ON THE CLIFFORD AND JORDAN-WIGNER ALGEBRAS

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SUMMARY:

The theory of the Clifford algebras is discussed in connection with the finite-dimensional Jordan-Wigner algebras and the affine algebra \( G_n \) of Schönberg. The relations of \( G_n \) with the \( n \)-dimensional antisymmetric tensors allow to see in a simple way the origin of some of the properties of the Clifford algebras and the spinors, especially the connection between \( 2p \)-dimensional spinors and \( p \)-dimensional antisymmetric tensors. On the other hand the theory of \( G_n \) is further extended. The comparison with the Jordan-Wigner algebra leads to relations between the Clifford algebras and the functions of two-valued variables. The spinors are shown to be functions of the sub-sets of finite sets and the Jordan-Wigner algebra is discussed in relation with such sets.

The null vectors in the Clifford algebra.

1. The Clifford algebra is a metric vector calculus of a real \( n \)-dimensional flat space endowed with a non-degenerate metric quadratic form \( g_{jk} x^j x^k \), the \( x \) denoting cartesian coordinates. To any vector \( \vec{V} \) corresponds an element \( \gamma_V \) of the Clifford algebra, the correspondence being characterized by two conditions

   \[
   \begin{align*}
   \text{I)} & \quad c_1 \vec{V}_1 + c_2 \vec{V}_2 \rightarrow c_1 \gamma_{V_1} + c_2 \gamma_{V_2} \quad [1 \text{ a}] \\
   & \quad \text{(the } c \text{ denote numbers)} \\
   \text{II)} & \quad \gamma^2_V = g_{jk} V^i_j V^k \, \mathbf{1}_C \quad [1 \text{ b}] \\
   & \quad \text{(} \mathbf{1}_C \text{ denotes the unity of the Clifford algebra)}
   \end{align*}
   \]

It follows from the conditions (I) and (II) that

\[
\gamma_{V_1} \gamma_{V_2} + \gamma_{V_2} \gamma_{V_1} = 2 g_{jk} V^i_j V^k \, \mathbf{1}_C \quad [2]
\]

The \( V^i \) denote the cartesian components of the contravariant vector \( \vec{V} \). The Clifford algebra \( C_n(\Omega) \) of the \( n \)-dimensional flat space
endowed with the metric $g_{jk}$ can be defined as the associative algebra with a unity $1_\mathbb{C}$ generated by the symbols $\gamma_\mathbf{v}$ of the contravariant vectors $\mathbf{V}$ that satisfy the conditions (I) and (II). We may obviously take as generators the symbols of $n$ linearly independent contravariant vectors. We shall denote by $\gamma_j$ the symbol of the $j$-th basic vector $\mathbf{I}_j$ of the cartesian coordinate system. The $n$ symbols $\gamma_j$ can be taken as the generators of $C_n(\mathbb{O})$, the multiplication rule being

$$\gamma_j \gamma_k + \gamma_k \gamma_j = 2 \ g_{jk} 1_\mathbb{C} \tag{3}$$

as a consequence of equation [2]. It will be convenient to take $C_n(\mathbb{O})$ over the field of the complex numbers.

We shall denote by $\mathbf{W}$ a generic null vector: $g_{jk} W^j W^k = 0$. The symbols $\gamma_{\mathbf{W}}$ are nilpotent: $\gamma_{\mathbf{W}}^2 = 0$. Let $\mathbf{W}_1, \ldots, \mathbf{W}_r$ be $r$ mutually orthogonal and linearly independent null vectors. The unity $1_\mathbb{C}$ and the $r$ symbols $\gamma_{\mathbf{W}_1}, \ldots, \gamma_{\mathbf{W}_r}$ generate a $r$-dimensional Grassmann algebra, which is simply the Grassmann algebra of the $r$-dimensional totally null vector space of the linear combinations of the $r$ null and mutually orthogonal vectors $\mathbf{W}_1, \ldots, \mathbf{W}_r$. In the euclidean case the $\mathbf{W}$ are always complex vectors, but in the case of an indefinite metric there are also real null (or isotropic) vectors.

It is well known that when $n$ is even, $n = 2p$, the maximal number of linearly independent mutually orthogonal null vectors is $p$. The totally null (or totally isotropic) linear space defined by $p$ linearly independent and mutually orthogonal null vectors is the linear space of the vectors of a maximal flat space of the null hypercone of the $n$-dimensional space, whose equation is

$$g_{jk} x^j x^k = 0.$$  

When $n$ is odd, $n = 2p + 1$, the maximal number of linearly independent and mutually orthogonal null vectors is also $p$ and the vector space they generate is the linear space of the vectors of a maximal flat space on the hypercone $g_{jk} x^j x^k = 0$. Thereby when $n > 1$, there is a Grassmann algebra of the order $[n/2]$, associated to any maximal flat space on the hypercone $g_{jk} x^j x^k = 0$, generated by the symbols of any set of $[n/2]$ linearly independent vectors of the flat space, $[n/2]$ denoting the largest integer contained in $n/2$. We shall from now on write $[n/2] = p$. 
Let the $J_a$ be a set of $2 \lfloor n/2 \rfloor = 2p$ mutually orthogonal real vectors, such that $g_{jk} J_a^i J_b^j = \delta_a^b$, with $\epsilon_a = 1$ or $-1$. We shall introduce $2p$ null vectors $\vec{W}_a$ and $\vec{W}_a^+$

$$
2 \vec{W}_a = \sqrt{\epsilon_a} \vec{J}_a - \sqrt{-\epsilon_{a+p}} \vec{J}_{a+p}, \quad 2 \vec{W}_a^+ = \sqrt{\epsilon_a} \vec{J}_a + \sqrt{-\epsilon_{a+p}} \vec{J}_{a+p}
$$

(a = 1, 2, \ldots, p) \tag{4}

We have obviously

$$
\vec{W}_a \cdot \vec{W}_b = 0, \quad \vec{W}_a^+ \cdot \vec{W}_b^+ = 0, \quad 2 \vec{W}_a \cdot \vec{W}_a^+ = \epsilon_{a,b}
$$

(a, b = 1, \ldots, p) \tag{5}

the point indicating the inner product. The linear combinations of the $\vec{W}_a$ constitute a $p$-dimensional totally isotropic vector space and the linear combinations of the $\vec{W}_a^+$ another such vector space. When the space is euclidean, $\vec{W}_a^+$ coincides with the complex conjugate $\vec{W}_a^*$ of $\vec{W}_a$, since $\epsilon_a = 1$ for all $a$. The $2p$ null vectors $\vec{W}_a$ and $\vec{W}_a^+$ are linearly independent. When $n = 2p$, the $2p$ vectors $\vec{W}_a$ and $\vec{W}_a^+$ constitute a basis of the linear space of the vectors $\vec{V}$.

We shall call null basis of a $2p$-dimensional metric space a set of $2p$ null vectors $\vec{W}_a$, $\vec{W}_a^+$ defined in the above way.

When $n = 2p$, we can express any vector $\vec{V}$ as a linear combination of the vectors of a null basis: $\vec{V} = \omega_\alpha \vec{W}_\alpha + \omega_\alpha^+ \vec{W}_\alpha^+$, with summation over the still indices $\alpha$. We shall denote the null vector $\omega_\alpha \vec{W}_\alpha$ by $\vec{\omega}_\alpha$ and the null vector $\omega_\alpha^+ \vec{W}_\alpha^+$ by $\vec{\omega}_\alpha^+$. Thus any $\vec{V}$ is the sum of two null vectors $\vec{\omega}_\alpha$ and $\vec{\omega}_\alpha^+$, the $\vec{\omega}_\alpha$ belonging to a totally isotropic $p$-dimensional vector space and the $\vec{\omega}_\alpha^+$ to another such vector space. We have $g_{jk} V^i V^j = \sum \omega_\alpha \omega_\alpha^+.$

In the $2p$-dimensional euclidean space the null vectors $\vec{\omega}_\alpha$ and $\vec{\omega}_\alpha^+$ of a real vector $\vec{V}$ are complex conjugate. Hence there is a one-one linear correspondence between the real vectors $\vec{V}$ and the vectors $\vec{\omega}$ of the $p$-dimensional totally isotropic vector space of the complex linear combination of the $\vec{W}_a$. The euclidean geometry in the
real space is associated to the unitary geometry of the $\mathbf{V}$-space since $g_{ij} V^i V^j = \Sigma |\omega_{rs}|^2$. In the case of $p = 1$ we get the well-known correspondence between the real vectors of the Euclidean plane and the complex numbers.

We shall call maximally indefinite metric of a $2p$-dimensional flat space the metric whose canonical quadratic form has $p$ positive and $p$ negative squares. In this case, $p$ of the $e_a$ are equal to $1$ and the other $p$ are equal to $-1$. We may numerate the $\mathbf{J}_a$ in such a way that the $e_a$ be $1$ and the $e_{a+p}$ be $-1$. Thus all the $\mathbf{W}_a$ and $\mathbf{W}_a^+$ are real and we get two real totally isotropic $p$-dimensional vector spaces of $\omega_r$ and $\omega_r^+$ associated to the linear space of the real $\mathbf{V}$.

For other types of indefinite metric there are no real $p$-dimensional totally isotropic vector spaces. When the metric is maximally indefinite and $n$ is divisible by $4$, $p = 2q$, we may numerate the $\mathbf{J}_a$ in such a way that $e_1 = e_2 = \ldots = e_q = e_{p+1} = \ldots = e_{p+q} = 1$ and $e_{q+1} = \ldots = e_p = e_{p+q+1} = \ldots = e_{2p} = -1$, in order that $\mathbf{W}_r = \mathbf{W}_r^*$ for $r = 1, \ldots, q$ and $\mathbf{W}_s^+ = -\mathbf{W}_s^*$ for $s = q + 1, \ldots, p$.

Thus for a real $\mathbf{V}$, $\omega_r^{+*} = -\omega_r^*$ and $\omega_r^{+**} = -\omega_r^{*+*}$ and $g_{ij} V^i V^j = \Sigma |\omega_{rs}|^2 - \Sigma |\omega_{rs}|^2$, so that the indefinite metric of the $\mathbf{V}$-space is now associated to an indefinite hermitian metric in the linear space of the complex vectors $\omega$. It is easily seen in a similar way that the indefinite metric whose canonical quadratic form contains $2m_+$ positive squares and $2m_-$ negative ones can be associated to the hermitian metric of the complex $\omega$-space $\Sigma |\omega_{rs}|^2 - \Sigma |\omega_{rs}|^2$ with $b = 1, \ldots, m_+$ and $c = m_+ + 1, \ldots, m_+ + m_-$. The elements $\gamma_{\mathbf{W}_a}$ and $\gamma_{\mathbf{W}_a^+}$ will be denoted by $\mathbf{W}_a$ and $\mathbf{W}_a^+$, for the sake of simplicity. It follows from the equations [5] that

$$W_a W_b + W_b W_a = 0 , \quad W_a^+ W_b^+ + W_b^+ W_a^+ = 0 ,$$

$$W_a W_b^+ + W_b W_a^+ = \varepsilon_{a,b} 1_c$$

[6]

When $n = 2p$, the $\mathbf{W}_a$ and $\mathbf{W}_a^+$ are a set of generators of the Clifford algebra $C_p^{2p}$ of our $2p$-dimensional metric space. We may also interpret them as the generators of an associative algebra of a pair
of $p$-dimensional vector spaces, in which the symbols $\omega$ and $\omega^+$ have the multiplication rules

\[\begin{align*}
\omega_1 \omega_2 + \omega_2 \omega_1 &= 0, \\
\omega_1 \omega_2^+ + \omega_2 \omega_1^+ &= 0 \quad [7\ a] \\
\omega_1 \omega_2^+ + \omega_2 \omega_1^+ &= \sum \omega_1^\ast \omega_2^\ast 1_c \quad [7\ b]
\end{align*}\]

By taking $\omega = \omega^\ast W_a$ and $\omega^+ = \omega^+ W_a^+$, we see that the rules [7 a] and [7 b] are a consequence of [6]. Actually we have

$\omega = \gamma_\omega$ and $\omega^+ = \gamma_\omega^+$.

We may interpret the $\omega^+$ as the covariant vectors of a $p$-dimensional space whose contravariant vectors are the $\omega$. Thus the $\omega, \omega^+$ algebra appears as the algebra $G_p$ of a $p$-dimensional affine space introduced by Schöenberg (\textsuperscript{1}). The interpretation of the $\omega^+$ as covariant vectors of a $p$-dimensional space is quite natural, since any $\omega^+$ defines a linear functional on the $\omega$, namely that which associates to $\omega$ the number $\omega \cdot \omega^+$, and any linear functional can be obtained from one and only one vector of the $\omega^+$ space by means of the inner product of the $n$-dimensional metric.

The possibility of reinterpreting the Clifford algebra of a $2p$-dimensional real space as a kind of algebra of a $p$-dimensional complex space is particularly natural in the case of the $2p$-dimensional euclidean metric, in which there is a one-one correspondence between the real $\mathbb{R}$ and the complex $\mathbb{C}$. The euclidean $C_{2p}$ can be identified to an algebra of the unitary geometry of the complex $\mathbb{C}$ space. Now $W$ and $W_a^+$ are the symbols of complex conjugate vectors and it is convenient to replace the $W_a^+$ by new symbols $W_a^\ast$. Thus the multiplication rules [6] for the euclidean case become

\[\begin{align*}
W_a W_b + W_b W_a &= 0, \\
W_a^\ast W_b^\ast + W_b^\ast W_a^\ast &= 0, \\
W_a W_b^\ast + W_b^\ast W_a &= \xi_{a,b} 1_c \quad [8]
\end{align*}\]

The multiplication rules [8] are analogous to those of the creation and anihilation operators of the quantum theory of a system of particles obeying the Fermi statistics, developed by Jordan and Wigner (\textsuperscript{2}). Now there is only a finite number $p$ of $W_a$, whereas there are infinite $W_a$ in the Jordan-Wigner formalism. Thereby it is reasonable to call the $W_a, W_a^\ast$ algebra a $p$-dimensional Jordan-
Wigner algebra. The original Jordan-Wigner formalism may be viewed as a geometric algebra of a separable complex Hilbert space of infinite dimensionality.

The choice of the totally isotropic $\omega$ and $\omega^+$ vector spaces depends on the choice of a null basis $\tilde{W}_a$, $\tilde{W}_a^+$, but it does not depend on that of the basic vectors $\vec{l}_j$ of the cartesian frame of coordinates, when the $\vec{l}_a$ are not related to the vectors $\vec{I}_j$. When $n = 2p$, it is possible to take a set of $\tilde{W}_a$ and $\tilde{W}_a^+$ as basic vectors $\vec{I}_j$. This is particularly convenient in the case of the maximally indefinite metrics in which we may take $2p$ real $\tilde{W}_a$ and $\tilde{W}_a^+$, as we have already seen.

The affine algebra $G_n$.

2. Let the $\vec{l}_j$ and $I^i$ be the basic contravariant and covariant vectors of a $n$-dimensional affine space. We shall denote the contravariant vectors by $\vec{V}$ and the covariant ones by $U$. Thus $\vec{V} = V^i \vec{l}_j$ and $U = U^i J_j$. The invariant $V^i U_j$ will be denoted by $\langle V, U \rangle$. The elementary vector calculus involves an affine product of contravariant vectors $\vec{V} \wedge \vec{V}'$ and a metric product $\vec{V} \times \vec{V}' = g_{jk} V^i V'^j$. That calculus does not distinguish the contravariant and covariant vectors. Such a procedure is not inconvenient as long as one deals only with an euclidean space and uses only orthogonal cartesian coordinates.

The Grassmann algebra is an affine calculus of contravariant vectors, closely related to the ordinary vector calculus. In the Grassmann algebra the generators are the symbols $I^i$ of the $\vec{l}_j$ and a unity (1) with the anticommutative multiplication rule $I^i I^k + I^k I^i = 0$. It is also possible to build a Grassmann algebra of the covariant vectors generated by a unity (1) and the symbols $I^i$ of the $I^i$, with the rule $I^i I^k + I^k I^i = 0$. Schöenberg (1) introduced the algebra $G_n$ of the contravariant and covariant vectors, with a unity (1), generated by the $2n$ elements $I^i$ and $I^j$ with the multiplication rules

$$I^j I^k + I^k I^j = 0, \quad I^i I^k + I^k I^i = 0, \quad I^j I^k + I^k I^j = \delta^j_k (1) \quad [1]$$
In the algebra $G_n$ the vectors $\mathbf{v}$ and $\mathbf{u}$ are represented by symbols $V$ and $U$ with the multiplication rules
\[
V V' + V' V = 0, \quad U U' + U' U = 0,
\]
\[
V U + U V = \langle V, U \rangle (1)
\]

Let us consider the direct sum of the $\mathbf{v}$ and $\mathbf{u}$ linear spaces. We may take the $\mathbf{i}_j$ and $\mathbf{j}_i$ as a basis of the sum space. Thus any vector $\mathbf{\Omega}$ of the sum-space can be represented as follows:
\[
\mathbf{\Omega} = V_1 \mathbf{i}_j + U_j \mathbf{j}_i = \mathbf{v} + \mathbf{u}.
\]

The affine geometry of the $n$-dimensional space leads naturally to the introduction of a metric in the $\mathbf{\Omega}$ - space, for which the square of the length of $\mathbf{\Omega}$ is $\langle V, U \rangle$ and the inner product of $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ is $1/2 (\langle V, U' \rangle + \langle V', U \rangle)$. The $\mathbf{v}$ and $U$ may be regarded as special kinds of null vectors of the $\mathbf{\Omega}$ - space constituting two totally isotropic $n$-dimensional vector spaces. The $2n$ vectors $\mathbf{i}_j$ and $\mathbf{j}_i$ constitute a null basis of the $\mathbf{\Omega}$ - space endowed with the above $\langle \mathbf{v}, \mathbf{u} \rangle$ metric.

Schönberg pointed out (1) that the affine algebra $G_n$ is the Clifford algebra $C_{2n}$ of the real $2n$-dimensional $\mathbf{\Omega}$ - space endowed with the maximally indefinite metric $\langle V, U \rangle = V^j U_j$. The approach to the theory of $C_{2n}$ given in section 1 is obviously suggested by the theory of $G_n$ as a $C_{2n}$. There, the $2p$-dimensional $\mathbf{v}$ - space was essentially treated as the direct sum of the two totally isotropic $\mathbf{w}$ and $\mathbf{w}^+$ vector-spaces. When the vectors $\mathbf{w}_a$ and $\mathbf{w}_a^+$ of a null basis are taken as basic cartesian vectors of the $2p$-dimensional cartesian space, the expression of the square of the length of $\mathbf{v}$ is $\sum \omega_{r}^* \omega_{r}^{*+}$, analogous to the expression $V^j U_j$ of the square of the length of a $2n$-dimensional vector $\mathbf{\Omega}$.

$G_n$ may be regarded as the basic elementary vector calculus of the $n$-dimensional affine geometry. It involves only the affine vector product, as the Grassmann algebra, but it deals symmetrically with the covariant and contravariant vectors, whilst the
Grassmann algebra deals only with one of the two kinds of vectors. The symmetry with respect to the two kinds of vectors gives to $G_n$ an extremely simple algebraic structure: $G_n$ and the algebra of the $2^n \times 2^n$ matrices taken over the same field of characteristic zero are isomorphic. Schönberg (3) showed that $G_n$ is the algebra of the linear operators on the vectors $\xi$ of the linear space of dimensionality $2^n$ obtained by the direct sum of the linear spaces of the covariant antisymmetric tensors of the orders $0, 1, 2, \ldots, n$ of the $n$-dimensional affine space. It follows from the symmetry of $G_n$ with respect to the covariant and contravariant vectors that it is also the algebra of the linear operators of the vector space obtained by the direct sum of the linear spaces of the antisymmetric contravariant tensors of the orders $0, 1, \ldots, n$ of the $n$-dimensional affine space. We shall denote by $\eta$ the vectors of the latter $2^n$-dimensional space.

When $G_n$ is treated as the algebra of the linear operators on the $\xi$, the $\xi$ appear as contravariant spinors and the $\eta$ as covariant spinors of the $2^n$-dimensional space endowed with the maximally indefinite metric $<V, U>$. When $G_n$ is regarded as the algebra of the linear operators on the $\eta$, they appear as the contravariant spinors and the $\xi$ as the covariant spinors of the $2^n$-dimensional space with the metric $<V, U>$.

There is a set of $n$ commutable idempotent elements $N_j = I_j I_j$ associated to the basic cartesian vectors. The $\hat{N}_j = I_j I_j$ are also idempotent and commutable. The elements $P = N_1 \ldots N_n$ and $Q = N_1 \ldots N_n$ do not depend on the choice of the $I_j$ and are also idempotent. It is easily seen that $I_j P = 0, P I_j = 0, I_j Q = 0$ and $Q I_j = 0$. Let us introduce the elements $P_{j_1, j_2, \ldots, j_n} = P I_{j_1} I_{j_2} \ldots I_{j_n}$ and $P^{i_1, i_2, \ldots, i_n} = I_{i_1} I_{i_2} \ldots I_{i_n} P$; we shall take $j_1 < j_2 < \ldots < j_n$ or $j_1'. It is easily seen that $P_{(j)} P_{(k)} = \xi_{(j)}^{(k)}$ (1). By means of the $P_{(j)}$ and $P^{(k)}$ we build the $2^{2n}$ elements $P_{j_1, j_2, \ldots, j_n}$ with all the ordered sets of values $j_1, j_2, \ldots, j_n$.

It follows from the orthogonality relations [3] that the $2^{2n}$ elements $P_{j_1, j_2, \ldots, j_n}$ are linearly independent and can be taken as a basis of $G_n$. This choice of the basis gives to $G_n$ the form of the algebra of the linear operators on the above $\xi$ spinors. In order to give to $G_n$ the
form of the algebra of the linear operators on the $\sigma$ it suffices to take as basis of $G_n$ the $2^{2n}$ elements $Q_{(j)}^{(k)} = Q_{(j)} Q_{(k)}$ with $Q_{(j_1,\ldots,j_r} I_{j_{r+1}} \ldots I_{j_r)} Q$ and $I_{k_1,\ldots,k_s} = Q I_{k_{s+1}} \ldots I_{k_s}$. Since

$$Q_{(j)}^{(k)} Q_{(k)}^{(j)} = 2^{(k)} Q_{(j)}^{(j)}$$

the $2^{2n}$ elements $Q_{(j)}^{(k)}$ corresponding to all the $2^{2n}$ different pairs of ordered sets $(j), (k)$ are linearly independent.

Let us consider the left ideal $G_n P$ constituted by the products $\Gamma P$ of the elements $\Gamma$ of $G_n$ by the idempotent $P$. $G_n P$ is a vector space of dimensionality $2^n$ having as basis the $2^n$ elements $P_{(j)}$ with all the distinct ordered sets $(j)$. $G_n P$ is essentially the direct sum of the linear spaces of the covariant antisymmetric tensors of all orders, endowed with a product that renders it an algebra. The $\Gamma$ may be regarded as the linear operators on the elements $\Gamma P$ of $G_n P$ that transform $\Gamma P$ into $\Gamma \Gamma' P$. Thus $G_n$ appears as the algebra of the linear operators on the $\xi$.

The linear space $G_n Q$ admits as a basis the set of the $2^n$ elements $Q_{(j)}$ corresponding to all the distinct ordered sets $(j)$. $G_n Q$ is essentially the direct sum of the linear spaces of the antisymmetric contravariant tensors of all orders, endowed with a product that renders it an algebra. The elements of $G_n$ can be associated to linear operators on the vectors of $G_n Q$. Thus $G_n$ appears as the algebra of the linear operators on the $\eta$.

Matrix representations of $G_n$ and $C_{2^p}$.

3. The representation of the elements of $G_n$ by linear operators on the $\xi$, discussed in section 2, gives a natural representation of the elements of $G_n$ over the real numbers by real $2^n \times 2^n$ matrices. In that representation the matrix of $I_j$ is the transpose of that of $I_j$ and the matrices of the $n$ commutable idempotent elements $N_j = I_j^2$ are diagonal. The matrices of the elements $J_j = I_j + I_j^2$ are therefore symmetrical and those of the $J_{j+n} = I_j - I_j^2$ are antisymmetrical. The $J_{j}$ are the symbols of $2^n$-dimensional orthogonal vectors $J_{\ast}^n$.

The generators $w_a$ and $w_a^+$ of a $C_{2^p}^{(j)}$ over the complex numbers introduced in section 1 have the same multiplication rules as the generators $I_a$ and $I_a^+$ of a $G_p$. We may therefore represent any $C_{2^p}$ over the complex field by $2^p \times 2^p$ complex matrices, in such a way that for all $a$ the matrices of $w_a$ and $w_a^+$ be both
real and mutually transposed, hence adjoint, by means of the above representation of $G_p$. In the present representation of $C_{(1)}^{2p}$ the matrices of the real $V$ are not always hermitian when $g_{ij} V^i V^j > 0$ and anti-hermitian when $g_{ij} V^i V^j < 0$. The matrices of the elements $w_a^+ w_a$ are diagonal and real.

The present representation of $G_n$ gives also a representation of the $n$-dimensional Jordan-Wigner algebra over the real numbers by $2^n \times 2^n$ real matrices, which is analogous to the well known representation of the quantum field theory of the fermions, in which the occupation-number operators of the fermion states are diagonalized.

The above vectors $\vec{J}_a, \vec{J}_j = \vec{I}_j + \vec{I}_i$ and $\vec{J}_{j+n} = \vec{I}_j - \vec{I}_i$, constitute an orthonormal basis of the $2^n$-dimensional $\mathbb{H}$-space with the metric $<V, U>$. The $n$-vectors $\vec{J}_{a_1 \ldots a_n} = \vec{J}_{a_1} \wedge \vec{J}_{a_2} \wedge \ldots \wedge \vec{J}_{a_n}$ are associated to the elements $J_{a_1 \ldots a_n} = J_{a_1} \ldots J_{a_n}$. The set of those elements constitutes a symbolic antisymmetric tensor of the order $n$ of the $\mathbb{H}$-space. It is easily seen that the square of any of those elements is either (1) or $-1$. and that

$$J_{a_1 \ldots a_n} J_{a_1 \ldots a_n}^{-1} = \epsilon^i_{a_1 \ldots a_n} I^i \text{ with } \epsilon^i_{a_1 \ldots a_n} = 1 \text{ or } -1 \quad [1]$$

We have $\epsilon^i_{a_1 \ldots a_n} = (-1)^{n-1}$ and $\epsilon^i_{a_1 \ldots a_n} = (-1)^n$. We shall take when $n$ is odd $J = J_1 \ldots J_n$ and when $n$ is even $J = J_{n+1} \ldots J_{2n}$ in order that for all $n$

$$J I_j J^{-1} = I_i \quad \text{and} \quad J I_i J^{-1} = I_j \quad [2]$$

The inner automorphism $\Gamma \rightarrow J \Gamma J^{-1}$ of $G_n$ corresponds to the transposition of the matrices of the elements associated to the $\mathbb{H}$, in the above representation of $G_n$.

Since $J N_j J^{-1} = \hat{N}_j$, we have $J P J^{-1} = Q$ and

$$J P^{k_1 \ldots k_r} J^{-1} = Q^{l_1 \ldots l_r} \quad [3]$$

The equations [2] and [3] are not covariant for a change of the basic vectors $\vec{I}_j$. This results from the fact that $J$ depends on the choice of the $\vec{I}_j$. 
We shall denote $J_{n+1,...,j_n}$ by $C$. It is easily seen that
\[
C \, P^{j_1,...,j_r} = (-1)^{\sigma_r} \frac{1}{(n-r)!} \varepsilon_{j_1,...,j_n} \, P^{j_{r+1},...,j_n} \tag{4}
\]
\[
2 \, g_r = (2n + r) (r - 1)^k \tag{5}
\]
Hence
\[
C \, \{ (r!)^{-1} \varepsilon_{j_1,...,j_r} \, P^{j_1,...,j_r} \} = \sum \{ (n-r)! \}^{-1} \eta^{i_1,...,i_{n-r}} \, P^{j_1,...,j_n} \tag{6}
\]
with
\[
(r!) \eta^{i_{r+1},...,i_n} = (-1)^{\sigma_r} \varepsilon^{j_1,...,j_n} \, \varepsilon_{i_1,...,i_r} \tag{7}
\]
Equations (6) and (7) show that the linear operator on the $\xi$ corresponding to the element $C$ transforms $\xi$ into the $\eta$ whose components are defined by (7) with $r = 0, 1, ..., n$. The element $C$ is therefore related to the duality of the antisymmetric tensors of the $n$-dimensional space.

**Alternative construction of $G_n$.**

4. The tensor product of the linear spaces of the $\mathbf{V}$ and $\mathbf{U}$ gives the linear space of the mixed tensors $A_j^k$. We can associate to $A_j^k$ the linear operator $A$ on the $\mathbf{V}$ defined as follows: $A \mathbf{V} = A_j^k \mathbf{V}^l \mathbf{I}_k$. Thus the space of the tensors $A_j^k$ becomes an associative algebra, the product of the tensors $A_j^k$ and $B_j^k$ taken in a given order being the tensor associated to the product of the operators $A$ and $B$ taken in the same order: $(A_j^k) (B_j^k) = (C_j^k)$ with $C_j^k = A_j^k B_j^k$.

The tensor $A_j^k$ may also be associated to a linear operator $\mathbf{A}$ on the $\mathbf{U}$ defined as follows: $\mathbf{A} \mathbf{U} = U_k A_j^k \mathbf{I}$. The space of the $A_j^k$ is now endowed with the structure of an associative algebra in which the product of $A_j^k$ and $B_j^k$ is the tensor of the product of the corresponding operators on the vectors $U \, \{ A_j^k \} \, \{ B_j^k \} = \{ A_j^k B_j^k \}$.

The present algebra of the $A_j^k$ is the reciprocal of that corresponding to the linear operators on the contravariant vectors $\mathbf{V}$.

The above considerations can obviously be applied to any pair of dual spaces of finite dimensionality. The linear spaces of the $\xi$ and $\eta$ of section 2 are dual ones. Since $G_n$ is the algebra of the linear operators on the $\xi$, it is the algebra obtained from the tensor product of the $\xi$ and $\eta$ spaces that corresponds to the operators on the $\xi$. $G_n$ is also the algebra of the linear operators on the $\eta$ obtained
from the tensor product of the $\xi$ and $\eta$ spaces. To any set of $(n + 1)^2$ tensors $A_{j_1j_2}^{k_1k_2}$ ($r$, $s = 0, 1, \ldots, n$), antisymmetrical with respect to the indices $j$ and also with respect to the indices $k$, correspond two elements $\sum' A_{j_1j_2}^{k_1k_2} F_{j_1j_2}^{i_1i_2}$ and $\sum' A_{j_1j_2}^{k_1k_2} Q_{i_1i_2}^{k_1k_2}$ of $G_n$, the former corresponding to a linear operator on the $\xi$ and the latter to a linear operator on the $\eta$, (the sign $\sum'$ indicates that the summation is taken only over the ordered sets with $j_1 < j_2 < \ldots < j_r$ and $k_1 < k_2 < \ldots < k_s$).

The double role of $G_n$ as operator algebra on the $\xi$ and $\eta$ is related to the tensor duality, in which $A_{j_1j_2}^{k_1k_2}$ corresponds to $B_{k_1k_2}^{i_1i_2} = \varepsilon_{i_1i_2}^{j_1j_2} A_{j_1j_2}^{k_1k_2} (r \neq s)^{-1}$, $\varepsilon$ denoting the generalized Kronecker delta. This follows from the fact for any values of the $j$ and $k$

$$Q_{i_1i_2}^{k_1k_2} = (-1)^r (r-1)^2 + s (s-1)^2 \left((n-r)! (n-s)\right)^{-1} (-1)^r \varepsilon_{i_1i_2}^{j_1j_2} P_{j_1j_2}^{k_1k_2}$$  

[1]

The passage from the $P$ basis of $G_n$ to its $Q$ basis corresponds therefore to the replacement of the tensors $A$ by their dual $B$.

The present definition of $G_n$ as the algebra of the linear operators of the $\xi$ — space leads immediately to the introduction of the $P_{G(b)}^{(q)}$, which correspond to the dyadics associated to the basic vectors of the $\xi$ — space. The element (1) corresponds to the operator unity. Hence we have

$$(1) = \sum_r (r!)^{-1} P_{j_1j_2}^{i_1i_2}$$  

[2]

We can now define the $I_i$ and $I^i$ in terms of the $P_{G(b)}^{(q)}$

$$I_i = \sum_r (r!)^{-1} P_{j_1j_2}^{j_1j_2} , \quad I^i = \sum_r (r!)^{-1} P_{j_1j_2}^{j_2j_1}$$  

[3]

Since $C_{2p}^{(q)}$ is the $G_p$ of the totally isotropic $p$-dimensional $\omega$-space introduced in section 1, we may define $C_{2p}^{(q)}$ as the algebra of the linear operators on the vectors of the direct sum of the linear spaces of the covariant antisymmetric tensors of all orders of the $\omega$-space. Those $2^p$-dimensional vectors are spinors of the $2$ $p$-dimensional metric $\vec{V}$-space. Thus we have a direct simple definition of the $2$ $p$-dimensional spinors and also a definition of the Clifford algebra $C_{2p}^{(q)}$ by means of those spinors.
It is important to note that in the above definition of $G_n$ we may replace the $\xi$-space by the direct sum of the linear spaces of the antisymmetric covariant relative tensors of weight $\alpha$ and the $\eta$-space by the direct sum of the linear spaces of the contravariant antisymmetric relative tensors of weight $-\alpha$. The choice $\alpha = 0$ is actually not the most interesting one from the point of view of the tensor duality: by taking $\alpha = -1/2$ we get in the linear space of the $\xi$ the invariant bilinear forms

$$
(\xi, \xi')_f = \sum_{r=0}^{n} \left\{ \frac{r!}{(n-r)!} \right\} \xi_{j_1, \ldots, j_r} \xi'_{j_1', \ldots, j_r', \ldots, j} f_r,
$$

where $\xi_{j_1, \ldots, j_r}$ denotes the well known Ricci tensor density and the $f_r$ arbitrary numerical coefficients. The existence of those forms follows from the duality

$$
\xi_{j_1, \ldots, j_r} = (r!)^{-1} \xi_{j_1, \ldots, j_r}
$$

The choice of the weight $\alpha = -1/2$ is however not possible in the real domain, because a real relative tensor of weight $-1/2$ in a cartesian system has imaginary components in another system with opposite orientation.

The equation $(\xi, \xi')_f = 0$ is invariant with respect to the changes of cartesian coordinates in the $n$-dimensional affine space. It does not depend on the choice of the weight $\alpha$ given to the $\xi_{j_1, \ldots, j_r}$. Let us interpret the components $\xi'_p$ of $\xi$ as homogeneous projective coordinates in a space of dimensionality $2^n - 1$, and the $\xi'_p$ in a similar way. The equation $(\xi, \xi')_f = 0$ associates a hyperplane of the projective $\xi$-space to a point of the $\xi$-space and conversely.

The fundamental polarity in the projective $\xi$-space

5. We shall now treat the components of $\xi$ as homogeneous projective coordinates of a point in a $2^n - 1$ dimensional space and the components of $\eta$ as homogeneous hyperplane coordinates in the same space. We shall write the equations of the tensor duality as follows

$$
\Lambda_{\eta_{j_1, \ldots, j_r}} = (-1)^r \xi_{j_1, \ldots, j_r} \xi'_{j_1', \ldots, j_r'} (r!)^{-1}
$$

$$
2 g_r = (2n + r)(r - 1)
$$
The set of equations \[1\] for \(r = 0, 1, \ldots, n\) defines a correlation \(\eta^s = C^s \xi_s\) in the \(\xi\)-space. Since \(C^s = (-1)^{n(n+1)/2} C^s\) the correlation is involutoric. There is a fundamental polarity in the projective \(\xi\)-space associated to the duality of the antisymmetric tensors. The bilinear equation of the polarity is \((\xi, \xi') = 0\).

\[
(\xi, \xi')_{k_1, \ldots, k_n} = \sum_r \{ r! (n - r)! \}^{-1} \xi_{k_1, \ldots, k_n} \xi'_{j_1, \ldots, j_n} (-1)^{s_r} \quad [3]
\]

\[
(\xi, \xi') = (-1)^{n(n+1)/2} (\xi, \xi') \quad [4]
\]

When \(n(n+1)/2\) is even, we have a polarity with respect to the hyperquadric \((\xi, \xi) = 0\). When \(n(n+1)/2\) is odd, the form \((\xi, \xi')\) is antisymmetric, so that the polarity is taken with respect to a linear complex.

The \(\xi\)-polarity corresponds to the well known polarity of the spinor theory. The above discussion shows that the spinor polarity of a 2\(p\)-dimensional metric space is essentially the polarity of the projective \(\xi\)-space of a \(p\)-dimensional affine space, arising from the duality of the \(p\)-dimensional antisymmetric tensors. It is known from the spinor theory that the spinor polarity is invariant for the linear transformations of the spinors induced by the 2\(p\)-dimensional rotations of the metric space. Since the \(\xi\) are the spinors of the 2\(n\)-dimensional space of the \(\tilde{\Omega} = \tilde{V} + U\) endowed with the maximally indefinite metric \(<V, U>\), we see that the \(\xi\)-polarity must be invariant for the linear transformations of the \(\xi\) induced by the 2\(n\)-dimensional rotations of the \(<V, U>\) metric, which constitute a group larger than the \(n\)-dimensional linear group.

The \(\xi\)-polarity is obviously related to the transformation of the linear space of the \(\xi\) defined by the element \(C\) of \(G_n\) introduced in section 3. We have \(\eta = C_{op} \xi, C_{op}\) denoting the linear operator corresponding to the element \(C\). It follows from the equations \([3]\) and \([1]\) that

\[
(\xi, \xi')_{k_1, \ldots, k_n} = \varepsilon_{k_1, \ldots, k_n} \sum_r (r!)^{-1} (C_{op} \xi)_{j_1, \ldots, j_r} \xi'_{j_1, \ldots, j_r} \quad [5]
\]

The above coefficients \(C^{s, \tau}\) are simply matrix elements of \(C_{op}\).

The \(\xi\) as functions of \(n\) two-valued variables.

6. Let us consider \(n\) variables \(N'_j\) which take only the values 0 and 1. The linear space of the real functions \(F(N')\) is of dimen-
sionality $2^n$. It is possible to establish a one-one linear correspondence between the $\xi$ and the $F(N')$ by associating to $\xi$ the function $\xi(N') = \sum_j \xi_{j_1, \ldots, j_r} F_{j_1, \ldots, j_r}(N')$, the summation being taken only over the ordered sets of values $j_1 < j_2 < \ldots < j_r$ and $F_{j_1, \ldots, j_r}(N') = \hat{\xi}(N'_j - 1) \ldots \hat{\xi}(N'_r - 1) \hat{\xi}(N'_{j_1}) \ldots \hat{\xi}(N'_{j_r})$, with $\hat{\xi}(u) = \tilde{\xi}_{u, u}$.

We may write $P_{j_1, \ldots, j_r} = P(N', N'')$ with $F_{j_1, \ldots, j_r}(N') F_{j_1, \ldots, j_r}(N'') = 1$, $j_1 < \ldots < j_r$ and $k_1 < \ldots < k_q$. There is a one-one linear correspondence between the elements $\Gamma$ of $G_n$ and the functions

$$\Phi(N', N'') : \Gamma \rightarrow \Phi_{\Gamma}(N', N''),$$

with

$$\Gamma = \sum_{N', N''} \Phi_{\Gamma}(N', N'') P(N', N'').$$

The values of the function $\Phi_{\Gamma}$ are the matrix elements of $\Gamma$ in the representation of $G_n$ discussed in section 3.

The two-valued variable $N'_j$ is closely related to the element $N_j$, since $N_j P_{j_1, \ldots, j_r} = N'_j P_{j_1, \ldots, j_r}$ and $P_{j_1, \ldots, j_r} N_j = N'_j P_{j_1, \ldots, j_r}$, with $N_j' = 1$ when $j$ coincides with one of the indices $j_1, \ldots, j_r$ and $N_j' = 0$ when this does not happen. There is an idempotent element $N_{\Omega} = <V, U>^{-1} U V$ associated to any non-isotropic direction of the space of the space of the $\Omega$, since $N_{\Omega}$ is not changed when $\Omega$ is replaced by a $\widetilde{\Omega}$, a being a non-null number. $N_j$ is the $N_{\Omega}$ of the direction of $I_j + I'$ The correspondence between the $N_{\Omega}$ and the non-isotropic directions of the $\Omega$ — space is not one-one, for the same $N_{\Omega}$ is associated to all the $a \mathbf{V} + b U$ with $ab \neq 0$. There is another idempotent element $\widehat{N}_{\Omega} = <V, U>^{-1} V U = (1) - N_{\Omega}$ associated to the $a \mathbf{V} + b U$ with $ab \neq 0$.

We can establish a one-one linear correspondence between the $\gamma$ and the functions of the $n$ two-valued variables $\widehat{N}'_j = 1 - N'_j$, which are related to the elements $\widehat{N}_j$ of $G_n$. There is also a one-one linear correspondence between the elements $\Gamma$ of $G_n$ and the functions $\Psi(\widehat{N}', \widehat{N}'')$ of the two-valued variables $\widehat{N}'_j$ and $\widehat{N}'''_j$.

It may be more convenient to use the variables $2 N'_j - 1 = N'_j - \widehat{N}'_j$, that take the values 1 and $-1$, instead of the $N'_j$.
or $\hat{N}_j$ with the values 0 and 1. Since $2 N_j - (1) = (I_j + I^i) (I_j - I^i)$, the $2 N_j - 1$ correspond to indices of inner orientation of two dimensional manifolds parallel to the pairs of vectors $\vec{I}_j + I^i$ and $\vec{I}_j - I^i$ of the $\mathbb{R}^n$ space. The $n$-dimensional geometrical interpretation of those indices is not quite obvious.

The close relation between the theory of the antisymmetric tensors of an affine space and that of the functions of two-valued variables is very interesting, especially in the four dimensional case, because of its physical importance. The existence of such a relation indicates that there are fundamental discrete properties of space and space-time, as yet not well understood, that may play an important part in the theory of the elementary particles.

The identification of the $\xi$ with the functions of $n$ two-valued variables $N_j'$ corresponds to the description of the states of a quantized field of fermions by functions of an enumerable infinity of two-valued variables analogous to the above $N_j'$. The states of the quantized fermion field are vectors of a linear representation-space of the Jordan-Wigner algebra of the emission and absorption operators, which is isomorphic to a $G_\infty$. The field variables analogous to our $N_j'$ give the numbers of fermions in the different particle-states. They have the values 0 and 1, because in each particle state there may be at most one fermion, as a consequence of the Pauli principle.

The possibility of using either the $\hat{N}_j$ or the $N_j'$ in our formalism corresponds to the well known symmetry of the formalism of the quantized fermion field with respect to the particles and the “holes”. The exchange of the $N_j'$ and the $\hat{N}_j'$ corresponds to that of the particles and the “holes”, because we have $\hat{N}_j' = 1 - N_j'$, so that the value 1 of $N_j'$ corresponds to the value 0 of $\hat{N}_j'$ and conversely.

$\xi$ may be regarded as a function of the sub-sets of the set of the $n$ basic vectors $\vec{I}_j$, whose value for the sub-set $(\vec{I}_{j_1}, \ldots, \vec{I}_{j_r})$ with $j_1 < j_2 < \ldots < \ldots < j_r$ is $\xi_{j_1, \ldots, j_r}$. The antisymmetric covariant tensors of the order $r$ are thus associated to the sub-sets with $r$ elements of the set of the $\vec{I}_j$. In particular the scalars correspond to the functions of the empty sub-set.
The $2p$-dimensional spinors.

7. The spinors of a $n$ dimensional metric space are the vectors of a linear space in which $C_n^{(2)}$ has an irreducible representation. The identity of $C_n^{(2p)}$ and the algebra generated by the $W_a$ and $W_a^+$ allows to obtain in a simple way the spinors of $C_n^{(2p)}$, as we shall show. Let us introduce the idempotent elements $\hat{N}_a = W_a W_a^+$. The $\hat{N}_a$ are all commutable, so that $P = \hat{N}_1 \ldots \hat{N}_p$ is also idempotent: $P^2 = P$. Since $W_a \hat{N}_a = \hat{N}_a W_a^+ = 0$ and both $W_a$ and $W_a^+$ are commutable with $\hat{N}_b$ when $b \neq a$, we have

$$W_a P = P W_a^+ = 0$$  \[1\]

Let us introduce the elements

$$P_{a_1, \ldots, a_r, b_1, \ldots, b_s} = W_{a_1}^+ \ldots W_{a_r}^+ P W_{b_1} \ldots W_{b_s}$$  \[2\]

which are antisymmetrical with respect to the $r$ indices $a$ and with respect to the the $s$ indices $b$ too. The $P_{(a); (b)}$ will be taken with $a_1 < a_2 < \ldots < a_r$, and $b_1 < b_2 \ldots < b_s$. We have

$$P_{(a); (b)} = P_{(a);(b)} P_{(b);(a)} P_{(a);(b)} P_{(a);(b)} = 0$$  \[3\]

Hence

$$P_{(a); (b)} = \zeta_{(a); (b)} P_{(a); (b)}$$  \[4\]

It follows from equation [4] that the elements $P_{(a); (b)}$ are all linearly independent and, since there are $2^{2p}$ of them, any element of $C_n^{(2p)}$ must be a linear combination of the $P_{(a); (b)}$, the number of linearly independent elements of a $C_n^{(2p)}$ being also $2^{2p}$. Moreover $C_n^{(2p)}$ must be equivalent to the algebra of the matrices with $2^p$ lines and $2^p$ columns, because of the multiplication rule [4]. $C_n^{(2p)}$ is the algebra of the linear operators of the vector space $C_n^{(2p)}$, constituted by the right products of its elements by $P$) since the left multiplication of any element of $C_n^{(2p)}$ by an element of $C_n^{(2p)}$ is again an element of that vector space.

The general form of the elements of $C_n^{(2p)}$ is $A P + A^* w_a^+ P + \ldots + \Sigma' A_{a_1 \ldots a_r} w_{a_1}^+ w_{a_2}^+ \ldots w_{a_r}^+ P + \ldots + A_{1 \ldots r}^* w_1^+ \ldots w_p^+ P$, the numerical coefficients $A_{a_1 \ldots a_r}$ being the components of a $p$-dimensional antisymmetrical contravariant tensor of order $r$. $\Sigma'$ denotes that the summation is taken only for sets $(a)$ such that $a_1 < a_2 < a_3 < \ldots$. The structure of the elements
of $C_2^{(q)}$ shows that the $2^p$ components of a spinor of $C_2^{(q)}$ constitute a set of antisymmetric tensors of the orders 0, 1, 2, ..., $p$ of the $p$-dimensional $\omega$ - space.

The existence of relations between the spinors of a $2p$-dimensional metric flat space and the $p$-dimensional antisymmetric tensors is well known from the Cartan approach to the theory of spinors (4). The above discussion is essentially the same given by Schönberg (4) in the theory of $G_n$. Cartan defines firstly the spinor in connection with the flat manifolds of maximal dimensionality lying on the null hypercone $g_{jk}x^jx^k = 0$ of a space of odd dimensionality $n = \parallel 2p + 1$ and later introduces the Clifford algebra. We start from the Clifford algebra and obtain the spinor from a left-ideal.

Let us consider now the right ideal $PC_2^{(q)}$ constituted by the products $P$, the $\Gamma$ being the elements of $C_2^{(q)}$. The elements of $C_2^{(q)}$ may be viewed as linear operators on the vectors of the $2p$-dimensional linear space $PC_2^{(q)}$ since to any $\Gamma$ corresponds the linear transformation $PC_2^{(q)} \rightarrow PC_2^{(q)} \Gamma$. Thus we get a linear representation of the reciprocal algebra of $C_2^{(q)}$, because the operator corresponding to the product $\Gamma_1 \Gamma_2$ is the product of the operators corresponding to $\Gamma_1$ and $\Gamma_2$ taken in the reversed order. The general form of the elements of the right ideal $PC_2^{(q)}$ is $P + P W B + \sum P W B W b_1 b_2 + \cdots + \sum P W b_1 W b_2 W b_3 + \cdots + P W b_1 \cdots b_n$ the $b_1 \cdots b_n$ being numerical coefficients antisymmetrical with respect to the indices $b_i$ which are components of antisymmetrical tensors of the $\omega^+ - \text{space}$. The numbers $B, B^a, B^{b_1 b_2}, \cdots, B^{b_1 \cdots b_n}$ are the components of a covariant spinor of $C_2^{(q)}$.

The $n$ elements $N_a = W_a^+ W_a$ are idempotent and commutable as the $N_a$. By means of them we build $Q = N_1 \cdots N_n$, which satisfies the conditions $W_a^+ Q = 0$, $Q W_a = 0$ and $Q^2 = Q$. Let us introduce the $Q(\varphi) ; (a) = W_{b_1} \cdots W_{b_n} Q W_{a_1}^+ \cdots W_{a_r}^+$, analogous to the $P(a) ; (b)$, taking them with $b_1 < b_2 < \cdots < b_n$ and $a_1 < a_2 < < \cdots < a_r$. We have

$$Q(\varphi) ; (a) = Q(\varphi) ; (c) Q(a) ; (a) \quad Q(\varphi) ; (a) Q(\varphi) ; (c) = \hat{z}(\varphi) , (a) Q$$ \[5\]

$$Q(\varphi) ; (a) Q(\varphi) ; (a) = \hat{z}(a) ; (a) Q(\varphi) ; (a)$$ \[6\]

It follows from [6] that the $Q(\varphi) ; (a)$ are a set of $2^p$ linearly independent elements.
The left ideal $C_2^{(0)} Q$ can be taken as a vector space whose linear operators are the elements of $C_2^{(1)}$. It is easily seen that the left ideal $C_2^{(0)} Q$ is related to the antisymmetric tensors of the $\omega^+-$space in the same way as $C_2^{(0)} P$ to those of the $\omega^- -$space. The properties of $C_2^{(0)}$ must obviously be completely symmetrical with respect to the $\omega$ and $\omega^+$ vector spaces. The $W_\alpha^+$ play in the $Q$ formalism the role played by the $W_\alpha$ in the $P$ formalism and conversely.

The $P$ and $Q$ formalisms are covariant for a change of basic vectors in the $\omega^- -$space and the associate change in the $\omega^+ -$space, for

$$P = W_1 \ldots W_n W_n^+ \ldots W_1^+, \quad Q = W_1^+ \ldots W_n^+ W_n \ldots W_1 [7]$$

The $w$-algebra of a finite set.

8. Let us consider a set $S_p$ constituted by $p$ objects $E_\alpha$. We shall associate to the $E_\alpha$ the symbols $w_\alpha$ and $w_\alpha^+$ of the generators of a $p$-dimensional Jordan-Wigner algebra, which will be regarded as an algebra of the set $S_p$ and called the $w$-algebra of the set. It follows from the considerations at the end of section 6 that the $w$-algebra is equivalent to that of the linear operators on the numerically-valued functions of the sub-sets of $S_p$. The formalism of the spinors of a $C_2^{(0)}$ developed in section 7 can be applied to the numerically-valued functions of the sub-sets of $S_p$.

We have regarded the Jordan-Wigner algebra of a quantized fermion field as an algebra of a separable Hilbert space, but it is also possible to view it as an algebra of the numerically-valued functions of the finite sub-sets of an enumerable infinite set of objects, namely the functions of a complete orthonormal set for the states of a fermion of the field. Thus the Jordan-Wigner algebra of the field appears as the infinite-dimensional analogue of the above $w$-algebras of finite sets.

The $w$-algebra over the complex numbers of a set with a single object is isomorphic to the Pauli-algebra of the spin operators, i.e. to the algebra of the linear operators of a two-dimensional complex vector space. This results from the fact that a $S_1$ possesses only two sub-sets: the empty sub-set and $S_1$ itself. Since the Pauli algebra is a fundamental geometric algebra of the three dimensional euclidean space and the four-dimensional minkowskian space-time:
the geometry of the flat space-time is fundamentally related to the
w-algebra of a single object.

The w-algebra of a \( S_2 \) taken over the complex numbers is iso-
morphic to a \( C_4 \) over the complex numbers, hence to the Dirac
algebra of the \( \gamma \)-matrices of the leptons and baryons. The above
results indicate that the two simplest sets \( S_1 \) and \( S_2 \) are likely to
play an important part in the theory of the structure of the world.

The Pauli algebra as an algebra of the space-time is that of the
half-spinors, which are related to a distinguished screw-orientation
of space. The w-algebra of \( S_1 \) is therefore associated to a minkows-
ian space-time with a preferred screw-orientation of space, the
actual situation for the neutrino. The Dirac algebra includes the
two screw-orientations of space. This suggests that the above \( S_1 \)
and \( S_2 \) could be the sets of one and two screw-orientations, respec-
tively.

REFERENCES

Cimento, VI, suppl. n. 1, 356, 1957.
(2) P. Jordan and E. Wigner, Zs. f. Physik, 47, 231, 1928.
(3) E. Cartan, Leçons sur la théorie des spineurs II, Paris 1938.