ON THE THEORY OF INTERPOLATION SPACES

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INTRODUCTION. In recent years various interpolation methods (i. e. constructions of interpolation spaces) have been given by many authors (see the bibliography, in particular [11], [12], [13], [14], [2], [5], [6], [9], [10]). In this article, which is based on three lectures given at the Universidad de Buenos Aires in May 1963, we consider two quite general interpolation methods called K-and J-methods, the introduction of which was suggested by the "equivalence theorem" of Lions-Peetre [16], [17], combined with some considerations in Peetre [21]. The K-and J-methods thus generalize the methods studied there. (It turns also out that K-methods are equivalent with the method of Gagliardo [6].) A preliminary account of the theory of K-and J-methods was given in [22]. In order to avoid unnecessary repetition we shall below concentrate on further developments not explicitely included in [22]..

The are two parts. In Part I we establish several interpolation theorems for K- and J- spaces give also an extention of the above mentioned "equivalence theorem" to these spaces. Theorems 1-5 are essentially contained in [22] while theorems 6-8 are new. As an application we obtain the interpolation theorems of M. Riesz [26] and Marcinkiewicz [18] as well as an extention of these theorems to Orlicz space. In Part II we consider more general spaces called n - and M - spaces. Some of the results of Part I can be easily carried over to the more general situation. The motivation for the introduction of N- and M-paces is that in this way we obtain a unified approach to K- and J-spaces on one hand and the "approximation spaces" of [21], [22] on the other hand. In particular we obtain general results (theorems 6-9) which cointain as a special case the "reiteration theorem" of [22] (which again generalizes the "reiteration theorem" of Lions-Peetre [16], [17]), as well as its analogue for the "approximation spaces" in [21], [22]. The enumeration of formulas etc. in the two Parts is independent.

We warn the reader that we are very negligant what concerns all questions of convergence, concentrating instead mainly on establishing the inequalities involved. It is of course clear that this is no serious limitation of the value of the theory established; in most cases the reader should have no difficulties in supplying missing details.

PART I

Some interpolation theorems for K- and J- spaces

Let A_0 and A_1 be two normed spaces both contained in one and the same complete normed space $\mathcal{A}$, the injection of A_i into A being continuos, $A_i \subset \mathcal{A}\,(i=0,\ 1)$. We can then form the *sum* $A_0 + A_1$ of A_0 and A_1 and the *intersection* $A_0 \cap A_1$ of A_0 and A_1. Each of these spaces is linear. In $A_0 + A_1$ we consider the family of (equivalent) norms

(1) $$K(t, a) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\,\|a_1\|_{A_0}) \quad (0 < t < \infty)$$

and in $A_0 \cap A_1$ the family of (equivalent) norms

(2) $$J(t, a) = \max\,(\ \|a\|_{A_0}\ ,\ t\|a\|_{A_1}) \quad (0 < t < \infty)$$

Fixing t (e.g. $t = 1$) they become normed spaces.

Let moreover $\Phi = \Phi[\phi]$ be a *function norm,* i.e. a positive (finite or infinite) functional defined in the set m_+ of all positive (finite or infinite) functions on $(0, \infty)$ measurable with respect to $\frac{dt}{t}$ such that the following axioms hold:

α) $\Phi[\phi] = 0 \leftrightarrow \phi(t) = 0$ a.e.; $\Phi[\phi] < \infty \rightarrow \phi(t) < \infty$ a.e.

β) $\Phi[a\phi] = \Phi[\phi] \quad (a > 0)$

γ) $\phi(t) \leqq \sum_{\nu=1}^{\infty} \phi_\nu(t)$ a.e. $\rightarrow \Phi[\phi] \leqq \sum_{\nu=1}^{\infty} \Phi[\phi_\nu]$

We say that Φ is of genus $\leqq f$ where $f = f(t)$ is a positive function if and only if the following inequality holds:

(3) $$\Phi[\phi(\lambda t)] \leqq f(\lambda) \quad \Phi[\phi(t)].$$

We denote by $(A_0, A_1)_{\Phi}^{K}$ the set of elements $a \in A_0 + A_1$ such that

(4) $$\Phi[K(t, a)] < \infty$$

and by $(A_0, A_1)^J_\Phi$ the set of elements $a \in A_0 + A_1$ such that there exists a measurable with respect to $\frac{dt}{t}$ function $u = u(t)$ with values in $A_0 \cap A_1$ such that

(5) $$a = \int_0^\infty u(t) \frac{dt}{t} \quad (\text{in } A_0 + A_1), \; \Phi[J(t, u(t))] < \infty.$$

Each of these spaces is linear. They become normed spaces if we introduce the norms

(6) $$\|a\|_{(A_0, A_1)^K_\Phi} = \Phi[K(t, a)]$$

and

(7) $$\|a\|_{(A_0, A_1)^J_\Phi} = \inf \Phi[J(t, u(t)].$$

We may call these spaces K- and J-spaces.
Let us set

(8) $$c_K = (\Phi[\min(1, t)])^{-1}$$

and

(9) $$c_J = \sup_{\Phi[\phi]=1} \int_0^\infty \min\left(1, \frac{1}{t}\right) \phi(t) \frac{dt}{t}$$

Then we have the following theorem.

Theorem 1. If $c_K < \infty$, *then* $(A_0, A_1)^K_\Phi \subset A_0 + A_1$ *and, if* $c_K > 0$, *then* $A_0 \cap A_1 \subset (A_0, A_1)^K_\Phi$. *If* $c_J < \infty$, *then* $(A_0, A_1)^J_\Phi \subset A_0 + A_1$ *and , if* $c_J > 0$, *then* $A_0 \cap A_1 \subset (A_0, A_1)^J_\Phi$.
All injections are continuous.

The proof may be found in [22].

We now turn to the following important interpolation theorem. Let Φ be of genus $\leqq f$.

Let, besides A_0 and A_1, B_0 and B_1 be another two normed spaces contained in one and the same normed space $\mathcal{B}$, the injection of B_i into $\mathcal{B}$ being continuous: $B_i \subset \mathcal{B}$ $(i=0, 1)$.

Theorem 2. Let Π *be a linear continuous mapping from* $A_0 + A_1$ *into* $B_0 + B_1$ *such that*

$$\|\Pi a\|_{B_i} \leqq M_i \|a\|_{A_i}, \; a \in A_i \qquad (i=0,1) \tag{10}$$

where M_0 *and* M_1 *are constants. Then*

$$\|\Pi a\|_B \leqq \gamma M_0 f\left(\frac{M_1}{M_0}\right) \|a\|_A, \; a \in A, \tag{11}$$

where 1º $A=(A_0, A_1)^K_\Phi$, $B=(B_0, B_1)^K_\Phi$, $\gamma=1$

or 2º $A=(A_0, A_1)^J_\Phi$, $B=(B_0, B_1)^J_\Phi$, $\gamma=1$

or 3º $A=(A_0, A_1)^J_\Phi$, $B=(B_0, B_1)^K_\Phi$,

$$\gamma = \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda}.$$

Proof: We note the following inequalities, which follow at once from (10):

$$K(t, \Pi a) \leqq M_0 K\left(\frac{M_1 t}{M_0}, \; a\right), \tag{12}$$

$$J(t, \Pi a) \leqq M_0 J\left(\frac{M_1 t}{M_0}, \; a\right), \tag{13}$$

$$K(t, \Pi a) \leqq \min\left(1, \frac{t}{s}\right) M_0 J\left(\frac{M_1 s}{M_0}, \; a\right). \tag{14}$$

Case 1º: Using (12) we get, in view of (3):

$$\|\Pi a\|_{(B_0, B_1)^K_\Phi} = \Phi[K(t, \Pi a)] \leqq M_0 \Phi\left[K\left(\frac{M_1 t}{M_0}, \; a\right)\right] \leqq$$

$$\leqq M_0 f\left(\frac{M_1}{M_0}\right) \Phi[K(t, a)] = M_0 f\left(\frac{M_1}{M_0}\right) \| a \|_{(A_0, A_1)^K_\Phi}.$$

Case 2º: We note that $\Pi a = \int_0^\infty \Pi u \left(\frac{M_1 t}{M_0}\right) \frac{dt}{t}$. Using (13) we get, in view of (3):

$$\|\Pi a\|_{(B_0,\ B_1)_K^J} \leqq \Phi[J(t, \Pi u \left(\frac{M_1 t}{M_0}\right))] \leqq$$

$$\leqq M_0\ \Phi[J\left(\frac{M_1 t}{M_0},\ \Pi u\left(\frac{M_1 t}{M_0}\right)\right)\ ] \leqq M_0 f\left(\frac{M_1}{M_0}\right) \Phi[J(t, u(t))]$$

and the last term tends to

$$M_0 f\left(\frac{M_1}{M_0}\right) \|a\|_{(A_0,\ A_1)_\Phi^J}$$

if u is chosen conveniently.

Case 3º: We note again that $\Pi a = \int_0^\infty \Pi u \left(\frac{M_1 t}{M_0}\right) \frac{dt}{t}$.

Using (14) we get:

$$\mathrm{K}(t, \Pi a) \leqq \int_0^\infty K(t, \Pi u \left(\frac{M_1 s}{M_0}\right)) \frac{ds}{s} \leqq$$

$$\leqq \int_0^\infty \min\left(1, \frac{t}{s}\right) M_0\ J\left(\frac{M_1 s}{M_0},\ u\left(\frac{M_1 s}{M_0}\right)\right) \frac{ds}{s} =$$

$$= \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) M_0\ J\left(\frac{M_1 t\lambda}{M_0},\ u\left(\frac{M_1 t\lambda}{M_0}\right)\right) \frac{d\lambda}{\lambda}$$

so that, in view of (3):

$$\|\Pi a\|_{(B_0,\ B_1)_\Phi^K} = \Phi[K(t, \Pi a)] \leqq$$

$$\leqq \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) M_0\ \Phi[J\left(\frac{M_1 t\lambda}{M_0},\ u\left(\frac{M_1 t\lambda}{M_0}\right)\right)\ ] \frac{d\lambda}{\lambda} \leqq$$

$$\leqq \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} M_0 f\left(\frac{M_1}{M_0}\right) \Phi[J(t, u(t))]$$

and the last term tends to

$$\int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} M_0 f\left(\frac{M_0}{M_1}\right) \|a\|_{(A_0,\ A_1)_\Phi^K}$$

if u is chosen conveniently.

The proof is complete.

Taking $A_0 = B_0$, $A_1 = B_1$, $\Pi =$ identity mapping we get as a concequence.

Theorem 3. We have $(A_0, A_1)^J_\Phi \subset (A_0, A_1)^K_\Phi$ *, with continuous injection, provided*

$$\int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} < \infty. \tag{15}$$

Indeed we have then the inequality

$$\| a \|_{(A_0, A_1)^K_\Phi} < \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} \| a \|_{(A_0, A_1)^J_\Phi}, \quad a \in (A_0, A_1)^J_\Phi. \tag{16}$$

The following theorem is a sort of converse.

Theorem 4. We have $(A_0, A_1)^K_\Phi \subset (A_0, A_1)^J_\Phi$ *, with continuous injection, provided* $c_K < \infty$ *and*

$$\min\left(1, \frac{1}{\lambda}\right) f(\lambda) \to 0 \quad \textit{as} \quad \lambda \to 0 \quad \textit{or} \quad \infty. \tag{17}$$

Indeed we have the inequality

$$\| a \|_{(A_0, A_1)^J_\Phi} \leqq 4 \| a \|_{(A_0, A_1)^K_\Phi}, \quad a \in (A_0, A_1)^K_\Phi. \tag{18}$$

This follows easily from the proof of theorem 1 (cf. [22]) and the following lemma.

Lemma 1. Let $a \in A_0 + A_1$ *be such that*

$$\min\left(1, \frac{1}{t}\right) K(t, a) \to 0 \quad \textit{as} \quad \lambda \to 0 \quad \textit{or} \quad \infty. \tag{19}$$

Then there exists a measurable with respect to $\frac{dt}{t}$ *function* $u = u(t)$ *with values in* $A_0 \cap A_1$ *such that*

$$a = \int_0^\infty u(t) \frac{dt}{t} \text{ (in } A_0 + A_1), \quad J(t, u(t)) < 4 K(t, a). \tag{20}$$

For details we refer to [22].

Remark 1. Note that (15) and (17) are fulfilled in the important special case $f(\lambda) = \lambda^\theta$, $0 < \theta < 1$. This leads in view of [21], [23], to the "equivalence theorem" of Lions-Peetre [16], [17] mentioned in the Introduction.

With the aid of theorem 4 we can give the following complement to theorem 2.

Theorem 5. Assume that (17) *holds true. Then the conclusion of theorem 2 holds also in the following case:* 4º $A = (A_0, A_1)^K_\Phi$, $B = (B_0, B_1)^J_\Phi$, $\gamma = 4$.

Let us now observe that, in view of the definition (1), $K(t,a)$ is *concave* considered as a function of t. Therefore $K(t, a)$ can be represented in the form

$$K(t, a) = \int_0^t k(s, a)\, ds \tag{21}$$

where $k(t, a)$ is *non-increasing* considered as a function of t, provided we impose also some auxiliary condition which assures that $K(t,a) \to 0$ as $t \to 0$. (This is always the case in example 1 below).

Theorem 6. We have $a \in (A_0, A_1)^K_\Phi$ *if and only if* $\Phi[t\, k(t, a)] < \infty$, *provided*

$$\int_0^1 f(\lambda) \frac{d\lambda}{\lambda} < \infty. \tag{22}$$

Proof: i) Since $K(t, a) \geqq t\, k(t, a)$ we get

$$\| a \|_{(A_0,\ A_1)^K_\Phi} = \Phi[K(t, a)] \geqq \Phi[t\, k(t, a)]$$

and the "only if" part follows.

ii) Let us make a change of variable in the integral (21):

$$K(t, a) = \int_0^1 t\, \lambda\, k(t\lambda, a) \frac{d\lambda}{\lambda}. \tag{23}$$

Therefore

$$\| a \|_{(A_0,\ A_1)^K_\Phi} = \Phi[K(t, a)] \leqq \int_0^1 \Phi[t\lambda k(t\lambda, a)] \frac{d\lambda}{\lambda} \leqq$$

$$\leqq \int_0^1 f(\lambda) \frac{d\lambda}{\lambda} \Phi[t\, k(t, a)]$$

and the "if" part follows.

We illustrate the above results in a concrete case.

Example 1. Let $A_0 = L_1$, $A_1 = L_\infty$ (with respect to some positive measure on some locally compact space). Then one can prove (cf. [22]) that $k(t,a) = a^*(t)$ where $a^*(t)$, as customary, denotes the non-increasing rearrangement of a on $(0, \infty)$ with the measure dt, i.e. a^* and a are equimeasurable (cf. e.g. [7]).

a) (Lebesgue spaces) Let us take

$$\Phi[\phi] = \left(\int_0^\infty \left(\frac{\phi(t)}{t}\right)^p dt\right)^{\frac{1}{p}} = \left\| \frac{\phi}{t} \right\|_{L_p}$$

One sees easily that Φ is of genus $\leqq \lambda^{1-\frac{1}{p}}$. Then

$$\Phi[t\, k(t, a)] = \| a^* \|_{L_p} = \| a \|_{L_p}$$

so that by theorem 6 $(L_1, L_\infty)^K_\Phi = L_p$ provided $p > 1$. Applying theorem 2 one gets as a special case the interpolation theorem of M. Riesz [26].

b) (Orlicz spaces). Let $M(\lambda)$ be a positive, non-decreasing convex function and let $\xi(\lambda)$ be a positive increasing function such that $M(\lambda\mu) \leqq \xi(\lambda)\, M(\mu)$. Let us take

$$\Phi[\phi] = \inf_{r>0} r \max\left(\int_0^\infty M\left(\frac{\phi(t)}{r\,t}\right) d\,t, \frac{1}{r}\right) = \left\| \frac{\phi}{t} \right\|_{L_M}$$

(which is Luxemburg's definition of the norm in Orlicz space, cf. e.g. [8]). One sees easily that Φ is of genus $\leqq \frac{\lambda}{\xi^{-1}(\lambda)}$. Then

$$\Phi[t\, k(t, a)] = \| a^* \|_{L_M} = \| a \|_{L_M}$$

so that by theorem 6 $(L_1, L_\infty)^K_\Phi = L_M$ provided $\int_0^1 \frac{d\lambda}{\xi^{-1}(\lambda)} < \infty$.

Applying theorem 2 one gets as a special case a sort of generalization to Orlicz space of the interpolation theorem of M.. Riesz [26].

Remark 2. A quite different approach to such interpolation theorems can be based on an idea in Cotlar [3], p. 197.

We discuss next some extentions of theorem 2 in the case 3º.

Theorem 7. Let Π *be a continuous linear mapping from* $A_0 + A_1$ *into* $B_0 + B_1$ *such that*

$$K(t, \Pi a) \leqq Q\left(\frac{t}{s}\right) M_0 J\left(\frac{M_0 s}{M_1}, a\right) \tag{24}$$

where $Q(\lambda)$ *is a positive function and* M_0 *and* M_1 *are constants. Then* (11) *holds with*

$$A = (A_0, A_1)^J_\Phi \ , \ B = B_0, B_1)^K_\Phi \ , \gamma = \int_0^\infty Q\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda}.$$

Proof: Identical with the proof of theorem 2 (case 3º).

Theorem 8. Assume, instead of (24), *that* Π *satisfies*

$$t\, k(t, \Pi a) \leqq q\left(\frac{t}{s}\right) M_0 \ J\left(\frac{M_1 s}{M_0}, a\right). \tag{25}$$

where $q(\lambda)$ *is a positive function and* M_0 *and* M_1 *are constants. Then* (24) *holds with*

$$Q(\lambda) = \int_0^1 q(\lambda\mu) \frac{d\mu}{\mu}. \tag{26}$$

Therefore hold also the conclusions of theorem 7.

Proof: Using (23) we get at once

$$K(t, a) \leqq \int_0^1 q\left(\frac{t\lambda}{s}\right) \frac{d\lambda}{\lambda} M_0 \ J\left(\frac{M_1 s}{M_0}, a\right)$$

and (24), with Q defined by (26), follows.

Example 2. An important special case is $q(\lambda) = \min(1, \lambda)$. Then $Q(\lambda) = \lambda$ if $\lambda \leqq 1$, $= 1 + \log\lambda$ if $\lambda > 1$.

Example 3. Let A_0, A_1,ϕ be as in example 1 and $q(\lambda)$ as in example 2. Applying theorem 8 we can now get as a special case the interpolation theorem of Marcinkiewicz [18] as well as a generalization of it to Orlicz space.

Remark 3. We conclude Part I with a few observations of heuristic nature intended to facilate the proper understanding of the above results. First we wish to point out that theorem 2 in the case 3º and theorem 7 are related to each other roughly as the theorems of M. Riesz and Marcinkiewicz. We also wish to point out that the special case $f(\lambda) = \lambda^{\theta}$ (thus essentially the case considered in Lions-Peetre [16], [17] is related to the general case roughly in a similar way as Lebesgue spaces L_p to Orlicz spaces L_M.

PART II

A general reiteration theorem.

Let $\mathcal{A}$ be a complete normed space. We consider two arbitrary families of norms (1) in $\mathcal{A}$, $N(t, a)$ and $M(t, a)$ $(0 < t < \infty)$. Let Φ be a function norm (see Part I). We denote then by F^{N}_{Φ} the set of elements $a \in A$ such that

$$(1) \qquad \Phi[N(t, a)] < \infty$$

and by E^{M}_{Φ} the set of elements $a \in A$ such that there exists a measurable with respect to $\frac{dt}{t}$ function $u = u(t)$ with values in $\mathcal{A}$ such that

$$(2) \qquad a = \int_0^{\infty} u(t) \frac{dt}{t} \text{ (in } \mathcal{A}\text{)}, \ \phi[M(t, u(t))] < \infty.$$

Each of these spaces is linear. They become normed spaces if we introduce the norms

$$(3) \qquad \|a\|_{F^{N}_{\Phi}} = \Phi[N(t,a)]$$

and

$$(4) \qquad \|a\|_{E^{M}_{\Phi}} = \inf \Phi[M(t, u(t))].$$

We may call this spaces N- and M-spaces.

(1) We use the word norm in a very wide sense including in this concept also what is usually called semi-norm (the value 0 is permited) and pseudonorm (the value ∞ is permited).

Let us discuss the two principal examples of N - and M-spaces.

Example 1. Let A_0 and A_1 be two normed spaces both contained in $\mathcal{A}$, the injection of A_i into $\mathcal{A}$ being continuous $(i = 0,1)$. We may take $N(t, a) = K(t, a)$, $M(t, a) = J(t, a)$. Then wen have

$$F^{N}_{\Phi} = (A_0, A_1)^{K}_{\Phi} \ , \ E^{M}_{\Phi} = (A_0, A_1)^{J}_{\Phi} \ .$$

Example 2. Let W_n $(n = 0, 1, 2, \ldots)$ be a family of linear subspaces of $\mathcal{A}$ such that $0 = W_0 \subset W_1 \subset W_2 \subset \ldots$ We may take

(5) $$N(t,a) = \inf_{w \in W_n} \| a - w \|_{\mathcal{A}}$$

and

(6) $$M(t, a) = \| a \|_{\mathcal{A}} \text{ if } a \in W_n, e^{-n} \leqq t < e^{-n-1} \text{ or } t > 1$$

$$= \infty \text{ if } a \in W_n, e^{-n} \leqq t < e^{-n-1}$$

We will start with some straight forward generalizations of certains results of Part I.

Theorem 1. Let Π *be a continuous linear mapping from* $\mathcal{A}$ *into* $\mathcal{A}$ *such that*

(7) $$M(t, \Pi a) \leqq Q\left(\frac{t}{s}\right) M_0 N\left(\frac{M_1 s}{M_0}, a\right)$$

where $Q(\lambda)$ *is positive function and* M_0 *and* M_1 *are constants. Suppose* Φ *is of genus* $\leqq f$. *Then*

(8)

$$\| \Pi a \|_{F^{N}_{\Phi}} \leqq \int_0^\infty Q\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} M_0 f\left(\frac{M_1}{M_0}\right) \| a \|_{E^{M}_{\Phi}} , \ a \in E^{M}_{\Phi} .$$

Proof: Identical with the proof of theorem I. 2 (Case 3°).

If $\Pi =$ identity mapping, we get as a consequence.

Theorem 2. Assume that

(9) $$M(t, a) \leqq Q\left(\frac{t}{s}\right) N(s,a)$$

where $Q(\lambda)$ is a positive function. Then we have $E_{\Phi}^{M} \subset F_{\Phi}^{N}$, with continuous injection, provided

$$(10) \qquad \int_0^{\infty} Q\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} < \infty.$$

Indeed we have the inequality

$$(11) \qquad \| a \|_{F_{\Phi}^{N}} \leqq \int_0^{\infty} Q\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} \| a \|_{E_{\Phi}^{M}}, \; a \in E_{\Phi}^{M}.$$

Example 3. In the case of example 1 we may take $Q(\lambda) = \min(1, \lambda)$ and in the case of example 2 $Q(\lambda) = 0$ if $\lambda \leqq 1$, 1 if $\lambda < 1$.

Let us denote by $\overset{0}{\mathcal{A}}$ the space of elements $a \in \mathcal{A}$ such that there exists a constant R and a measurable with respect to $\frac{dt}{t}$ function function $u = u(t)$ with values in $\mathcal{A}$ such that

$$(12) \qquad a = \int_0^{\infty} u(t) \frac{dt}{t}, \; M(t, u(t)) \leqq R\, N(t, a).$$

Example 4. In the case of example 7 $a \in \overset{0}{\mathcal{A}}$ with $R = 4$ provided (see Lemma I.1)

$$(13) \qquad \min\left(1, \frac{1}{t}\right) K(t, a) \to \text{as } t \to 0 \text{ or } \infty$$

and in the case of example 2 $a \in \overset{0}{\mathcal{A}}$ with $R = 2$ provided (cf. [22])

$$(14) \qquad N(t, a) \to 0 \text{ as } t \to 0.$$

We can now give a converse of theorem 2.

Theorem 3. If $a \in F_{\Phi}^{N}$ implies $a \in \overset{0}{\mathcal{A}}$ with R independent of a, then $F_{\Phi}^{N} \subset E_{\Phi}^{M}$, with continuous injection. Indeed we have the inequality

(15) $$\| a \|_{E_\Phi^M} \leqq R \| a \|_{F_\Phi^N}, \quad a \in F_\Phi^N .$$

Example 5. In the case of example 1 it suffices that f satisfies (see theorem I.4)

(16) $$\min\left(1, \frac{1}{\lambda}\right) f(\lambda) \to 0 \text{ as } \lambda \to 0 \text{ or } \infty.$$

In the case of example 2 it suffices that

(17) $$f(\lambda) \to 0 \text{ as } \lambda \to 0.$$

Let $f = f(\lambda)$ be any positive function. Let A be a normed space contained in $\mathscr{A}$.

Definition 1. We say that A is of class $\mathcal{D}_f^N$ if and ouly if

(18) $$N(t, a) \leqq Df(t) \; \| a \|_A$$

where D is a constant, and that A is of class $\mathcal{C}_f^M$ if and only if

(19) $$\| a \|_A \leqq C \, f\left(\frac{1}{t}\right) M(t, a)$$

where C is a constant.

Example 6. Assume that Φ is of genus $\leqq f$. In the case of example 1, $(A_0, A_1)_\Phi^K$ is of class $\mathcal{D}_f^K$ provided $c_K < \infty$ and of class $\mathcal{C}_f^J$ provided $c_K > 0$; $(A_0, A_1)_\Phi^J$ is of class $\mathcal{D}_f^K$ provided $c_J < \infty$ and of class $\mathcal{D}_f^J$ provided $c_J > 0$. (Here c_K and c_J are as in (I.8) and (I.9)!) This follows easily from the proof of theorem I.1 (cf. [22]).

Spaces of classes $\mathcal{D}_f^N$ and $\mathcal{C}_f^N$ are characterized by the following theorems.

Theorem 4. A *is of class* $\mathcal{D}_f^N$ *if and only if*

(20) $$A \subset F_{\Phi}^{N}, \quad \Phi[\phi] = \sup \frac{\phi(t)}{f(t)}.$$

Theorem 5. A *is of class* $\mathcal{C}_f^M$ *if and only if*

(21) $$A \supset E_{\Phi}^{M}, \quad \Phi[\phi] = \int_0^{\infty} f\left(\frac{1}{t}\right) \phi(t) \frac{dt}{t}.$$

The proof of these theorems is obvious.

Let us from now on assume that $f(\lambda)$ is of the from λ^{α}. We shall write $\mathcal{D}_{\alpha}^{N}$ and $\mathcal{C}_{\alpha}^{M}$ instead of $\mathcal{D}_{\lambda^\alpha}^{N}$ and $\mathcal{C}_{\lambda^\alpha}^{M}$. Let $\alpha_0 < \alpha_1$ be given. If Φ is any function norm we define Ω by

(22) $$\Omega\, [\phi] = \Phi\, \left[t^{\alpha_0} \varphi\, (t^{\alpha_1 - \alpha_0})\right].$$

If Φ is of genus $\leqq f$ then Ω is of genus $\leqq r$ where r is given by

(23) $$r(\lambda) = \lambda^{-\frac{\alpha_0}{\alpha_1 - \alpha_0}} f\left(\lambda^{\frac{1}{\alpha_1 - \alpha_0}}\right).$$

We can now announce our main results.

Theorem 6. Let A_i *be of class* $\mathcal{D}_{\alpha_i}^{N}$ $(i = 0,1)$. *Then*

(24) $$\| a \|_{F_{\Phi}^{N}} \leqq D_0\, r\left(\frac{D_1}{D_0}\right) \| a \|_{(A_0, A_1)_{\Phi}^{K}}, \quad a \in (A_0, A_1)_{\Phi}^{K}$$

so that $(A_0, A_1)_{\Phi}^{K} \subset F_{\Phi}^{N}$ *with continuous injection.*

Theorem 7. Let A_i *be of class* $\mathcal{C}_{\alpha_i}^{M}$ $(i = 0,1)$. *Then*

(25) $$\| a \|_{(A_0, A_1)_{\Phi}^{J}} \leqq \frac{1}{\alpha_1 - \alpha_0}\, C_0\, r\left(\frac{C_1}{C_0}\right) \| a \|_{F_{\Phi}^{M}}, \quad a \in E_{\Phi}^{M}$$

so that $E_{\Phi}^{M} \subset (A_0, A_1)_{\Phi}^{J}$.

Since the proof of theorem 7 is similar though slightly longer (cf. [22] for details) we shall only indicate the proof of theorem 6.

Proof of theorem 6: Let $a = a_0 + a_1$. Then we have

$$N(t,a) \leqq N(t, a_0) + N(t, a_1) \leqq D_0 \, t^{\alpha_0} \, \| a_0 \|_{A_0} + D_1 \, t^{\alpha_1} \, \| a_1 \|_{A_1} =$$
$$= D_0 \, t^{\alpha_0} \left(\| a_0 \|_{A_0} + \frac{D_1}{D_0} t^{\alpha_1 - \alpha_0} \| a_1 \|_{A_1} \right).$$

Making vary a_0 and a_1 we get

$$N(t, a) \leqq D_0 \, t^{\alpha_0} \, K \left(\frac{D_1}{D_0} t^{\alpha_1 - \alpha_0} , a \right)$$

from which the result easily follows by (22) and (23).

Theorem 8. Let A_i be of class $\mathcal{D}^N_{\alpha i}$ and of class $\mathcal{C}^M_{\alpha i}$ $(i = 0,1)$.

Then $E^M_\Phi \subset (A_0, A_1)^J_\Phi \subset (A_0, A_1)^K_\Phi \subset F^N_\Phi$, with continuous injections, provided

$$\int_0^\infty \min \left(\frac{1}{\lambda^{\alpha_0}}, \frac{1}{\lambda^{\alpha_1}} \right) f(\lambda) \frac{d\lambda}{\lambda} < \infty . \tag{26}$$

Proof: Apply theorem 2 (or theorem I.3).

Theorem 9. Let again A_i be of class $\mathcal{D}^N_{\alpha i}$ and of class $\mathcal{C}^M_{\alpha i}$ $(i = 0,1)$. Suppose that the assumptions of theorem 3 are fulfilled. Then $F^N_\Phi = E^M_\Phi = (A_0, A_1)^K_\Phi = (A_0, A_1)^J_\Phi$, with continuous injections, provided (26) *holds*

Proof: Apply theorem 3.

Example 7. Consider the case of example 1. Let $\widetilde{A}_i$ be of class $\mathcal{D}^K_{\alpha i}$ and of class $\mathcal{C}^J_{\alpha i}$ $(i = 0,1)$. Then $(A_0, A_1)^K_\Phi = (A_0, A_1)^J_\Phi =$

$= (\widetilde{A}_0, \widetilde{A}_1)^K_\Omega = (\widetilde{A}_0, \widetilde{A}_1)^J_\Phi$ provided (26) and (16) hold. This is the "reiteration theorem" of [22]. (A "reiteration theorem" of somewhat different nature connected with the "complex variable" methods of [2], [9],]14[was recently found by Lions [15].)

Remark 1. With the aid of the reiteration theorem we can also extend the results of example I.1 to the case $A_0 = L_{p_0}$, $A_1 = L_{p_1}$.

Example 8. Consider the case of example 2. Let A_i be of class $\mathcal{D}^K_{\alpha_i}$ and of class $\mathcal{C}^J_{\alpha_i}$ $(i = (0,1)$. Then $F^N_\Phi = E^M_\Phi = (A_0, A_1)^K_\Omega =$ $= (A_0, A_1)^J_\Omega$ provided (26) and (17) hold. This is analogue of the "reiteration theorem" for the "approximation spaces" (cf. [21], [22]).

We conclude by applying example 8 in a concrete case.

Example 9. Let $\mathcal{A}$ be L_p with respect to the Haar measure dx on the additive group of real numbers, i.e. the intervall $(-\infty, \infty)$. Denote W^m_p the space of functions a whose generalized derivatives up to order m are in L_p: $\left(\frac{d}{dx}\right)^j a \in L_p$ if $o \leqq j \leqq m$. Let W_n be the space of functions a in L_p such that the generalized Fourier transform vanishes outside $(-e^n, e^n)$;i.e. a is entire of exponential type e^n. Then (trivial) L_p is of class $\mathcal{D}^M_0$ and of class $\mathcal{C}^N_0$ and (using Fourier transforms) W^m_p is of class $\mathcal{D}^M_m$ and of class $\mathcal{C}^N_m$. Therefore $F^N_\Phi = E^M_\Phi = (L_p, W^m_p)^K_\Omega = (L_p, W^m_p)^J_\Omega$ where Ω is given by (22) with $\alpha_0 = 0$, $\alpha_1 = m$ provided $\int_0^\infty \min\left(1, \frac{1}{\lambda^m}\right) f(\lambda) \frac{d\lambda}{\lambda} < \infty$ and $f(\lambda) \to 0$ as $\lambda \to 0$. Let us specialize to $\Phi[\phi] =$ $= \sup \frac{\phi(t)}{t^\alpha}$, $0 < \alpha < m$. Then we may take $f(\lambda) = \lambda^\alpha$ so the above assumptions of f are fulfilled. On the order hand it is known (cf. [12], [17], [25], [22]) that in this case $(L_p, W^m_p)^K_\Omega =$ $= (L_p, W^m_p)^J_\Omega$ is the space of functions $a \in L_p$ satisfying the following Hölder type condition: $\sup_h h^{-\alpha} \| (\Delta(h))^m a \|_{L_p} < \infty$ where $\Delta(h)$ is the operation of taking differences of increment h: $\Delta(h)a(x) =$ $= a(x+h) - a(x)$. In this way we are lead to the classical theorems of Jackson and Bernstein in the constructive theory of functions (cf. e g. [1]). One can also consider the case of ν variables $(\nu \geq 1)$, in which way we obtain various results found in recent years by Nikolskij and his school (cf. e. g. [19]), as well as other extentions. The full details will be published in a forthcoming paper.

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