SPACES OF DIFFERENTIABLE FUNCTIONS AND DISTRIBUTIONS, WITH MIXED NORM.

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INTRODUCTION. The aim of the present paper is to study certain results about operators and distribution spaces related with the spaces L^P with mixed norm (*). These last spaces are treated in [2], later we shall give their definition. It is our intention to translate to the spaces L^P most of the results of [3]. Some theorems here may be proved in the same way as in [3], and we shall not give the proofs in these cases. In other theorems, which follow the same lines as in [3], we emphasize only that parts of the proof which are not obvious translations of similar results in [3].

This paper is divided into three parts. In the first, the spaces $L^P{}_u$ are introduced. which are spaces of tempered distributions. There we are also dealing with Bessel potential operators and derivation acting on $L^P{}_u$. For this it is necessary to consider an extension of a theorem of Mihlin. The first part concludes with an interpolation theorem between the spaces $L^P{}_u$.

In the second part we consider a similar result to a theorem of Sobolev an Krylov (for the spaces $L^P{}_u$ are related to the spaces $H^p{}_n$ of Sobolev).

In the third part we deal with Hölder continuity of the functions belonging to $L^P{}_u$, $0 < u \leqslant 1$. The main result of this part is stronger than its analogous and an alternative proof is given.

NOTATIONS. $x = (x_1, \ldots, x_n)$, $y = y_1, \ldots, y_n)$ denote points of the n - dimensional euclidean space E^n, $P = (p_1, \ldots, p_n)$, Q, R, stand for n - tuples of generalized real numbers, $(1 \leqslant p_i \leqslant \infty)$. We introduce the mixed norm, $\| f \|_P$, for a measurable function $f(x)$ on E^n as

$$\| f \|_P = \| \ldots \| f \|_{p_1/x_1} \ldots \|_{p_n/x_n} \quad \text{(cfr. [2])}$$

(*) This paper is part of the author's thesis.

and we denote with $L^P(E^n) = L^P$ the class of measurable functions f on E^n such that $\| f \|_P < \infty$.

$\alpha = (\alpha_1, \ldots, \alpha_n)$, β, γ, stand for n-tuples of non-negative integers and D^α, denotes the derivative $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$, *while* $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. We suppose further that the reader is familiar with Schwartz's spaces (S), (S'), (D), (D') and (O^M). For $f \in (S')$, $\hat{f}$ (or $(f)^\wedge$) indicates the Fourier transform of f (for $f \in (S)$, $f(x) = \int_{E^n} \exp\,(-2\pi \mathrm{i} x . y)\, f(y)\, dy$).

C indicates absolute constants, dependant of the dimension n. In different formulae it may take different values. Special constants which mantain their values throughout a proof we denote with M.

If $F(x, y, z)$ is a relation between the real variables x, y and z, then $F(P, Q, R)$ stands for the n relations $F(p_i, q_i, r_i)$, $i = 1, \ldots, n$.

1. Let J^z be the Bessel transform defined by

$$(J^z f)^\wedge = (1 + 4\pi^2 |x|^2)^{-z/2} \hat{f}$$

for $f \in (S')$ and z an arbitrary complex number.

Each J^z defines an isomorphism on (S'), since

$$(1 + 4\pi^2 |x|^2)^{-z/2} \in (O_M)$$

for every z, and the family $\{J^z\}$ is an additive group in the index z. Furthermore, if $Re(z) > 0$

$$(1 + 4\pi^2 |x|^2)^{-z/2} = (G_z)^\wedge \qquad \text{, where } G_z(x) \in L^1(E^n)$$

and $G^z(x) = (2\pi)^{(1-n)/2} \,.\, 2^{-z/2} . \left[\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{n-z+1}{2}\right)\right]^{-1}$

$$\int_0^\infty \exp\,(-|x|\,(1+t))\,.\left(t + \frac{t^2}{2}\right)^{(n-z-1)/2} . \, dt$$

for $0 < Re(z) < n + 1$.

A consequence of this formula is

Theorem 1. Let $Re(z) > 0$ and $1 \leqslant P \leqslant \infty$. Then J^z transforms $L^P(E^n)$ continuously into L^P, and if z is real, with norm less than one.

In what follows we need an extension of a theorem of Mihlin (Theorem 3). This will be a consequence of the following Theorem 2 about singular integrals (proved in [1]).

Theorem 2. Let $K(x)$ be a function on E^n which verifies

a) *$K(x)$ is locally integrable*

b) $$\int_{|x| \geq 2 . |y|} |K(x-y) - K(x)| \, dx \leqslant M_1 < \infty .$$

If for some q, $1 < q < \infty$, $\| K * f \|_q \leqslant M_2 \| f \|_q$ holds, for every f which is bounded and has bondéd support, then for every P, $1 < P < \infty$, the inequality $\| K * f \|_P \leqslant C_{P,q}(M_1 + M_2) \| f \|_P$ holds.

Theorem 3. Let K be the operator on (S) defined by

$$(Kf)^{\wedge}(x) = k(x) \, . \, \hat{f}(x),$$

where $k(x)$ verifies

a) *$k(x)$ has continuous derivatives up to order* $\kappa = \left[\dfrac{n+2}{2} \right]$

b) $$\int_{t/2 \leq |x| \leq 2t} |D^{\alpha} k(x)|^2 \, dx < M^2 . \, t^{n-2|\alpha|} \text{ for every real } t,$$

$0 < t < \infty$ *and* $|\alpha| \leqslant \kappa$.

Then $\| Kf \|_P \leqslant C_P . M \| f \|_P$ for $1 < P < \infty$ and $f \in (S)$.

Proof. Let $\phi(x) \in (D)$ be such that its support is contained in

$$\{x ; \frac{1}{2} < |x| < 2\}$$

and $\sum_{m=-\infty}^{\infty} \phi(2^m x) = 1$ for $x \neq 0$.

If $k_j(x) = \phi(2^j x) \, k(x)$, we define

$$K_N(x) = \Big(\sum_{j=-N}^{N} k_j(x) \Big)^{\vee} .$$

Hörmander proved ([4], Theorem 2.5) that

$$|(K_N)^{\wedge}(x)| \leqslant C.M \qquad (1)$$

and

$$\int_{|x|\geq 2|y|} |K_N(x-y) - K_N(x)| \;\; dx \leqslant C.M.$$

Applying Theorem 2, taking into account that (1) implies

$\|K_N * f\|_2 \leqslant C.M\, \|f\|_2$, we obtain

$$\|K_N * f\|_P \leqslant C_P.M\, \|f\|_P \qquad \text{for every } P,\ 1 < P < \infty \qquad (2).$$

But from (1) it also follows that $K_N * f \to Kf$ in L^2 and therefore a subsequence $K_{N_i} * f \to Kf$ a.e.

Applying Fatou's lemma to (2) we obtain

$$\|Kf\|_P \leqslant C_P.M\; \|f\|_P \qquad \text{q.e.d.}$$

Corollary. Theorem 3 remains true if condition b is replaced by

b') $\quad |D^{\alpha} k(x)| \leqq M.|x|^{-|\alpha|} \qquad$ *for* $|\alpha| \leqslant \kappa$

since b') implies b).

The following theorems we state without proofs, since their proofs in [3] only use the theorems of Young and Mihlin, and may be carried over without change on L^P.

Theorem 4. *The operator* $D^{\alpha} J^z$, *where* $|\alpha| \leqslant Re(z)$, *is continuous from* L^P to L^P, $1 < P < \infty$. *In the case* $z = iv$

$$\|J^z f\|_P \leqslant C_P\;(1 + |\nu|)^n.\|f\|_P.$$

Definition 1. *For u a real number,* $1 \leqslant P \leqslant \infty$, *we define* $L^P{}_u$ *as the image of* L^P *under* J^u. *That is* $L^P{}_u = \{f;\, f = J_g{}^u$ with $g \in L^P\}$. *The norm in* $L^P{}_u$ *we define as* $\|f\|_{P,u} = \|J^{-u} f\|_P$.

Theorem 5. *a) The spaces* $L^P{}_u$ *are isometric to* L_P.

b) *If* $1 < P < \infty$, J^z *is an isomorphism from* $L^P{}_u$ *onto* $P^P{}_{u+Re(z)}$.

c) *If z is rea lthe preceding isomorphism is an isometry, even if* $1 \leqslant P \leqslant \infty$.

d) *If* $u < v$, *then* $L_u^P \supset L_v^P$ *and for* $f \in L^P{}_v$ *we have* $\| f \|_{P,u} \leqslant \| f \|_{P,v}$.

e) *If* $1 < P < \infty$, *then* D^α *transforms* $L^P{}_u$ *continuously into* $L^P{}_{u - |\alpha|}$.

Definition 2. If u *is a non-negative integer, we call* $H^P{}_u$, $1 < P < \infty$, *the Banach space of all functions of* L^P, *which admit derivatives in the sense of Schwartz up to order* u, *in* L^P. *TRhe norm in* $H^P{}_u$ *defined by* $|f|_{P,u} = \sum_{|\alpha| \leq u} \|D^\alpha f\|_P$.

Theorem 6. a) If $1 < P < \infty$ *and* u *is a non-negative integer, then* $L^P{}_u = H^P{}_u$ *and*

$$C_{P,u} \cdot \| f \|_{P,u} \leqslant | f |_{P,u} \leqslant C_{P,u} \| f \|_{P,u}$$

b) *If* u *is a non-positive integer,* $1 < P < \infty$, *then* $f \in L^P{}_u$ *if and only if* $f = \sum_{|\alpha| \leq -u} D^\alpha g_\alpha$, *where* $g_\alpha \in L^P$.

Further, there exists a choice of g_α *such that*

$$C_{P,u} \| f \|_{P,u} \leqslant \sum_{|\alpha| \leq -u} \|g_\alpha\|_P \leqslant C_{P,u} \| f \|_{P,u}.$$

Note. If u is a non-negative integer, t a positive real number, then $L^P{}_{u+t} \subset H^P{}_u \subset L^P{}_{u-t}$ for every P, $1 \leqslant P \leqslant \infty$, the inclusions being continuous.

The proof runs the same way as that of Theorem 6, but instead of Theorem 4, it uses the fact that, if $u > |\alpha|$ then $D^\alpha G_u(x) \in L^1$ and therefore $D^\alpha J^u$ transforms continuously L^P into L^P for $1 < P < \infty$.

Theorem 7. Let f *and* $g \in (S)$, $1 \leqslant P \leqslant \infty$. *Then* $< f, g > = \int f(x) g(x)\, dx$ *verifies* $< f, g > \leqslant \| f \|_{P,u} \cdot \| g \|_{P',-u}$, *where* $1/P' = 1 - (1/P)$, *and* $< f, g >$ *admits a continuous extension to* $L^P{}_u \otimes L^{P'}{}_{-u}$.

If $1 \leqslant P < \infty$, *then every continuous linear functional on* $L^P{}_u$ *has the form* $1(f) = < f, g >$ *with certain* $g \in L^{P'}{}_{-u}$.

Next we give an interpolation theorem between the spaces $L^P{}_u$.

Theorem 8. Let A be an operator defined on $(S(E^n))$ *with values in* $(S'(E^m))$, *continuous from* $L^{P_i}_{u_i}(E^n)$ *into* $L^{Q_i}_{v_i}(E^m)$, $i = 0,1$ $1 < P_i, Q_i < \infty$.

That is $\| Af \|_{Q_i, v_i} \leqslant M_i \| f \|_{P_i, u_i}$ *for* $f \in (S)$.

Let P, Q, u and v be defined by $1/P = t/P_1 + (1-t)/P_0$; $1/Q = t/Q_1 + (1-t)/Q_0$, $u = t.u_1 + (1-t).u_0$, $v = t.v_1 + (1-t).v_0$, *where* $0 < t < 1$.

Then for $f \in (S(E^n))$,

$$\| Af \|_{Q,v} \leqslant C_{M_i, P_i, Q_i, u_i, v_i} \| f \|_{P,u}.$$

Proof. We suppose without loss of generality that $v_1 \geqslant v_0$. Let K *be a mollifier in* E^n, $K = \epsilon^{-n} \phi \left(\frac{x}{\epsilon}\right)*$, *where* $\phi \in (D)$, $\phi \geqslant 0$, $\int_{E^n} \phi \, dx = 1$. Further, let J_1^z and J_2^z be the Bessel transforms in E^n and E^m respectively, and $1(z) = (u_1 - u_0).z + u_0$, $L(z) = (v_0 - v_1)\, z - v_0$. Finally let $B_z = J_2^{L(z)} A.K J_1^{1(z)} f$ for f belonging to the class H of simple functions, constant on rectangles (see [2], p. 313), and $0 \leqslant Re(z) \leqslant 1$.

$K J_1^{1(z)} f = J_1^{1(z)} K f \in (S) \subset L^{P1}{}_{u1}$, therefore $A K J_1^{1(z)} \in L^{Q1}{}_{v1}$, and since $Re(L(z)) \geqslant - v_1$, it follows from Theorem 4 that $J_2^{L(z)} A K J_1^{1(z)} f \in L^Q 1(E^m)$.

We shall prove now that the operator B_z verifies the hypotesis of the interpolation theorem in [2], p. 313.

Lemma. The operator B_z *verifies*

a) $\| B_z f \|_{Qj} \leqslant C_{Pj, Qj, Mj, uj, vj} (1 + |z|)^{n+m} \| f \|_{Pj}$ for $z = j + iy$, $j = 0,1$.

b) *For* $f \in H(E^n)$ *and* g *bounded with bounded support, measurable on* E^m, *we have.*

i) $\int (B_z f).g \, dy$ *is analytic in* $0 < Re(z) < 1$, *and continuous in* $0 \leqslant Re(z) \leqslant 1$.

$$ii) \quad \int_{|y| \leq N} | B_z f | \, dy \leqslant C_{N,f} (1 + |z|)^{m+n}.$$

Proof of the lemma. Let $z = j + iy$ where $j = 0,1$.

Then $Re(l(z)) = u_j$, $Re(L(z)) = - v_j$, and

$Im(1(z)) = y(u_1 - u_0)$, $Im(L(z)) = y.(v_0 - v_1)$. Applying Theorem 4, we obtain:

$\| B_z f \|_{Qj} \leqslant C \, (1 + |z|)^{n+m} . M_j \| Kf \|_{Pj}$, and since $\| Kf \|_{Pj} \leqslant \| f \|_{Pj}$, i) follows.

In the case $z = x + iy$, $0 \leqslant x \leqslant 1$, we have $Re\,(l(z)) \geqslant u_1 + |u_1 - u_0|$, $Re\,(L(z)) \geqslant - v_1$. Using again Theorem 4,

$$\| B_z f \|_{Q1} \leqslant C.(1 + |z|)^{n+m} \| Kf \|_{P, -|u_1 - u_0|}, \tag{3}$$

and iii) follows.

To prove ii), we observe that for $f, g \in (S)$, $I(f, g, z) = \int_{Em} (J_2^{-v1} A J_1^{1(z)} f) . (J_2^{(v1-v0)z} g) \, dx$ is an entire function of z. In analogy to Theorem 7 we also have

$$\int (B_z f) \, g \, dx = I(Kf, g, z) \quad \text{for } f \in H(E^n), g \in L^{Q'}{}_1 \tag{4}$$

Now if $g_k \in (S)$ and $g_k \to g$ in $L^{Q'}{}_1$, then (3) implies that $I(Kf, g_k, z)$ converges quasi-uniformly to $I(Kf, g, z)$ on the closed strip $0 \leqslant Re(z) \leqslant 1$ (i. e. it converges uniformly on compact subsets). ii) follows.

We proceed now with the proof of Theorem 8.

From the interpolation theorem of [2], using the preceding lemma, we obtain

$$\| B_t f \|_Q = \| J_2^{-v} A J_1^u Kf \|_Q \leqslant C \| f \|_P , \tag{5}$$

t, u, v, P and Q being the numbers and be n- and m tuples defined in the statement of this theorem.

Taking $\epsilon < \epsilon_0(f)$ in the definition of K, me have $\| f \|_P \leqslant$ $\leqslant 2 \| Kf \|_P$, and we get from (5), $\| Ag \|_{Q,v} \leqslant C \| g \|_{P,u}$, where $g = J_1^u Kf$.

Since the set $\{g = J_1{}^u Kf;\ f \in H,\ \varepsilon < \varepsilon_0(f)\}$ is dense in (S) with respect to the norm of L^{Pu}, the theorem is established.

2. We shall establish now results which correspond to theorems of Sobolev and Krylov (see Theorem 2 below). To do this we need a stronger form of Sobolev's theorem for mixed norm (cfr. [2]).

Theorem 1. Let $L = (l_1, \ldots, l_n)$ a n-tulpe *or real numbers,* $0 \leqslant l_i \leqslant 1$, $0 < l_n$. *If* P *and* Q *verify* $1/P - 1/Q = L$, $1 \leqslant p_i \leqslant {}^1/l_i$, $1 < p_n < 1/l_n$, *then*

$\| f * | x |^{l-n} \|_Q \leqslant c_{P,Q} \| f \|_P$ *for every* $f \in L^P$, *where* $l = l_1 + \ldots + l_n$.

Proof. We must prove that

$$I = \left| \int_{E^n \times E^n} f(x) \, g(y) \, | x - y |^{l-n} \, dx \, dy \right| \leqslant C \| f \|_P . \| g \|_{Q'}$$

where $1/Q' = 1 - (1/Q)$.

Let us consider the $(n-1)$-tuples $\overline{P} = (p_1, \ldots, p_{n-1})$, $\overline{Q} = (q_1, \ldots, q_{n-1})$, $\overline{L} = (l_1, \ldots, l_{n-1})$ and $\overline{R} = (r_1, \ldots, r_{n-1})$, the latter defined by $1/\overline{R} = 1 + (1/\overline{Q}) - (1/\overline{P}) = 1 - \overline{L}$.

The function $h(x_1, \ldots, x_{n-1}, a) = h(\overline{x}, a) = (|\overline{x}|^2 + a^2)^{(l-n)/2}$ satisfies $\| h(\overline{x}, a) \|_{\overline{R}} = |a| \sum_{i=1}^{n-1} (^1/r_i) \| (h(a\overline{x}, a) \|_{\overline{R}} = |a|^{l-n+\sum_{i=1}^{n-1}(^1/r_i)} . \| h(\overline{x}, 1) \|_{\overline{R}}$.

Now, $\| h(\overline{x}, 1) \|_{\overline{R}}$ is finite as it is easy to see, so

$$\| h(\overline{x}, a) \|_{\overline{R}} = |a|^{ln-1} . C_{\overline{R}} \tag{1}$$

Fixing x_n and y_n we obtain from Young's theorem and (1)

$$|\int g(y)\, f(x)\, |x - y|^{l-n} dx_1, \ldots, dx_{n-1}\, dy_1 \ldots dy_{n-1}| \leqslant \tag{2}$$

$$\leqslant \| g \|_{\overline{Q}^1} (E^{n-1}) . \| f \|_{\overline{P}} (E^{n-1}) . \| h(\overline{x}, x_n - y_n) \|_{\overline{R}}$$
$$= C . |x_n - y_n|^{ln-1} \| g \|_{\overline{Q}}, \| f \|_{\overline{P}} .$$

Calling $F(x_n) = \| f \|_{\overline{P}}$, $G(x_n) = \| g \|_{\overline{Q}}$, from (2) we obtain $I \leqslant C . \int F(x_n)\, G(y_n)\, |x_n - y_n|^{ln-1} dx_n\, dy_n \leqslant$ (by Sobolev's theorem) $\leqslant C \| F \|_{pn} \| G \|_{q'n} = C \| f \|_P . \| g \|_{Q'}$ q.e.d.

To establish Theorem 2 we still need three auxiliary lemmas. There $\triangle_h f(x) = f(x + h) - f(x)$.

Lemma 1. If $f \in L^P(E^n)$ *and* $\| \triangle^2_h f \|_P \leqslant c . |h|^s$ *with* $0 < s < 2$, *then*

$$\| \triangle_h f \|_P \leqslant C_s . \begin{cases} |h|^s & \text{if} \quad s < 1 \\ |h| \; . \left(\log^+ \left(\frac{1}{|h|}\right) + 1\right) & \text{if } s = 1 \\ |h| & \text{if} \quad s > 1 . \end{cases}$$

Proof. From formula

$$\triangle_h f(x) = -(1/2) \sum_{k=0}^{N-1} 2^{-k} \triangle^2_{2^k h} f(x) + 2^{-N} \triangle_{2^N h} f(x)$$

we obtain by using the hypothesis,

$$\| \triangle_h \mathrm{f} \|_P \leqslant C |h|^s \sum_{k=0}^{N-1} 2^{k(s-1)} + 2^{-N} C \tag{3}$$

In the case $s < 1$, the series $\sum 2^{k(s-1)}$ converges, so for $N \to \infty$ we get our result.

If $s = 1$, from (3) we obtain

$$\| \Delta_h f \|_P \leqslant C(|h| N + 2^{-N}),$$

and if $s > 1$

$$\| \Delta_h f \|_P \leqslant C (|h|^s 2^{N(s-1)} + 2^{-N}).$$

Taking in these cases N such that

$\log^+ \left(\frac{1}{|h|}\right) < N \leqslant \log^+ \left(\frac{1}{|h|}\right) + 1$ we obtain the thesis.

Lemma 2. Let $F_u(\rho) = \dfrac{1}{\Gamma\left(\frac{n-u+1}{2}\right)} \displaystyle\int_0^\infty \exp(-\rho(1+t)) \left(t + \frac{t^2}{2}\right)^{(n-u-1)/2} dt$

for real u, $u < n + 1$, ρ *being a real positive variable. Then,*

a) $$\frac{d F_u(\rho)}{d\rho} = -\frac{\rho}{2} F_{u-2}(\rho)$$

b) $$F_u(\rho) \leqslant C \cdot \begin{cases} \rho^{u-n} \exp\left(-\frac{\rho}{2}\right) & \text{if } u < n \\ \left(\log^+ \frac{1}{\rho} + 1\right) \exp(-\rho) & \text{if } u = n \\ \exp(-\rho) & \text{if } u > n \end{cases}$$

Proof. a) can be obtained by differentiating under the integral sign and integrating by parts.

b) For $\rho \geqslant 1$ $F_u(\rho) \leqslant C \int_0^\infty \exp(-t-\rho) \left(t + \frac{t^2}{2}\right)^{(u-n-1)/2} dt \leqslant C \cdot \exp(-\rho)$.

Now if $\rho < 1$, we separate three cases.

$$u < n: F_u(\rho) \leqslant C\, \rho^{u-n} \int_0^\infty \exp(-t) \left(\rho t + \frac{t^2}{2}\right)^{(n-u-1)/2} dt \leqslant C\, \rho^{u-n}.$$

$u = n$: $F_n(\rho) = (1/2) \int_\rho^\infty r . F_{n-2}(r)\, dr \leqslant$ (by the preceding case) $\leqslant C \int_\rho^1 \frac{dr}{r} + C = C\, (\log \left(\frac{1}{\rho}\right) + 1)$.

$$u > n: F_u(\rho) \leqslant C . \exp(-\rho) \int_0^\infty \left(t + \frac{t^2}{2}\right)^{(n-u-1)/2} dt = = C.\ exp\,(-\rho), \text{q.e.d.}$$

Lemma 3. If $G_u(x) = [(1 + 4\pi^2 \,|\, x\,|^2)^{-u/2}]^\vee$, *we have for* $1 \leqslant R \leqslant \infty$ *and* $u > 0$,

a) $G_u(x) \in L^R$ *if* $\Sigma\ (1/r_i) > n - u$

b) $\| \Delta_h G_u \|_R \leqslant C\ .\ |\, h\,|^{\Sigma (1/r_i) - n + u}$ *if* $1 > \Sigma\ (1/r_i) - n + u > 0$

c) *For* $2 > \Sigma\ (1/r_i) - n + u > 0$ *we have* $\| \Delta^2_u G_h \|_R \leqslant C . |\, h\,|^{\Sigma (1/r_i) - n + u}$.

Proof. a) If $0 < u < n+1$, $G_u(x) = C\, F_u(\,|\,x\,|\,)$ and from Lemma 2, b), follows a) in case $u < n + 1$.

If $u \geqslant n+1$, there exists $v < n + 1$, such that $\Sigma\ (1/r_i) > n - v$, and we have $\| G_u \|_R = \| G_v * G_{u-v} \|_R \leqslant \| G_v \|_R \| G_{u-v} \|_1 = = \| G_v \|_R$, q.e.d.

b) and c). We shall show that

i) b) with $u < n$ implies c) with $u < n + 1$

ii) c) with $u < n + 1$ implies b)

iii) b) implies c).

In fact, suppose b) is true for $u < n$, and let $u < n + 1$, $2 > \Sigma\ (1/r_i) - n + u > 0$. Now we define P by $\frac{1}{P} = \frac{1}{2} + \frac{1}{2R}$.

Then P verifies

$$\frac{1}{P}+\frac{1}{P}=1+\frac{1}{R} \text{ and } 1 > \Sigma\,(1/p_i) - n + \frac{u}{2} > 0,$$

and we have

$$\triangle^2_h G_u = \triangle^2_h (G_{\frac{u}{2}} * G_{\frac{u}{2}}) = (\triangle_h G_{\frac{u}{2}}) * (\triangle_h G_{\frac{u}{2}}).$$

Applying Young's inequality

$$\| \triangle^2_h\, G_u \|_R \leqslant \| \triangle_h G_{\frac{u}{2}} \|^2_P \leqslant (\text{because } \tfrac{u}{2} < n) \leqslant$$

$$\leqslant C \mid h \mid^{2(\Sigma 1/p_i - n + \frac{u}{2})} = c \mid h \mid^{\Sigma(1/r_i) - n + u}. \text{ This proves i)}$$

ii) is consequence of Lemma 1, since the hypotesis of b) imply $u < n + 1$.

iii) may be proved in the same way as i).

From i), ii), and iii) it follows that to prove b) and c) it is enough to prove b) in the case $u < n$.

Applying the mean value theorem and Lemma 2, a), we obtain $| \triangle_h G_u(x) | = C \mid \triangle_h F_u(|\, x\, |) | \leqslant C \mid h \mid F'_u(r) = C \mid h \mid . r .\, F_{u-2}(r)$ where $r = | x + \theta h |$ with certain θ, $0 < \theta < 1$.

From Lemma 2, b), we then have

$$| \triangle_h G_u(x) | \leqslant \begin{cases} C \mid h \mid r^{u-n-1} \leqslant C \mid h \mid . \mid x \mid^{u-n-1} & \text{if } \mid x \mid \geqslant 2h \\ C \mid x + h \mid^{n-u} + \mid x \mid^{n-u} & \text{if } \mid x \mid \leqslant 2 \mid h \mid \end{cases} \quad (4)$$

If we call

$$G_1(x;a) = \begin{cases} a \,.\, \mid x \mid^{u-n-1} & \text{if } \mid x \mid \geqslant 2a > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$G_2(x;a) = \begin{cases} \mid x \mid^{n-u} & \text{if } \mid x \mid \leqslant 3a \\ 0 & \text{elsewhere,} \end{cases}$$

it follows from (4) that

$$\| \triangle_h G_u \|_R \leqslant C \,[\, \| G_1(x; | h |) \|_R + \| G_2(x; | h |) \|_R] =$$
$$= C \mid h \mid^{\Sigma(1/r_i)} [\, \| G_1(\mid h \mid x; \mid h \mid) \|_R + \| G_2 (\mid h \mid x; \mid h \mid) \|_R \,] =$$
$$= C \mid h \mid^{\Sigma(1/r_i) + n - u} [\, \| G_1 (x; 1) \|_R + \| G_2 (x; 1) \|_R \,] .$$

Since $\| G_i(x;1) \|_R < \infty$ we thus have proved b) in case $u < n$, q.e.d.

Theorem 2. Let P and Q, $1 \leqslant P, Q \leqslant \infty$ *be such that* $1/P \geqslant 1/Q$, $1 > 1/p_n > 1/q_n > 0$ *and let u and v be real numbers which satisfy* $\Sigma\ (1/p_i - 1/q_i) = u - v$.

Then $L^P_u \subset L^Q_v$, *the inclusion being continuous.*

If $1 > u - \Sigma\,(1/p_i) > 0$, $1 \leqslant P \leqslant \infty$, *then every function* $f \in L^P_u$ *coincides a.e. with a ocntinuous function* $\bar{f}$, *and*

$$|\bar{f}| \leqslant C \, \| f \|_{P,u} \quad , \quad |\bar{f}(x+h) - \bar{f}(x)| \leqslant C \, |h|^{u - \Sigma \frac{1}{p}} \, . \, \| f \|_{P,u}.$$

Proof. Let us call $L = (l_1, \ldots, l_n)$ the n - tuple $L = \dfrac{1}{P} - \dfrac{1}{Q}$;

then $0 < \Sigma\, l_i = u - v < n$, and for $f \in L^P$, we have by Lemma 2, b), $|(J^{u-v}f)(x)| = |(G_{u-v} * f)(x)| \leqslant C\,|x|^{u-v-n} * |f|$.
Now applying Theorem 1, $\| J^{u-v} f \|_Q \leqslant C_{P,Q} \| f \|_P$, or, what is the same, $\| J^u f \|_{Q,v} \leqslant C_{P,Q} \| J^u f \|_{P,u}$. This proves the first part. For the second part, since $u > \Sigma\,(1/p_i) = n - \Sigma\,(1/p'_i)$, we have from Lemma 3, a), that $G_u(x) \in L^{P'}$.

Now if $f \in L^P_u$, then $f = G_u * g$ with $g \in L^P$. Then Young's theorem implies the continuity of f and also that

$$\| f \|_\infty \leqslant \| G_u \|_{P'} \, . \, \| g \|_P = C \, . \, \| f \|_{P,u}.$$

Finally $f(x+h) - f(x) = \triangle_h G_u * g$, and again by Lemma 3, b), and Young's theorem,

$$|f(x+h) - f(x)| \leqslant \| \triangle_h G_u \|_{P'} . \| f \|_{P,u} \leqslant$$

$$\leqslant C\,|h|^{u - \Sigma \frac{1}{p_i}} . \| f \|_{P,u} \quad \text{q..e.d.}$$

Note 1. If in the first part of the preceding theorem we had $u - v > \Sigma\ (1/p_i - 1/q_i)$, the same conclusion holds for $1 \leqslant P, Q \leqslant \infty$, $1/P \geqslant 1/Q$.

In fact then by Lemma 3, a), $G_{u-v} \in L^R$, $\dfrac{1}{R} + \dfrac{1}{P} = \dfrac{1}{Q} + 1$, and $\| G_{u-v} f \|_Q \leqslant C \| f \|_P$ follows from Young's theorem.

Note 2. If in the second part we had $u - \Sigma\,(1/p_i) = 1$, the conclusion would change only in the fact that then

$$|\bar{f}(x+h) - \bar{f}(x)| \leqslant C\,|h|\,(\log^+\left(\frac{1}{|h|}\right) + 1)\ \| f \|_{P,u}.$$

Indeed, from Lemma 3, c), and Lemma 1, we obtain

$$\| \triangle_h G_u \|_{P'} \leqslant C \, | h | \; \left(\log^+ \left(\frac{1}{|h|}\right) + 1\right) \text{ in this case.}$$

3. In this part we shall study Hölder continuity properties of functions beloging to a class $L^P{}_u$.

Definition. Let u be a real number and r the greatest integer less than u. We denote with $\wedge^P{}_u$*, the class of distributions f,* $f \in L^P{}_r$*, such that*

$$\| \triangle^2{}_h \, J^{-r} f \|_P \leqslant c_f \, | h |^{u-r}.$$

The norm in $\wedge^P{}_u$ *we define as* $\| f \|^P{}_{,u}$*, plus the least constant* C_f *that satisfies the preceding inequality. We denote it with* $/f/_{P,u}$.

Theorem 1. a) If u is not an integer, then $\| \triangle^2{}_h J^{-r} f \|_P \leqslant$ $\leqslant C \, | h |^{u-r}$ *is equivalent to* $\| \triangle_h J^{-r} f \|_P \leqslant C \, | h |^{u-r}$.

b) J^v *is an isomorphism from* $\wedge^P{}_u$ *onto* $\wedge^P{}_{u+v}$ *for every pair of real numbers u and v and* $1 \leqslant P \leqslant \infty$.

c) *If* $u < v$ *and* $1 \leqslant P \leqslant \infty$*, then* $\Lambda_u{}^P \supset L_u{}^P \supset \Lambda_v{}^P$*, the incusions being continuous.*

To prove Theorem 1 we need certain auxiliary results, which we state next.

Lemma 1. Let $g(x) \in (D)$ *have support contained in* $\{x; \frac{1}{2} < |x| < 2\}$, *and let t, v, be real numbers,* $0 < t < 1$. *If the function* $G_{v,t}(x) \in (S)$ *is defined by*

$$(G_{v,t})^\wedge \, (x) = g(tx) \, . \, (1 + 4 \, \pi^2 \, | x |^2)^{\frac{v}{2}},$$

then $\| G_{v,t} \|_1 \leqslant C_{g,v} \, t^{-v}$.

Proof. $$G_{v,t}(x) = \frac{1}{(-2\pi i x)^\alpha} \int_{E^n} \exp \; (2 \pi i \, (x, y))$$

$$D^\alpha \, [g(ty) \, . \, (1 + 4 \pi^2 \, | y |^2)^{\frac{v}{2}} \,] dx$$

the integrand having support in $\left\{ y; \frac{1}{2t} < | y | < \frac{2}{t} \right\}$

and being bounded by $C_{v,g} \;\; t^{|\alpha| - v}$ there.

Therefore $| G_{v,t}(x) | \leqslant C_{v,g} \,.\, t^{s-v-n} \; | x |^{-s}$ for $s = 0,1,\ldots$ and in particular

$$| G_{v,t}(x) | \leqslant C. \begin{cases} t^{-n-v} & \text{for } | x | \leqslant t \\ t^{1-v} \,.\, | x |^{-n-1} & \text{for } | x | \geqslant t. \end{cases}$$

Integrating these bounds we get our thesis.

In Lemma 2, x, y, z will denote one dimensional non - negative variables.

Lemma 2. Let r, s and t be real numbers, $r \geqslant 0$, $0 < t < 1$, $s < 1$. If $g(x) \in C^\infty\,(0, \infty)$, being zero for $x \geqslant 4\pi$, and $\dfrac{dg(x)}{dx} \in L^1(0, \infty)$ *then* $| I | = | \int_0^\infty g(x) \exp\left(\dfrac{i\,x\,y}{t}\right) . x^r . (t^2 + x^2)^{-\frac{(r+s)}{2}} dx \,|$

$$\leqslant C_s \,\| g' \|_1 \left(\frac{t}{y}\right)^{s_1}$$

where s_1 is a certain real number $0 < s_1 < 1$.

Proof. We call $F(x) = \int_0^x \exp\left(\dfrac{izy}{t}\right) z^r \; (t^2 + z^2)^{-\frac{(r+s)}{2}} dz.$

Then $I = \int_0^\infty g(x) . F'(x)\, dx = - \int_0^\infty g'(x) . F(x)\, dx$, and we have

$$| I | = \| g' \|_1 . \sup_{0 < x < 4\pi} | F(x) | \,.$$

We shall show that

$$\sup_{0 < x < 4\pi} | F(x) | \leqslant C_s \left(\frac{t}{y}\right)^{s_1} \qquad (1)$$

having this way proved the lemma.

For this we decompose $F(x)$ into real and imaginary parts and we shall find bounds for each part.

$$Re(F(x)) = \left\{ \int_0^{\frac{\pi t}{2y}} + \int_{\frac{\pi t}{2y}}^{\frac{3\pi t}{2y}} + \ldots + \int_{\frac{(2m-1)\pi t}{2y}}^{x} \right\}$$

$$\cos\left(\frac{z\,y}{t}\right) z^r (t^2 + z^2)^{\frac{-r-s}{2}} dz = c_0 + c_1 + \ldots + c_m.$$

The sequence c_i is of alternating sign, and if the function $z^r(t^2+z^2)^{\frac{-r-s}{2}}$ is monotonous in $\left(\frac{(2j-1)\pi t}{2y}, \frac{(2k+1)\pi t}{2y}\right)$ then the sequence $|c_j|, \ldots, |c_k|$, is also monotonous. Therefore we can associate $c_0+\ldots+c_m)$ in at most five terms $(c_0+\ldots+c_{i_1})+\ldots+(c_{i_4}+\ldots+c_m)$ such that in each parenthesis the moduli of the summands are monotonous.

Then
$$|c_0+\ldots+c_m| \leqslant 10 . \max_{0\leqq i\leqq m} |c_i| \tag{2}$$

Further we have

$$z^r(t^2+z^2)^{-(r+s)/2} \leqq \begin{cases} (1+z)^{-s} & \text{if } s<0 \\ z^{-s} \text{ if } 0\leqslant s<1 \end{cases}$$

so that from (2) it follows

$$\sup_{0<x<4\pi} |Re(F(x))| \leqslant C_s . \begin{cases} (t/y)^{1-s} & \text{if } 0\leqslant s<1 \\ (t/y) & \text{if } s<0 \end{cases} \tag{3}$$

Similar bounds may be obtained for $|Im(F(x))|$.

Formula (3) implies the thesis for $s \geqslant 0$ with $s_1=(1-s)$.

If $s\leqslant 0$, $|F(x)| \leqslant \int_0^{4\pi}(1+z)^{-s}\,dz = C_s$, and from (3) we obtain

$$\sup_{0<x<4\pi} |F(x)| \leqslant C_s \min\,(1, t/y) \leqslant C_s \min\,(1, (t/y)^{s1}) \leqq$$
$\leqslant C_s\,(t/y)^{s1}$ for any s_1, $0<s_1<1$, q.e.d.

Now we shall generalize in certain sense Lemma 1; x, y are again points of E^n.

Lemma 3. If $v \geqslant 0$, *then Lemma 1 holds for any* $g \in (D)$ *support of* $g \subset \{x; |x| \leqq 2\}$.

Proof. Let $v>0$.

$$G_{v,t}(y) = (-2\pi i\, y_k)^{-j}\int_{E^n} \exp\,(2\pi i(x,y)\frac{\partial^J}{\partial x_k{}^J}$$
$$[g(tx)\,(1+4\pi^2|x|^2)^{\frac{v}{2}}]\,dx$$

$= y_k{}^{-j}$ times a finite sum of terms T_j (4)
with

$$T_j = \int_{E^n} \exp(2\pi i(x,y))\ g_s(tx)\ (1+4\pi^2|x|^2)^{(v-j-r+s)/2} x_k{}^r\, t^s\, dx.$$

Here $g_s(x)$ are linear combinations of derivatives of $g(x)$, having thus the same support as $g(x)$, $g_s(x) \in (D)$; $0 \leqslant s \leqslant j$, $0 \leqslant r \leqslant j$.

We shall find bounds for T_j in two cases.

i) $j = 0$.

$$|T_0(y)| \leqslant C_g \cdot \int_{|x| \leq 2/t} (1+4\pi^2|x|^2)^{\frac{v}{2}} dx \leqslant C_{g,v} \cdot t^{-v-n} \qquad (5)$$

ii) $j = n$. Changing variables $z = 2\pi tx$,

$$T_n(y) = t^{-v} \int_{E^n} \exp\left(i\,|y|\,|z|\,\frac{\cos \widehat{yz}}{t}\right) g_s\left(\frac{z}{2\pi}\right) (t^2 + |z|^2)^{(v-n+s-r)/2}\, z_k{}^r\, dz.$$

Writing this integral in polar coordinates we obtain

$$T_n(y) = t^{-v} \int_{\Sigma} \left(\frac{z_k}{|z|}\right)^r d\Sigma \int^{\infty} \exp\left(i\,|y|\,\rho\,\frac{\cos\gamma}{t}\right) g_s\left(\frac{\rho}{2\pi}, \frac{z}{|z|}\right) (t^2+\rho^2)^{(v-n+s-r)/2} \cdot \rho^{n+r-1}\, d\rho$$

an applying Lemma 2 to the integral in ρ, we get

$$|T_n| \leqslant C_v\, t^{-v} \left(\frac{t}{|y|}\right)^{s_1} \int_{\Sigma} |\cos\gamma|^{-s_1} \left\| \frac{\partial}{\partial\rho} g^s\left(\frac{\rho}{2\pi}, \frac{z}{|z|}\right) \right\|_{1/p} d\Sigma = C_{v,g}\, t^{-v} \left(\frac{t}{|y|}\right)^{s_1} \qquad (6)$$

From (4), (5) and (6) it follows that

$$|G_{v,t}{}^{(y)}| \leqslant C_{v,g} \cdot \begin{cases} t^{-v-n} & \text{for} \quad |y| \leqslant t \\ t^{-v} \cdot \left(\dfrac{t}{|y|}\right)^{s_1} \cdot |y|^{-n} & \text{for} \quad |y| \geqslant t \end{cases}$$

and integrating these bounds we obtain the thesis for $v > 0$.

If $v = 0$, it is immediate that $G_{0,t}(y) = t^{-n} G_{0,1}(y/t)$, so that

$$\| G_{0,t} \|_1 = \| G_{0,1} \|_1 \qquad \text{q.e.d.}$$

Lemma 4. There exists a function $g(x) \in (D(E^1))$ *such that support of* $g \subset \{x; \frac{1}{2} < x < 2\}$ *and* $\sum_{m=-\infty}^{\infty} g^2(2^{-m}x) = 1$ *for* $x > 0$.

Proof. There are standard constructions (e.g. see [4]) for a $g_1(x)$ belonging to $(D(E^1))$, support of $g_1 \subset \{x; \frac{1}{2} < x < 2\}$,

such that $\sum_{m=-\infty}^{\infty} g_1(2^{-m}x) = 1$ for $x > 0$.

Now

$$\sum_{m=-\infty}^{\infty} g_1{}^2(2^{-m}x) = g_2(x) \quad \text{verifies}$$

$$g_2 \in C^\infty(0, \infty),\ g_2(x) > 0 \text{ for } x > 0, \text{ and } g_2(2^m x) = g_2(x).$$

Therefore we may take $g(x) = g_1(x) \,.\, g_2^{-1/2}(x)$ q.e.d.

Proof of Theorem 1. a) follows from Lemma 1, section 2.

b) Since for v an integer b) is true by the definition of $\Lambda^P{}_u$, one needs only to prove b) for $0 < v < 1$ and consider the case $0 < u \leqslant 1$. Let $f \in \Lambda^P{}_u$.

Case 1: $u + v \leqslant 1$ (therefore $u < 1$).

$$\| \Delta^2_h J^v f \|_P = \| \Delta_h G_v * \Delta_h f \|_P \leqslant \| \Delta_h G_v \|_1 \| \Delta_h f \|_P$$

Using a) and Lemma 3, b) of section 2, we get

$$\| \Delta^2_h J^v f \|_P \leqslant C_{v,u} \; | h |^{u+v} /f/_{P,u} \tag{7}$$

Further $\| J^v f \|_P \leqslant \| G_v \|_1 \; \| f \|_P = \| f \|_P \leqslant /f/_{P,u}$ (8)

From (7) and (8) it follows

$$/J^v f/_{P,u+v} \leqslant C_{u,v} \; /f/_{P,u} \qquad \text{q.e.d.}$$

Case 2: $1 < u + v < 2$.

Let $\{\Gamma_i\}$, $i = 1, \ldots, N$, be a family of open, circular cones, with vertexes at the origin, such that they cover the surface Σ of the unit sphere, and the angle between two generatrices of each $\{\Gamma_i\}$ is not greater than $\pi/2$. We call e_i the unit vector direction coincides with that of the axis of $\{\Gamma_i\}$.

Let $g_i(x)$, $i = 1, \ldots, N$, functions of (D), with support of $g_i \subset \Gamma_i$, such that

$$\sum_{i=1}^{N} g_i^2(x) = 1 \quad \text{for} \quad x \in \Sigma.$$

Let $g(x)$ be the function of Lemma 4. We define $g_i^*(x) = g_i\left(\frac{x}{|x|}\right) \cdot g(|x|)$, $i = 1, \ldots, N$. Then $g_i^* \in (D)$, support of $g_i^* \subset \Gamma_i \cap \left\{ x;\ \frac{1}{2} < |x| < 2 \right\}$, and they satisfy

$$\sum_{m=-8}^{\infty} \sum_{i=1}^{N} g_i^{*2}(2^{-m}x) = 1 \quad \text{for} \quad x \neq 0.$$

But $\sum_{m=0}^{-\infty} \sum_{i=1}^{N} g_i^{*2}(2^{-m}x) = g^{**}(x) \in (D)$ if we put $g^{**}(0) = 1$ and it vanishes for $|x| \geqslant 2$.

With these definitions we thus have

$$g^{**}(x) + \sum_{m=1}^{\infty} \sum_{i=1}^{N} g_i^{*2}(2^{-m}x) = 1 \quad \text{for} \quad x \in E^n.$$

Then for any t, $0 < t \leqslant 1$,

$$(J^{v-1}f)^{\wedge}(x) = g^{**}(tx)\,(1 + 4\pi^2 |x|^2)^{(1-v)/2}.\hat{f} +$$
$$+ \sum_{m=1}^{\infty} \sum_{j=1}^{N} [g_j^*(2^{-m}tx)\,(1 + 4\pi^2 |x|^2)^{(1-v)/2}]$$
$$[g_j^*(2^{-m}tx)\,(1 - \exp(2\pi i 2^{-m}t(x, \bar{e}_j)))^{-2}] \cdot (\triangle^2_{2^{-m}t\bar{e}_j} f)^{\wedge}.$$

Using the notation of Lemmas 1 and 3, we may rewrite this as

$$J^{v-1}f = G_{1-v,t} * f + \sum_{m=1}^{\infty} \sum_{i=1}^{N} G^{*i}_{1-v,2^{-m}t} * G^{i}_{0,2^{-m}t} * \triangle^2_{2^{-m}t\bar{e}_i} f \tag{9}$$

and from the same Lemmas 1 and 3, taking $t = 1$, we get

$$\| J^{v-1}f \|_P \leqslant C_{v,g} \left\{ \| f \|_P + \sum_{m=1}^{\infty} 2^{m(1-v-u)} /f/_{P,u} \right\} \leqslant C_v . /f/_{P,u} \tag{10}$$

Analogously taking $t = |h|$ in (9), we obtain,

$$\| \triangle^2_h J^{v-1} f \|_P \leqslant C_v \left(\| \triangle^2_h f \|_P + \sum_{m=1}^{\infty} (2^{-m} |h|)^{u+v-1} /f/_{P,u} \right) \leqslant$$
$$\leqslant C_v |h|^{u+v-1} /f/_{P,u} \qquad (11)$$

(10) and (11) imply

$$/ J^v f /_{P,u-v} \leqslant C^v /f/_{P,u} \qquad \text{q.e.d.}$$

c) From the very definition, $\Lambda_{v-u}{}^P \subset L_0{}^P = L^P$, and $/f/_{P,v-u} \leqslant \| f \|_P$. Applying J^u to both spaces and using b), we obtain $L_u{}^P \supset \Lambda_v{}^P$, which is the second inclusion we had to prove.

To prove the first inclusion, we call $s = u - r$.
Then $\| \triangle^2_h J^s f \|_P = \| \triangle^2_h G_s * f \|_P \leqslant \| \triangle^2_h G_s \|_1 . \| f \|_P$, for $f \in L^P$.
Using Lemma 3, c), of section 2,

$\| \triangle^2_h J^s f \|_P \leqslant C_s |h|^s \| f \|_P$. Since further $\| J^s f \|_P \leqslant \| f \|_P$, we have $/ J^u f /_{P,u} \leqslant C^s \| J^u f \|_{P,u}$, q.e.d.

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