that α is f-symmetric (i.e., $f(\alpha(x),y) = f(x,\alpha(y))$). It is denoted by $(L, \alpha, f, \varepsilon)$ and f is called invariant scalar product.

Let $(L', [\cdot,\cdot]'_L, \beta, \varepsilon)$ be another hom-Lie color algebra. Two quadratic hom-Lie superalgebras $(L, f, \alpha, \varepsilon)$ and $(L', f', \beta, \varepsilon)$ are said to be isometric if there exists a hom-Lie color algebra isomorphism $\phi : L \to L'$ such that $f(x,y) = f'(\phi(x),\phi(y))$, $\forall x,y \in L$.

We consider the dual space L^* of L. Then L^* is a G-graded space, where $L^*_g = \{\beta \in L^* \mid \beta(x) = 0,\ \forall |x| \neq -g\}$. Moreover, L^* is a graded L-module.

The base field $\mathbb{K}$ itself can be considered as a G-graded space, if one sets $\mathbb{K}_0 = \mathbb{K}$, $\mathbb{K}_g = \{0\}$, for $g \neq 0$. Then as a trivial graded L-module, $C^n(L,\mathbb{K})$ ($n \geq 0$, $C^0(L,\mathbb{K}) = \mathbb{K}$) is the G-graded vector space spanned by all n-linear homogenous maps f of $L \times \cdots \times L$ into $\mathbb{K}$ satisfying

$$f(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) = -\varepsilon(x_i,x_{i+1})f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n),$$

where $C^n(L,\mathbb{K})_g = \{f \in C^n(L,\mathbb{K}) \mid f(x_1,\ldots,x_n) = 0, \text{ if } |x_1| + \cdots + |x_n| + g \neq 0\}$.

Let $(L = \oplus_{g\in G} L_g, \alpha, \varepsilon)$ be a Lie color algebra over a field $\mathbb{K}$, $L^* = \oplus_{g\in G} L^*_g$ be its dual space, and w be a homogeneous bilinear map: $L \times L \to L^*$ satisfying $|w| = 0$.

Since $L = \oplus_{g\in G} L_g$ and $L^* = \oplus_{g\in G} L^*_g$ are G-graded spaces, the direct sum

$$L \oplus L^* = \oplus_{g\in G}(L \oplus L^*)_g = \oplus_{g\in G}(L_g \oplus L^*_g)$$

is G-graded. In the sequel, we always consider $a + \alpha \in L \oplus L^*$ as a homogeneous element such that $a \in L$, $\alpha \in L^*$ and $|a + \alpha| = |a| = |\alpha|$; then $\varepsilon(a+\alpha, a'+\alpha') = \varepsilon(a,a')$.

Lemma 4.4 ([2]). *Let* ad *be the adjoint representation of a hom-Lie color algebra* $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$*, and let us consider the even linear map* $\pi : L \to \mathrm{End}(L^*)$ *defined by* $\pi(x)(f)(y) = -\varepsilon(x, f)(f \circ \mathrm{ad}(x)(y))$*,* $\forall x, y \in L$*. Then* π *is a representation of* L *on* $(L^*, \tilde{\alpha})$ *if and only if*

$$\mathrm{ad}(x) \circ \mathrm{ad}(\alpha(y)) - \varepsilon(x, y)\mathrm{ad}(y) \circ \mathrm{ad}(\alpha(x)) = \alpha \circ \mathrm{ad}([x, y]_L).$$

We call the representation π *the coadjoint representation of* L*.*

Lemma 4.5 ([2, Theorem 4.11]). *Under the above notations, let* $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ *be a hom-Lie color algebra, and* $\omega : L \times L \to L^*$ *be an even bilinear map. Assume that the coadjoint representation exists. The* G*-graded space* $L \oplus L^*$*, provided with the following bracket and a linear map defined respectively by*

$$[x + f, y + g]_{L\oplus L^*} = [x, y]_L + \omega(x, y) + \pi(x)g - \varepsilon(x, y)\pi(y)f, \tag{4.1}$$

$$\alpha'(x + f) = \alpha(x) + f \circ \alpha. \tag{4.2}$$

Then $(L \oplus L^*, [\cdot,\cdot]_{L\oplus L^*}, \alpha', \varepsilon)$ *is a hom-Lie color algebra if and only if* ω *is a 2-color-cocycle:* $L \times L \to L^*$*, i.e.,* $\omega \in Z^2(L, L^*)_{\bar{0}}$*.*

Clearly, L^* is an abelian hom-ideal of $(L \oplus L^*, [\cdot,\cdot]_{\alpha'}, \alpha', \varepsilon)$ and L is isomorphic to the factor hom-Lie color algebra $(L \oplus L^*)/L^*$. Moreover, consider the following ε-symmetric bilinear form q_L on $L \oplus L^*$ for all $x + f, y + g \in L \oplus L^*$:

$$q_L(x + f, y + g) = f(y) + \varepsilon(x, y)g(x).$$

Then we have the following lemma:

Lemma 4.6. *Let* L*,* L^**,* ω *and* q_L *be as above. Then the tuple* $(L \oplus L^*, q_L, \alpha', \varepsilon)$ *is a quadratic hom-Lie color algebra if and only if* ω *is colorcyclic in the following sense:*

$$w(x, y)(z) = \varepsilon(x, y + z)w(y, z)(x) \quad \textit{for all } x, y, z \in L.$$

Proof. If $x + f$ is orthogonal to all elements of $L \oplus L^*$, then $f(y) = 0$ and $\varepsilon(x, y)g(x) = 0$, which implies that $x = 0$ and $f = 0$. So the ε-symmetric bilinear form q_L is nondegenerate.

Now suppose that $x + f$, $y + g$, $z + h \in L \oplus L^*$; then

$$\begin{aligned} &q_L([x + f, y + g]_{L\oplus L^*}, z + h) \\ &\qquad = \omega(x, y)(z) - \varepsilon(x, y)g([x, z]_L) + f([y, z]_L) + \varepsilon(z, x + y)h([x, y]_L). \end{aligned}$$

On the other hand,

$$\begin{aligned} &q_L(x + f, [y + g, z + h]_{L\oplus L^*}) \\ &= f([y, z]_L) + \varepsilon(x, y + z))\omega(y, z)(x) + \varepsilon(z, x + y)h([x, y]_L) - \varepsilon(x, y)g([x, z]_L). \end{aligned}$$

Hence the lemma follows. □

Now, for colorcyclic 2-cocycle ω we shall call the quadratic hom-Lie color algebra $(L\oplus L^*, q_L, \alpha', \varepsilon)$ the T^*-extension of L (by ω) and denote the hom-Lie color algebra $(L \oplus L^*, [\cdot,\cdot]_{L\oplus L^*}, \alpha', \varepsilon)$ by $T^*_\omega L$.

Definition 4.7. Let L be a hom-Lie color algebra over a field $\mathbb{K}$. We inductively define a derived series

$$(L^{(n)})_{n\geq 0} : L^{(0)} = L, \quad L^{(n+1)} = [L^{(n)}, L^{(n)}],$$

a central descending series

$$(L^{n})_{n\geq 0} : L^{0} = L, \quad L^{n+1} = [L^{n}, L],$$

and a central ascending series

$$(C_n(L))_{n\geq 0} : C_0(L) = 0, \quad C_{n+1}(L) = C(C_n(L)),$$

where $C(I) = \{a \in L \mid [a, L] \subseteq I\}$ for a subspace I of L.

L is called solvable and nilpotent (of length k) if and only if there is a (smallest) integer k such that $L^{(k)} = 0$ and $L^k = 0$, respectively.

In the following theorem we discuss some properties of $T^*_\omega L$.

Theorem 4.8. *Let $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ be a hom-Lie color algebra over a field $\mathbb{K}$.*

(1) If L is solvable (nilpotent) of length k, then the T^*-extension $T^*_\omega L$ is solvable (nilpotent) of length r, where $k \leq r \leq k+1$ ($k \leq r \leq 2k-1$).
(2) If L is nilpotent of length k, so is the trivial T^*-extension $T^*_0 L$.
(3) If L is decomposed into a direct sum of two hom-ideals of L, so is the trivial T^*-extension $T^*_0 L$.

Proof. (1) Firstly we suppose that L is solvable of length k. Since $(T^*_\omega L)^{(n)}/L^* \cong L^{(n)}$ and $L^{(k)} = 0$, we have $(T^*_\omega L)^{(k)} \subseteq L^*$, which implies $(T^*_\omega L)^{(k+1)} = 0$ because L^* is abelian, and it follows that $T^*_\omega L$ is solvable of length k or $k+1$.

Suppose now that L is nilpotent of length k. Since $(T^*_\omega L)^n/L^* \cong L^n$ and $L^k = 0$, we have $(T^*_\omega L)^k \subseteq L^*$. Let $\beta \in (T^*_\omega L)^k \subseteq L^*$, $b \in L$, $x_1+f_1, \ldots, x_{k-1}+f_{k-1} \in T^*_\omega L$, $1 \leq i \leq k-1$; we have

$$\begin{aligned}&[[\cdots[\beta, x_1+f_1]_{L\oplus L^*}, \cdots]_{L\oplus L^*}, x_{k-1}+f_{k-1}]_{L\oplus L^*}(b)\\ &\qquad = \beta \mathrm{ad} x_1 \cdots \mathrm{ad} x_{k-1}(b) = \beta([x_1, [\cdots, [x_{k-1}, b]_L \cdots]_L]_L) \in \beta(L^k) = 0.\end{aligned}$$

This proves that $(T^*_\omega L)^{2k-1} = 0$. Hence $T^*_w L$ is nilpotent of length at least k and at most $2k-1$.

(2) Suppose that L is nilpotent of length k. Adopting the notations of the proof of part (1), for $x_k + f_k \in T_0^* L$, we have

$$
\begin{aligned}
&[x_1 + f_1, [\cdots, [x_{k-1} + f_{k-1}, x_k + f_k]_{L\oplus L^*} \cdots]_{L\oplus L^*}]_{L\oplus L^*} \\
&\qquad = [x_1, [\cdots, [x_{k-1}, x_k]_L \cdots]_L]_L \\
&\qquad\quad + \sum_{i=1}^{k} [x_1, [\cdots, [x_{i-1}, [f_i, [x_{i+1}, [\cdots, [x_{k-1}, x_k] \cdots]]]] \cdots]] \\
&\qquad = \mathrm{ad}x_1 \cdots \mathrm{ad}x_{k-1}(x_k) + f_1[\mathrm{ad}x_2, [\cdots, [\mathrm{ad}x_{k-1}, \mathrm{ad}x_k] \cdots]] \\
&\qquad\quad + \varepsilon k - 1 \prod_{i=1}^{k-1} \varepsilon(x_i, x_{i+1} + \cdots + x_k) f_k \mathrm{ad}x_{k-1} \cdots \mathrm{ad}x_1 \\
&\qquad\quad + \varepsilon k - 2 \prod_{i=1}^{k-2} \varepsilon(x_i, x_{i+1} + \cdots + x_k) f_{k-1} \mathrm{ad}x_k \mathrm{ad}x_{k-2} \cdots \mathrm{ad}x_1 \\
&\qquad\quad + \sum_{i=2}^{k-2} \prod_{j=1}^{i-1} (-1)^{i-1} \varepsilon(x_j, x_{j+1} + \cdots + x_k) \\
&\qquad\qquad\qquad \cdot f_i[\mathrm{ad}x_{i+1}, [\cdots, [\mathrm{ad}x_{k-1}, \mathrm{ad}x_k] \cdots]] \mathrm{ad}x_{i-1} \cdots \mathrm{ad}x_1,
\end{aligned}
$$

where we use the fact that $\mathrm{ad}[x, y] = [\mathrm{ad}x, \mathrm{ad}y]$, $\forall x, y \in L$. Note that

$$
\begin{aligned}
&\mathrm{ad}x_1 \cdots \mathrm{ad}x_{k-1}(x_k) \in L^k = 0, \\
&f_1[\mathrm{ad}x_2, [\cdots, [\mathrm{ad}x_{k-1}, \mathrm{ad}x_k] \cdots](L) \subseteq f_1(L^k) = 0, \\
&f_k \mathrm{ad}x_{k-1} \cdots \mathrm{ad}x_1(L) \subseteq \alpha_k(L^k) = 0, \\
&f_{k-1} \mathrm{ad}x_k \mathrm{ad}x_{k-2} \cdots \mathrm{ad}x_1(L) \subseteq f_{k-1}(L^k) = 0, \\
&f_i[\mathrm{ad}x_{i+1}, [\cdots, [\mathrm{ad}x_{k-1}, \mathrm{ad}x_k] \cdots]] \mathrm{ad}x_{i-1} \cdots \mathrm{ad}x_1(L) \subseteq f_i(f^k) = 0.
\end{aligned}
$$

Then the right hand side of the equation vanishes and hence $(T_0^* L)^k = 0$.

(3) Suppose that $0 \neq L = I \oplus J$, where I and J are two nonzero hom-ideals of $(L[\cdot,\cdot]_L, \alpha)$. Let I^* (resp. J^*) denote the subspace of all linear forms in L^* vanishing on J (resp. I). Clearly, I^* (resp. J^*) can be canonically identified with the dual space of I (resp. J) and $L^* \cong I^* \oplus J^*$.

Since $[I^*, L]_{L\oplus L^*}(J) = I^*([L, J]_L) \subseteq I^*(J) = 0$ and $[I, L^*]_{L\oplus L^*}(J) = L^*([I, J]_L) \subseteq L^*(I \cap J) = 0$, we have $[I^*, L]_{L\oplus L^*} \subseteq I^*$ and $[I, L^*]_{L\oplus L^*} \subseteq I^*$. Then

$$
\begin{aligned}
[T_0^* I, T_0^* L]_{L\oplus L^*} &= [I \oplus I^*, L \oplus L^*]_{L\oplus L^*} \\
&= [I, L]_L + [I, L^*]_{L\oplus L^*} + [I^*, L]_{L\oplus L^*} + [I^*, L^*]_{L\oplus L^*} \\
&\subseteq I \oplus I^* = T_0^* I.
\end{aligned}
$$

It is clear that $T_0^* I$ is a G-graded space, then $T_0^* I$ is a hom-ideal of L and so is $T_0^* J$ in the same way. Hence $T_0^* L$ can be decomposed into the direct sum $T_0^* I \oplus T_0^* J$ of two nonzero hom-ideals of $T_0^* L$. □

Lemma 4.9. *Let $(L, q_L, \alpha, \varepsilon)$ be a quadratic hom-Lie color algebra of even dimension n over a field $\mathbb{K}$ and I be an isotropic $n/2$-dimensional subspace of L. Then I is a hom-ideal of $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ if and only if I is abelian.*

Proof. Since $\dim I + \dim I^{\perp} = n/2 + \dim I^{\perp} = n$ and $I \subseteq I^{\perp}$, we have $I = I^{\perp}$.

If I is a hom-ideal of $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$, then $q_L(L, [I, I^{\perp}]) = q_L([L, I], I^{\perp}) \subseteq q_L(I, I^{\perp}) = 0$, which implies $[I, I] = [I, I^{\perp}] \subseteq L^{\perp} = 0$.

Conversely, if $[I, I] = 0$, then $f(I, [I, L]) = f([I, I], L) = 0$. Hence $[I, L] \subseteq I^{\perp} = I$. This implies that I is an ideal of $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$. □

Theorem 4.10. *Let $(L, q_L, \alpha, \varepsilon)$ be a quadratic hom-Lie color algebra of even dimension n over a field $\mathbb{K}$ of characteristic not equal to two. Then $(L, q_L, \alpha, \varepsilon)$ is isometric to a T^*-extension $(T^*_\omega B, q_B, \beta', \varepsilon)$ if and only if n is even and $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ contains an isotropic hom-ideal I of dimension $n/2$. In particular, $B \cong L/I$.*

Proof. ($\Longrightarrow$) Since $\dim B = \dim B^*$, $\dim T^*_\omega B$ is even. Moreover, it is clear that B^* is a hom-ideal of half the dimension of $T^*_\omega B$ and by the definition of q_B, we have $q_B(B^*, B^*) = 0$, i.e., $B^* \subseteq (B^*)^{\perp}$ and so B^* is isotropic.

($\Longleftarrow$) Suppose that I is an $n/2$-dimensional isotropic hom-ideal of L. By Lemma 4.9, I is abelian. Let $B = L/I$ and $p : L \to B$ be the canonical projection. Clearly, $|p(x)| = |x|$, $\forall x \in L_{|x|}$. Since $\operatorname{ch} \mathbb{K} \neq 2$, we can choose an isotropic complement subspace B_0 to I in L, i.e., $L = B_0 \dotplus I$ and $B_0 \subseteq B_0^{\perp}$. Then $B_0^{\perp} = B_0$ since $\dim B_0 = n/2$.

Denote by p_0 (resp. p_1) the projection $L \to B_0$ (resp. $L \to I$) and let q_L^* denote the homogeneous linear map $I \to B^* : i \mapsto q_L^*(i)$, where $q_L^*(i)(p(x)) := q_L(i, x)$; it is clear that $|q_L^*(x)| = |x|$, $\forall x \in L_{|x|}$. We claim that q_L^* is a linear isomorphism. In fact, if $p(x) = p(y)$, then $x - y \in I$, hence $q_L(i, x-y) \in q_L(I, I) = 0$ and so $q_L(i, x) = q_L(i, y)$, which implies q_L^* is well-defined and it is easily seen that q_L^* is linear. If $q_L^*(i) = q_L^*(j)$, then $q_L^*(i)(p(x)) = q_L^*(j)(p(x))$, $\forall x \in L$, i.e., $q_L(i, x) = q_L(j, x)$, which implies $i - j \in L^{\perp} = 0$, hence q_L^* is injective. Note that $\dim I = \dim B^*$, then q_L^* is surjective.

In addition, q_L^* has the following property:

$$\begin{aligned} q_L^*([x,i])(p(y)) &= q_L([x,i]_L, y) = -\varepsilon(x,i) q_L([i,x]_L, y) = -\varepsilon(x,i) q_L(i, [x,y]_L) \\ &= -\varepsilon(x,i) q_L^*(i) p([x,y]_L) = -\varepsilon(x,i) q_L^*(i)[p(x), p(y)]_L \\ &= -\varepsilon(x,i) q_L^*(i)(\mathrm{ad} p(x)(p(y))) = (\pi(p(x)) q_L^*(i))(p(y)) \\ &= [p(x), q_L^*(i)]_{L \oplus L^*}(p(y)), \end{aligned}$$

where $x, y \in L$, $i \in I$. A similar computation shows that

$$q_L^*([x,i]) = [p(x), q_L^*(i)]_{L \oplus L^*}, \quad q_L^*([i,x]) = [q_L^*(i), p(x)]_{L \oplus L^*}.$$

Define a homogeneous bilinear map

$$\begin{aligned} \omega : B \times B &\longrightarrow B^* \\ (p(b_0), p(b_0')) &\longmapsto q_L^*(p_1([b_0, b_0'])), \end{aligned}$$

where $b_0, b_0' \in B_0$. Then $|w| = 0$ and w is well-defined since the restriction of the projection p to B_0 is a linear isomorphism.

Now, define the bracket on $B \oplus B^*$ by (4.1) and (4.2); we have that $B \oplus B^*$ is a G-graded algebra. Let φ be the linear map $L \to B \oplus B^*$ defined by $\varphi(b_0 + i) = p(b_0) + q_L^*(i)$, $\forall\, b_0 + i \in B_0 \dotplus I = L$. Since the restriction of p to B_0 and q_L^* are linear isomorphisms, φ is also a linear isomorphism. Note that

$$\begin{aligned}
&\varphi([b_0 + i, b_0' + i']_L) = \varphi([b_0, b_0']_L + [b_0, i']_L + [i, b_0']_L) \\
&\quad = \varphi(p_0([b_0, b_0']_L) + p_1([b_0, b_0']_L) + [b_0, i']_L + [i, b_0']_L) \\
&\quad = p(p_0([b_0, b_0']_L)) + q_L^*(p_1([b_0, b_0']_L) + [b_0, i']_L + [i, b_0']_L) \\
&\quad = [p(b_0), p(b_0')]_L + \omega(p(b_0), p(b_0')) + [p(b_0), q_L^*(i')]_L + [q_L^*(i), p(b_0')]_L \\
&\quad = [p(b_0), p(b_0')]_L + \omega(p(b_0), p(b_0')) + \pi(p(b_0)(q_L^*(i')) - \varepsilon|b_0||b_0'|\pi(p(b_0')(q_L^*(i)) \\
&\quad = [p(b_0) + q_L^*(i), p(b_0') + q_L^*(i')]_{B\oplus B^*} \\
&\quad = [\varphi(b_0 + i), \varphi(b_0' + i')]_{L\oplus L^*}.
\end{aligned}$$

Then φ is an isomorphism of G-graded algebras, and so $(B \oplus B^*, [\cdot,\cdot]_{B\oplus B^*,\varepsilon}, \beta)$ is a hom-Lie color algebra. Furthermore, we have

$$\begin{aligned}
q_B(\varphi(b_0 + i), \varphi(b_0' + i')) &= q_B(p(b_0) + q_L^*(i), p(b_0') + q_L^*(i')) \\
&= q_L^*(i)(p(b_0')) + \varepsilon(b_0, b_0')q_L^*(i')(p(b_0)) \\
&= q_L(i, b_0') + \varepsilon(b_0, b_0')q_L(i', b_0) \\
&= q_L(b_0 + i, b_0' + i'),
\end{aligned}$$

then φ is isometric. The relation

$$\begin{aligned}
q_B([\varphi(x), \varphi(y)], \varphi(z)) &= q_B(\varphi([x, y]), \varphi(z)) \\
&= q_L([x, y], z) = q_L(x, [y, z]) = q_B(\varphi(x), [\varphi(y), \varphi(z)])
\end{aligned}$$

implies that q_B is a nondegenerate invariant ε-symmetric bilinear form, and so $(B \oplus B^*, q_B, \beta', \varepsilon)$ is a quadratic hom-Lie color algebra. In this way, we get a T^*-extension $T_\omega^* B$ of B and consequently, $(L, q_L, \alpha, \varepsilon)$ and $(T_\omega^* B, q_B, \beta', \varepsilon)$ are isometric as required. □

The proof of Theorem 4.10 shows that the homogeneous bilinear map ω depends on the choice of the isotropic G-graded subspace B_0 of L complement to the hom-ideal I. Therefore there may be different T^*-extensions describing the "same" quadratic hom-Lie color algebras.

Let $(L, [\cdot,\cdot]_L, \alpha, \varepsilon)$ be a hom-Lie color algebra over a field $\mathbb{K}$, and let $\omega_1 : L \times L \to L^*$ and $\omega_2 : L \times L \to L^*$ be two different colorcyclic 2-color-cocycles satisfying $|\omega_1| = |\omega_2| = 0$. The T^*-extensions $T_{\omega_1}^* L$ and $T_{\omega_2}^* L$ of L are said to be equivalent if there exists an isomorphism of hom-Lie color algebras $\phi : T_{\omega_1}^* L \to T_{\omega_2}^* L$ which is the identity on the hom-ideal L^* and which induces the identity on the factor hom-Lie color algebra algebra $T_{\omega_1}^* L/L^* \cong L \cong T_{\omega_2}^* L/L^*$. The two T^*-extensions $T_{\omega_1}^* L$ and $T_{\omega_2}^* L$ are said to be isometrically equivalent if they are equivalent and ϕ is an isometry.

Proposition 4.11. *Let L be a hom-Lie color algebra over a field $\mathbb{K}$ of characteristic not equal to 2, and ω_1, ω_2 be two colorcyclic 2-color-cocycles $L \times L \to L^*$ satisfying $|\omega_i| = 0$. Then we have:*

(i) $T^*_{\omega_1}L$ is equivalent to $T^*_{\omega_2}L$ if and only if there is $z \in C^1(L, L^*)_0$ such that

$$\omega_1(x, y) - \omega_2(x, y) = \pi(x)z(y) - \varepsilon(x, y)\pi(y)z(x) - z([x, y]_L), \quad \forall x, y \in L, \quad (4.3)$$

where $C^1(L, L^*)_0$ denotes $z \in C^1(L, L^*)$ and $|z| = 0$. If this is the case, then the colorsymmetric part z_s of z, defined by $z_s(x)(y) := \frac{1}{2}(z(x)(y) + \varepsilon(x, y)z(y)(x))$, for all $x, y \in L$, induces a colorsymmetric invariant bilinear form on L.

(ii) $T^*_{\omega_1}L$ is isometrically equivalent to $T^*_{\omega_2}L$ if and only if there is $z \in C^1(L, L^*)_0$ such that (4.3) holds, for all $x, y \in L$ and the colorsymmetric part z_s of z vanishes.

Proof. (i) $T^*_{\omega_1}L$ is equivalent to $T^*_{\omega_2}L$ if and only if there is an isomorphism of hom-Lie color algebras $\Phi : T^*_{\omega_1}L \to T^*_{\omega_2}L$ satisfying $\Phi|_{L^*} = \mathrm{Id}_{L^*}$ and $x - \Phi(x) \in L^*$, $\forall x \in L$.

Suppose that $\Phi : T^*_{\omega_1}L \to T^*_{\omega_2}L$ is an isomorphism of hom-Lie color algebras and define a linear map $z : L \to L^*$ by $z(x) := \Phi(x) - x$; then $z \in C^1(L, L^*)_0$ and for all $x + f, y + g \in T^*_{\omega_1}L$, we have

$$\begin{aligned}\Phi([x + f, y + g]_\Omega) &= \Phi([x, y]_L + \omega_1(x, y) + \pi(x)g - \varepsilon(x, y)\pi(y)f)\\ &= [x, y]_L + z([x, y]_L) + \omega_1(x, y) + \pi(x)g - \varepsilon(x, y)\pi(y)f.\end{aligned}$$

On the other hand,

$$\begin{aligned}[\Phi(x + f), \Phi(y + g)] &= [x + z(x) + f, y + z(y) + g]\\ &= [x, y]_L + \omega_2(x, y) + \pi(x)g + \pi(x)z(y)\\ &\quad - \varepsilon(x, y)\pi(y)z(x) - \varepsilon(x, y)\pi(y)f.\end{aligned}$$

Since Φ is an isomorphism, (4.3) holds.

Conversely, if there exists $z \in C^1(L, L^*)_0$ satisfying (4.3), then we can define $\Phi : T^*_{\omega_1}L \to T^*_{\omega_2}L$ by $\Phi(x + f) := x + z(x) + f$. It is easy to prove that Φ is an isomorphism of hom-Lie color algebras such that $\Phi|_{L^*} = \mathrm{Id}_{L^*}$ and $x - \Phi(L) \in L^*$, $\forall x \in L$, i.e., $T^*_{\omega_1}L$ is equivalent to $T^*_{\omega_2}L$.

Consider the colorsymmetric bilinear form $q_L : L \times L \to \mathbb{K}, (x, y) \mapsto z_s(x)(y)$ induced by z_s. Note that

$$\begin{aligned}\omega_1(x, y)(m) - \omega_2(x, y)(m) &= \pi(x)z(y)(m) - \varepsilon(x, y)\pi(y)z(x)(m) - z([x, y]_L)(m)\\ &= -\varepsilon(x, y)z(y)([x, m]_L) + z(x)([y, m]_L) - z([x, y]_L)(m)\end{aligned}$$

and

$$\begin{aligned}&\varepsilon(x, y + m)(\omega_1(y, m)(x) - \omega_2(y, m)(x))\\ &\quad = \varepsilon(x, y + m)(\pi(y)z(m)(x) - \varepsilon(y, m)\pi(m)z(y)(x) - z([y, m]_L)(x))\\ &\quad = \varepsilon(x + y, m)z(m)([x, y]_L) - \varepsilon(x, y)z(y)([x, m]_L) - \varepsilon(x, y + m)z([y, m]_L)(x).\end{aligned}$$

Since both ω_1 and ω_2 are colorcyclic, the right hand sides of the above two equations are equal. Hence

$$\begin{aligned}&-\varepsilon(x,y)z(y)([x,m]_L)+z(x)([y,m]_L)-z([x,y]_L)(m)\\&\quad=\varepsilon(x+y,m)z(m)([x,y]_L)-\varepsilon(x,y)z(y)([x,m]_L)-\varepsilon(x,y+m)z([y,m]_L)(x).\end{aligned}$$

That is,

$$z(x)([y,m]_L)+\varepsilon(x,y+m)z([y,m]_L)(x)=z([x,y]_L)(m)+\varepsilon(x+y,m)z(m)([x,y]_L).$$

Since $\mathrm{ch}\mathbb{K}\neq 2$, $q_L(x,[y,m])=q_L([x,y],m)$, which proves the invariance of the ε-symmetric bilinear form q_L induced by z_s.

(ii) Let the isomorphism Φ be defined as in (i). Then for all $x+f,\ y+g\in L\oplus L^*$, we have

$$\begin{aligned}q_B(\Phi(x+f),\Phi(y+g))&=q_B(x+z(x)+f,y+z(y)+g)\\&=z(x)(y)+f(y)+\varepsilon(x,y)(z(y)(x)+g(x))\\&=z(x)(y)+\varepsilon(x,y)z(y)(x)+f(y)+\varepsilon(x,y)g(x)\\&=2z_s(x)(y)+q_B(x+f,y+g).\end{aligned}$$

Thus, Φ is an isometry if and only if $z_s=0$. □

5. Abelian extensions of hom-Lie color algebras

In this section, we show that associated to any abelian extension, there is a representation and a 2-cocycle. We assume that the hom-Lie color algebra is multiplicative.

Definition 5.1. Let $(L,[\cdot,\cdot],\alpha)$, $(V,[\cdot,\cdot]_V,\alpha_V)$, and $(\hat{L},[\cdot,\cdot]_{\hat{L}},\alpha_{\hat{L}})$ be hom-Lie color algebras and $i:V\to\hat{L}$, $p:\hat{L}\to L$ be morphisms of hom-Lie color algebras. The following sequence of hom-Lie color algebras is a short exact sequence if $\mathrm{Im}(i)=\mathrm{Ker}(p)$, $\mathrm{Ker}(i)=0$ and $\mathrm{Im}(p)=L$:

$$0\longrightarrow V\stackrel{i}{\longrightarrow}\hat{L}\stackrel{p}{\longrightarrow}L\longrightarrow 0,$$

where $\alpha_V(V)=\alpha_{\hat{L}}(V)$.

In this case, we call $\hat{L}$ an extension of L by V, and denote it by $E_{\hat{L}}$. It is called an abelian extension if V is an abelian ideal of $\hat{L}$, i.e., $[u,v]_{\hat{L}}=0$ for all $u,v\in V$.

A section σ of $p:\hat{L}\to L$ consists of linear maps $\sigma:L\to\hat{L}$ such that $p\circ\sigma=\mathrm{id}_L$ and $\sigma\circ\alpha=\alpha_{\hat{L}}\circ\sigma$.

Definition 5.2. Two extensions of hom-Lie color algebras $E_{\hat{L}}:0\longrightarrow V\stackrel{i}{\longrightarrow}\hat{L}\stackrel{p}{\longrightarrow}L\longrightarrow 0$ and $E_{\tilde{L}}:0\longrightarrow V\stackrel{j}{\longrightarrow}\tilde{L}\stackrel{q}{\longrightarrow}L\longrightarrow 0$ are equivalent if there exists a morphism of hom-Lie color algebras $F:\hat{L}\to\tilde{L}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}0&\longrightarrow&V&\stackrel{i}{\longrightarrow}&\hat{L}&\stackrel{p}{\longrightarrow}&L&\longrightarrow&0\\&&\downarrow{\scriptstyle\mathrm{id}}&&\downarrow{\scriptstyle F}&&\downarrow{\scriptstyle\mathrm{id}}&&\\0&\longrightarrow&V&\stackrel{j}{\longrightarrow}&\tilde{L}&\stackrel{q}{\longrightarrow}&L&\longrightarrow&0.\end{array}$$

Let $\hat{L}$ be an abelian extension of L by V, and $\sigma : L \to \hat{L}$ be a section. Define maps $\theta : L \to \mathrm{End}(V)$ by

$$\theta(x)(v) = [\sigma(x), v]_{\hat{L}},$$

for all $x \in L$, $v \in V$.

Theorem 5.3. *Let (V, α_V) and (L, α) be multiplicative hom-Lie color algebras. With the above notations, (V, α_V, θ) is a representation of (L, α) and does not depend on the choice of the section σ. Moreover, equivalent abelian extensions give the same representation.*

Proof. First, if we choose another section $\sigma' : L \to \hat{L}$, then

$$p(\sigma(x) - \sigma'(x)) = x - x = 0 \Rightarrow \sigma(x) - \sigma'(x) \in V \Rightarrow \sigma'(x) = \sigma(x) + u,$$

for some $u_i \in V$.

Note that $[u, v]_{\hat{L}} = 0$ for all $u, v \in V$; this yields that

$$[\sigma'(x), v] = [\sigma(x) + u, v]_{\hat{L}} = [\sigma(x), v]_{\hat{L}}.$$

This shows that θ is independent of the choice of σ.

Second, we prove that (V, α_V, θ) is a representation of (L, α). For $x, y \in L$, $v \in V$, we have

$$\begin{aligned} \alpha_V(\theta(x)(v)) &= \alpha_V[\sigma(x), v] = [\alpha_V(\sigma(x)), \alpha_V(v)] \\ &= [\sigma(\alpha(x)), \alpha_V(v)] = \theta(\alpha(x))\alpha_V(v). \end{aligned}$$

Thus, we obtain that equation (3.2) holds

Since $[\sigma(x), \sigma(y)]_{\hat{L}} - \sigma([x, y]_L) \in V$ and V is an abelian ideal of $\hat{L}$, we have

$$\theta([x, y]) \circ \alpha_V(v) = [\sigma[x, y], \alpha_V(v)]_{\hat{L}} = [[\sigma(x), \sigma(y)], \alpha_V(v)]_{\hat{L}}.$$

On the other hand,

$$\begin{aligned} &\theta(\alpha(x))\theta(y)(v) - \varepsilon(x, y)\theta(\alpha(y))\theta(x)(v) \\ &\qquad = [\sigma(\alpha(x)), [\sigma(y), v]] - \varepsilon(x, y)[\sigma(\alpha(y)), [\sigma(x), v]] \\ &\qquad = [\alpha_{\hat{L}}(\sigma(x)), [\sigma(y), v]] - \varepsilon(x, y)[\alpha_{\hat{L}}(\sigma(y)), [\sigma(x), v]] \\ &\qquad = -\varepsilon(y, v)\varepsilon(x, v)[\alpha_V(v), [\sigma(x), \sigma(y)]] \\ &\qquad = [[\sigma(x), \sigma(y)], \alpha_V(v)]. \end{aligned}$$

Thus, we obtain that equation (3.1) holds.

Third, we will show that equivalent abelian extensions give the same θ.

Suppose that $E_{\hat{L}}$ and $E_{\tilde{L}}$ are equivalent abelian extensions, and $F : \hat{L} \to \tilde{L}$ is the hom-Lie color algebras morphism satisfying $F \circ i = j$, $q \circ F = p$. Choosing linear sections σ and σ' of p and q, we have $qF\sigma(x) = p\sigma(x) = x = q\sigma'(x)$, then $F\sigma(x) - \sigma'(x) \in \mathrm{Ker}(q) \cong V$. Moreover,

$$[\sigma(x), u]_{\hat{L}} = [F\sigma(x), u]_{\hat{L}} = [\sigma'(x), u]_{\hat{L}}.$$

The proof is complete. □

Let $\sigma : L \to \hat{L}$ be a section of the abelian extension. Define the following map:

$$\omega(x_1, x_2) = [\sigma(x_1), \sigma(x_2)]_{\hat{L}} - \sigma([x_1, x_2]_L), \tag{5.1}$$

for all $x_1, x_2, x_3 \in L$.

Theorem 5.4. *Let $0 \longrightarrow V \longrightarrow \hat{L} \longrightarrow L \longrightarrow 0$ be an abelian extension of L by V. Then ω defined by* (5.1) *is a 2-cocycle of L with coefficients in V, where the representation θ is given by* (3.1)*.*

Proof. Putting $n = 2$, $r = 0$ in (2.5), we have

$$\begin{aligned}\delta_0^2(f)(x,y,z) &= \varepsilon(\gamma,x)\rho(\alpha(x))f(y,z) - \varepsilon(\gamma+x,y)\rho(\alpha(y))f(x,z)\\ &\quad + \varepsilon(\gamma+x+y,z)\rho(\alpha(z))f(x,y) - f([[x,y],\alpha(z)])\\ &\quad + \varepsilon(y,z)f([[x,z],\alpha(y)]) + f([\alpha(x),[y,z]]).\end{aligned}$$

Setting $f = \omega$, $\rho = \theta$ and noting that $\theta(x)(u) = [\sigma(x), u]$, we obtain

$$\begin{aligned}\delta_0^2(\omega)(x,y,z) &= \varepsilon(\gamma,x)\theta(\alpha(x))\omega(y,z) - \varepsilon(\gamma+x,y)\theta(\alpha(y))\omega(x,z)\\ &\quad + \varepsilon(\gamma+x+y,z)\theta(\alpha(z))\omega(x,y) - \omega([[x,y],\alpha(z)])\\ &\quad + \varepsilon(y,z)\omega([[x,z],\alpha(y)]) + \omega([\alpha(x),[y,z]])\\ &= \varepsilon(\gamma,x)[\sigma(\alpha(x)),\omega(y,z)] - \varepsilon(\gamma+x,y)[\sigma(\alpha(y)),\omega(x,z)]\\ &\quad + \varepsilon(\gamma+x+y,z)[\sigma(\alpha(z)),\omega(x,y)] - \omega([[x,y],\alpha(z)])\\ &\quad + \varepsilon(y,z)\omega([[x,z],\alpha(y)]) + \omega([\alpha(x),[y,z]])\\ &= [\alpha_{\hat{L}}(\sigma(x)),[\sigma(y),\sigma(z)]] - [\alpha_{\hat{L}}(\sigma(x)),\sigma[y,z]]\\ &\quad - \varepsilon(x,y)[\alpha_{\hat{L}}(\sigma(y)),[\sigma(x),\sigma(z)]] + \varepsilon(x,y)[\alpha_{\hat{L}}(\sigma(y)),\sigma[x,z]]\\ &\quad + \varepsilon(x+y,z)[\alpha_{\hat{L}}(\sigma(z)),[\sigma(x),\sigma(y)]] - \varepsilon(x+y,z)[\alpha_{\hat{L}}(\sigma(z)),\sigma[x,y]]\\ &\quad - [\sigma([x,y]),\alpha_{\hat{L}}(\sigma(z))] + \sigma([[x,y],\alpha(z)])\\ &\quad + \varepsilon(y,z)[\sigma([x,z]),\alpha_{\hat{L}}(\sigma(y))] - \varepsilon(y,z)\sigma([[x,z],\alpha(y)])\\ &\quad + [\alpha_{\hat{L}}(\sigma(x)),\sigma([y,z])] - \sigma([\alpha(x),[y,z]])\\ &= 0,\end{aligned}$$

where the last equality follows from the ε-hom-Jacobi identity. Therefore, ω is a 2-cocycle. □

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