MAXWELL'S EQUATIONS FOR MULTIPOLES

J. J. Giambiagi
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In this note, we propose to write Maxwell's equations for singular multipoles with variable moment, using derivatives of the delta function.

For the case of a dipole in the origin, oriented in the positive direction of the x-axis, we have for $\vec{j}$ and ρ the following expressions

$$(1) \qquad \rho = -\delta'(x)\,\delta(y)\,\delta(z)\,F(t) \qquad j_x = \delta(x)\,\delta(y)\,\delta(z)\,F'(t)$$

where $F(t)$ is the dipole moment.

In order to get the solution of Maxwell's equations we apply Schwartz's definition of convolution [1] between two distributions

$$(2) \qquad (S * T)_x \,.\, \varphi(x) = (S_\xi \times T_\eta)\,\varphi(\xi + \eta).$$

The solution will be [2] the convolution of the second member with the Green function of the wave equation.

The condition for the existence of the convolution (2) —in the sense of the distribution theory— is that [3], being both supports non-compacts, $\xi \varepsilon A$, $\eta \varepsilon B$ (A, support of S; B support of T) $\xi + \eta$ cannot be at finite distance unless ξ and η are both at finite distance. This condition is always fulfilled if the velocity of the dipole is less than the velocity of light.

We will show that this solution reduces to the ordinary one [4]. Let us take the scalar potencial

$$A_0 \,.\, \varphi = -\frac{\delta(r-t)}{r} * F(t)\,\delta'(x)\,\delta(y)\,\delta(z) \,.\, \varphi$$

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(1) L. Schwartz, *Théorie des distributions*, II, p. 11.

(2) *Ibid.* II, p. 68.

(3) *Ibid.* II, p. 26.

(4) Frank-v. Mises, *Diff. Gl. der Physik*, II, p. 789.

$$-=\frac{\delta(r-t)}{r}F(\tau)\,\delta'(\xi)\,\delta(\eta)\,\delta(s)\,.\,\varphi(x+\xi,y+\eta,z+\zeta,t+\tau)=$$

$$=\frac{\delta(r-t)}{r}F(\tau)\,.\,\frac{\partial\varphi(x,y,z,t+\tau)}{\partial x}$$

$$=F(\tau)\,.\iiint\frac{1}{r}\frac{\partial\varphi(x,y,z,\tau+r)}{\partial x}dx\,dy\,dz$$

$$=\iiiint\frac{F(\tau)}{r}\frac{\partial\varphi(x,y,z,\tau+r)}{\partial x}dx\,dy\,dz\,d\tau$$

putting $\qquad \tau+r=t \qquad \tau=t-r$

$$A_0\,.\,\varphi=\iiiint dx\,dy\,dz\,dt\,\frac{F(t-r)}{r}\,\frac{\partial}{\partial x}\varphi(xy\,zt).$$

But, according to the definition of derivative of a distribution, this means

$$A_0=-\frac{\partial}{\partial x}\,\frac{F(t-r)}{r}.$$

In general, for a multipole in the x-direction in the origen, we have

$$P_x=(-1)^{n-1}\,\delta^{(n-1)}(x)\,\delta(y)\,\delta(z)\,F(t)$$

$$\vec{j}=\frac{\partial\vec{P}}{\partial t} \qquad \rho=-\operatorname{div}P \quad \text{with} \quad P_y=0 \quad P_z=0.$$

If the multipole is moving, but always having its moment in the x-direction, we simply replace the arguments x,y,z,t by $x-\xi(t)$; $y-\eta(t)$; $\zeta(t)$.

Of course, for practical purposes one does not need to work with distribution theory, but simply work with the deltas as ordinary functions.

It is possible to solve problems with temporal multipoles, and multipoles oriented arbitrarily in space.

If, instead of using the wave equation, we use the Klein Gordon equation, we could calculate the field of «nucleonic multipoles».

(5) L. Schwartz, *op. cit.*, I, p. 35.