Throughout this paper $R$ denotes an associative ring with identity. We shall study the following properties associated to $R$.

a) the purity of the inclusion $R=M$ of $R$ in an injective $R$-module $M$ containing it.

b) the algebraic closure of M. Hall, of submodules of free $R$-modules.

c) a weak injectivity property of $R$ as an $R$-module.

Section 2 contains the main results. In Section 3 we characterize von Neumann rings in terms of purity.

1. PRELIMINAIRES.

i) Purity. Let $M$ and $N$ be right $R$-modules. An exact sequence $0 \to N \to M$ of $R$-modules will be said pure if for every left $R$-module $A$, the induced sequence $0 \to N \otimes A \to M \otimes A$ is exact ($\otimes = \otimes_R$). If $N$ is a submodule of $M$, we say that $N$ is pure in $M$ if the exact sequence $0 \to N \to^i M$, where $i$ denotes the inclusion map, is pure. Let $N$ be a right $R$-module. Then the following conditions are easily seen to be equivalent (and we shall therefore say simply that $N$ is pure),

1) $N$ is pure in any injective module containing it

2) $N$ is pure in its injective hull

3) $N$ is pure in any module containing it.

ii) Conditions ($h^v$), ($c^v$), ($a^v$), ($b^v$). Let $A$ be a left (resp. right) $R$-module and $n \in N$. $A^n$ denotes the left (resp. right) $R$-module, direct sum of $n$ copies of $A$. If $a \in A^n$ we write $a = [a_1, \ldots, a_n]$ in terms of its coordinates. With $R^n$ (resp. $R^n$) we denote the previous situation for $A = R$. Let $A$ be a left $R$-module. We define a left pairing $R^n \times A^n \to A$ by $r.a = \sum_{i=1}^{n} r_i a_i$.

For any non-empty set $S \subseteq R^n$, $S^r$ denotes the right annihilator of $S$ in $R^n$, that is

$S^r = \{ r / r \in R^n \text{ and } s.r = 0 \text{ if } s \in S \}$

In analogous way we define the left annihilator $T^l = R^n$ of a non-empty set $T \subseteq R^n$.

According to M. Hall (1), a submodule $S$ of $R^n$ will be said to be closed if $S = (S^r)^l = S^r = S$. We can now state

CONDITION ($h^v$)$_1$: Every finitely generated submodule of $R^n$ is closed.
CONDITION \((c^0)_1\): Every finitely generated left ideal of \(R\) is closed.

CONDITION \((c^0)_1\) is the special case of \((h^0)_1\) when \(n = 1\).

Next we define the weak injectivity referred above. This is

CONDITION \((a^0)_1\): Every \(R\)-homomorphism of a finitely generated left ideal of \(R\), into \(R\), is realized by a right multiplication by an element of \(R\).

CONDITION \((b^0)_1\): Let \(U\) and \(T\) be left ideals of \(R\), then
\[
(U \cap TF = U^R + T^R)
\]
holds.

We also define analogous conditions for right objects, we write them \((h^0)_r\), \((c^0)_r\), etc. ...

On restricting the previous conditions to principal ideals or cyclic submodules we introduce conditions \((h^{0\circ})_1\), \((h^{0\circ})_r\), etc. ...

The following results will be used in the sequel.

**Proposition 1.1.** (Ikeda-Nakayama (2), Th. 1). The following implications hold in \(R\):

i) \((a^{0\circ})_1 \implies (c^{0\circ})_r\)

ii) \((a^0)_1 \implies (b^0)_1\)

iii) \((a^0)_1 \implies (c^0)_r\)

**Proposition 1.2.** (3) 1, §2, Exer. 24). Let \(M\) be a right \(R\)-module and \(M'\) a submodule of \(M\). Then \(M'\) is pure in \(M\) if and only if for any set of elements \(m_i \in M'\), \(x_j \in M\), \(r_{ij} \in R\) \((i=1,\ldots,m; j=1,\ldots,n)\) such that
\[
m_i = \sum_{j=1}^{n} x_j \cdot r_{ij}
\]
there exist elements \(x_j' \in M'\), \(j=1,\ldots,n\) satisfying
\[
m_i' = \sum_{j=1}^{n} x_j' \cdot r_{ij}
\]

As an immediate consequence of Prop. 1.2 we have the following

**Proposition 1.3.** Let \(R\) be an injective hull of \(R\), as right \(R\)-modules. Assume that \(R\) is pure in \(R\). Then any homomorphism \(\mu: U \to R\) of a finitely generated submodule \(U\) of \(R^n\) into \(R\) admits an extension to \(R^n\).

**Proof:** Clearly \(\mu\) admits an extension to \(\mu': R^n \to R\). Therefore if \(u_1,\ldots,u_m\) denote a set of generators of \(U\) and \(e_1,\ldots,e_n\) the canonical basis of \(R^n\), we have
\[
\mu(u_i) = \sum_{j=1}^{n} u'(e_j) \cdot r_{ij}
\]

By the purity of \(R\) in \(R\) there exist elements \(x_j'\), \(j=1,\ldots,n\) in \(R\).
satisfying
\[ u(u_i) = \sum_{j=1}^{n} x_j^i \cdot r_{ij} \quad i=1, \ldots, m \]

Consequently the mapping defined by
\[ e_j \rightarrow x_j \]
gives an extension of \( u \).

**PROPOSITION 1.4.** Let \( A \) be a left \( R \)-module. Then \( A \) is injective if and only if every homomorphism \( U \rightarrow A \) of a submodule \( U \) of \( R^n \) into \( A \) is realized by an element of \( A^n \), that is, there exists \( y \in A^n \) such that \( u(u) = u.y \) for all \( u \in U \).

2. MAIN RESULTS.

Let \( R_r \) denote an injective right \( R \)-module containing \( R \)

**THEOREM 1.** The following implications hold in \( R \):
\[
R \text{ is right pure in } R_r \iff (h^o)_r \iff (h^o 0)_r \iff (a^o)_r
\]

**Proof:** \( R \) is right pure in \( R_r \) \( \iff \) \((h^o)_r \)

Let \( H \) be a finitely generated submodule of \( R^n \) and let
\[ z_i = [z_{i1}, \ldots, z_{im}] \in R^n, \quad i=1, \ldots, m \]
be a set of generators of it. Let \( a = [a_1, \ldots, a_n] \in R^n \) be an element of \( H^n \), that is, such that
\[ u \in R^n, \ z_i.u = 0, \ i=1, \ldots, m \implies a.u = 0 \]

Let \( H^n \) be the submodule of \( R^n \) generated by the vectors
\[ z_i = [z_{i1}, \ldots, z_{im}] , \ i=1, \ldots, n \]
Then (1) says precisely that
\[ u: z_i \rightarrow a_i \]
defines a homomorphism
\[ u: H^n \rightarrow R \]

There exists then by Prop. 1.4, \( b = [b_1, \ldots, b_n] \in R_r \) satisfying
\[ a_i = u(z_i^1) = b.z_i^1 \quad i=1, \ldots, n \]

By the purity of \( R \) in \( R_r \) we find \( u \in R^n \) with
\[ a_i = u.z_i^1 \quad i=1, \ldots, m \]
that is
\[ a_i = \sum_{j=1}^{m} u_j \cdot z_{ij} \quad \text{or} \quad a = \sum_{j=1}^{m} u_j \cdot z_j \]
which amounts to saying that \( a \in \mathcal{H} \), as we wanted to prove.

\((h^o)_1 \Rightarrow R \) is right pure in \( R_r \).

Let \( a_i \in R, z_i \in R^n, u \in R^m, i=1,\ldots,m \) satisfy

\[(2) \quad a_i = u \cdot z_i \quad i=1,\ldots,m \]

If \( b \in R^m \) satisfies \( z_i^t \cdot b = 0 \), then by (2) we have \( a \cdot b = 0 \) and by condition \((h^o)_1 \) we have that there exist \( r_i \in R, i=1,\ldots,m \)

with \( a = \sum_{i=1}^{m} r_i \cdot z_i \]

that is \( a_i = r_i \cdot z_i \quad i=1,\ldots,m \)

with \( r = [r_1,\ldots,r_m] \). This proves our claim.

\((h^o)_1 \Rightarrow (a^o)_1 \) is trivial.

Finally we prove the equivalence \((a^o)_1 \iff (a^o)_r \)

\((h^o)_1 \Rightarrow (a^o)_r \)

Let \( I = \langle a_1, \ldots, a_n \rangle \) be a right ideal of \( R \) generated by \( a_1,\ldots,a_n \).

Let \( \phi: I \to R \) be a homomorphism of \( I \) into \( R \), as right \( R \)-modules.

Let \( b_i = \phi(a_i), i=1,\ldots,n \). Since \( \phi \) is a homomorphism, for any \( t_1,\ldots,t_n \in R \)

\[ \sum_{i=1}^{n} a_i \cdot t_i = 0 \Rightarrow \sum_{i=1}^{n} b_i \cdot t_i = 0 \]

This means that \( [b_1,\ldots,b_n] \in [a_1,\ldots,a_n]^r_1 = \langle[a_1,\ldots,a_n] \rangle \).

So there is \( k \in R \) satisfying

\[ [b_1,\ldots,b_n] = k \cdot [a_1,\ldots,a_n] \]

that is \( \phi(a_i) = b_i = k \cdot a_i \)

and this proves \((a^o)_r \).

\((a^o)_r \Rightarrow (h^o)_1 \)

This implication will be proved following the scheme of the proof of Th. 5.1 in (1). We recall that by PROP. 1.1 (or its dual),

\((a^o)_r \Rightarrow (h^0)_r, (c^o)_1 \) Let \( S \) be a submodule of \( R^n \) generated by \( a_1,\ldots,a_n \).

The proof will proceed by induction on \( n \). For \( n = 1 \), \( S \) is a principal left ideal of \( R \) and by \((c^o)_1 \) we have that \( S = \tilde{S} \). Let \( 2 \leq n \) and assume that every cyclic submodule of \( R^n(\text{gen} \backslash 2) \) is closed.

Let

\[ T_1 = \{[x_1^0,0,\ldots,0] \in R^n / a_1 \cdot x_1^0 = 0 \} \]

\[ T_2 = \{[0,x_2^0,\ldots,0] \in R^n / a_2 \cdot x_2^0 = 0 \} \]

Clearly \( T_1, T_2 \subseteq S^s \).

Then for every \( u = [u_1,\ldots,u_n] \in \tilde{S} \) we have \( u \in T_1 \), so \( u_1 \cdot x_1^0 = 0 \) and by the closeness of \( \langle a_1 \rangle \) we get \( u_1 = t \cdot a_1 \), \( t \in R \).
Now \( U - \{v_1, \ldots, v_n\} = [0, v_2, \ldots, v_n] = V \subseteq \mathcal{T}_2 \). By the closure of the principal left submodule generated by \([a_2, \ldots, a_n]\), we have

\[ [0, v_2, \ldots, v_n] = r[0, a_2, \ldots, a_n] \]

Let \( I_1 = \langle a_1 \rangle, I_2 = \langle a_2, \ldots, a_n \rangle \). Then \( w \in I_1 \cap I_2 \) if and only if there exist \( x_1, \ldots, x_n \in R \) such that

\[ w = a_1 x_1 = -(a_2 x_2 + \ldots + a_n x_n) \]

But

\[ V = [0, r a_2, \ldots, r a_n] \subseteq \mathcal{T} \]

and \( w \in I_1 \cap I_2 \) as above give

\[ 0 = r a_2 x_2 + \ldots r a_n x_n = -r a_1 x_1 \]

that is

\[ r \in (I_1 \cap I_2) \]

and since we have condition \((b^*)_1\), \( r \) can be written as

\[ r = m_1 + m_2, \quad m_i \in I_i \]

Hence

\[ V = [0, r a_2, \ldots, r a_n] = [0, m_1 a_2, \ldots, m_1 a_n] = m_1 [a_1, \ldots, a_n] \]

and

\[ U = V + t [a_1, \ldots, a_n] = (m_1 + t) [a_1, \ldots, a_n] \subseteq \mathcal{T} \]

Theorem is now proved.

AN EXAMPLE.

Let \( R \) be a right Ore domain (that is, a ring without zero divisors \( \neq 0 \) and with the right common multiple property). Then if \( h(R) \) is the injective hull of \( R \), \( h(R) \) carries a ring structure which makes it isomorphic to the left field of quotients of \( R \). Clearly \( R \) is right pure in \( h(R) \) if and only if \( h(R) = R \) is a division ring. More generally, for any \( n \in \mathbb{N} \), \( M_n(R) \) is right pure in \( M_n(h(R)) \) and only if \( R = h(R) \), \( M_n( \cdot ) \) denotes the full ring of matrices). In fact, if \( M_n(R) \) is right pure in \( M_n(h(R)) \), then by THEOREM 1, \( M_n(R) \) satisfies condition \((a^*)_R \). But this readily implies that condition \((a^*)_R \) holds in \( R \). We are done, since a ring without zero divisors \( \neq 0 \) and satisfying \((a^*)_R \) is necessarily a division ring.

THEOREM 2. Let \( R \) be a left semihereditary ring. Then

\[ (a^*)_1, (c^*)_1 \rightarrow (h^*)_1 \]
Proof: Let $S$ be submodule of $R^n$ generated by the vectors $A_i = [a_{i1}, \ldots, a_{in}]$ $i=1, \ldots, s$

Let $S'$ be the submodule of $S$ consisting of all vectors with 0 in the first component. Then

**Lemma 1. $S'$ is finitely generated**

Proof: Let

$A = [a_{11}, a_{21}, \ldots, a_{sn}] \in R^n$

and assume, for the time being, that the left annihilator of $A$ in $R^n$ be generated by

$g^i = [b_{i1}^1, \ldots, b_{in}^i] $ $i=1, \ldots, m$

Then if $x \in S'$ we have $r_1, \ldots, r_s \in R$ satisfying

$x = \sum_{i=1}^m r_i \cdot A_i = [0, r_{11} a_{11}, \ldots, r_{s1} a_{s1}]$

therefore

$[r_1, \ldots, r_s] = \sum_{j=1}^m t_j \cdot b_j^j$ $t_j \in R$

that is

$r_k = \sum_{j=1}^m t_j \cdot b_{kj}$ $k=1, \ldots, s$

But then

$x = \sum_{k=1}^s r_k \cdot A_k = \sum_{k=1}^s (\sum_{j=1}^m t_{kj} \cdot b_{kj}) \cdot A_k$

$= \sum_{j=1}^m t_j \cdot (\sum_{k=1}^s b_{kj}^j \cdot A_k)$

We now claim that

$A_j' = \sum_{k=1}^s b_{kj}^j \cdot A_k$ $j=1, \ldots, m$

generate $S'$. In fact, notice that $x$ was an arbitrary element of $S'$ and that the first component of $A_j'$ is $\sum_{k=1}^s b_{kj}^j a_{k1} = 0$

Our claim follows.

Now, in order to complete the proof of Lemma 1 we need to prove that we can assume that the left annihilator of $A$ in $R^n$ is finitely generated. For this we shall use the hypothesis that $R$ is a left semihereditary ring. Let $F$ be a free left $R$-module generated by $f_1, \ldots, f_n$ and $0 \to K \to F \xrightarrow{\phi} L \to 0$ be an exact sequence where $L$ is the left ideal of $R$ generated by $a_{11}, \ldots, a_{s1}$ and $\phi$ be the homomorphism defined by $f_j \to a_{j1}$. Notice that $K$ is isomorphic to the left annihilator of $A$ in $R^n$. Since $L$ is projective, that sequence splits and $K$ is then a direct summand of a finitely generated $R$-module, therefore is finitely generated. This ends the proof of Lemma 1.
We proceed the proof of THEOREM 2 by induction in the length of the vectors in \( S \). If \( n = 1 \), then \( S \) is a finitely generated left ideal of \( R \), and so by condition \((c^0)_1\) is closed. Let \( 2 \leq n \) and assume that every finitely generated submodule of \( R(\mathbf{n-1}) \) is closed. In particular, the submodule \( S \subset R(\mathbf{n-1}) \) associated to \( S^* \), dropping the first coordinate of the elements in \( S^* \), is closed. Next we need to prove another partial result

**LEMMA 2.** If \( [x_1^\prime, \ldots, x_n^\prime] \in S^* \), then there exists \( x_i \in R \) such that \( [x_1^\prime, x_2^\prime, \ldots, x_n^\prime] \in S^r \)

**Proof:** Let \( r_1, \ldots, r_s \in R \) satisfy \( \sum_{i=1}^s r_i \cdot a_{i1} = 0 \). Then

\[
\sum_{i=1}^s r_i \cdot A_i = [0, \sum_{i=1}^s r_i a_{i2}, \ldots, \sum_{i=1}^s r_i a_{in}] \in S^r
\]

and by the hypothesis we have

\[
0 = \sum_{k=2}^n (\sum_{i=1}^s r_i a_{ik}) \cdot x_k
- \sum_{i=1}^s r_i \cdot (\sum_{k=2}^n a_{ik} x_k)
\]

which says that

\[
\phi : a_{i1} \rightarrow \sum_{k=2}^n a_{ik} x_k'
\]

defines an \( R \)-homomorphism of the left ideal generated by \( a_{i1} \) \( i = 1, \ldots, s \) into \( R \). By property \((a^s)_1\) there is \( x_i \in R \) realizing \( \phi \), that is

\[
a_{i1} x_1 + a_{i2} x_2' + \ldots + a_{in} x_n' = 0 \quad i = 1, \ldots, s
\]

and this ends the proof of LEMMA 2.

To complete the proof of THEOREM 2 we follow the scheme of proof of THEOREM 5.2 of \( (1) \). Let \( U = [u_1, \ldots, u_n] \in S^t \) \( S^r \) contains all those vectors

\[
[x_1, 0, \ldots, 0] \text{ such that } a_{i1} x_1 = 0 \text{, } i = 1, \ldots, s
\]

Therefore

\[
x_1 \in \langle a_{11}, a_{21}, \ldots, a_{1s} \rangle^r
\]

\[
u_1 \in \langle a_{11}, a_{21}, \ldots, a_{1s} \rangle^{r_1} = \langle a_{11}, a_{21}, \ldots, a_{1s} \rangle
\]

(by condition \((c^s)_1\)),

\[
S, \quad u_1 = r_{i1} a_{i1} + \ldots + r_{is} a_{is}, \quad r_i \in R
\]

Let

\[
u' = \sum_{i=1}^s r_i A_i
\]

\( u' \) belongs to \( S \) and moreover \( v = u - u' = [0, v_2, \ldots, v_n] \in S^r \)
satisfies

\[ v_2x_2 + \cdots + v_nx_n = 0 \]

for any \([x_1, x_2, \ldots, x_n] \in S^R\).

Let \(\mathcal{D}\) denote the submodule of \(\mathbb{R}^{\text{dim}(S) - 1}\) of all elements \(x_2, \ldots, x_n\)
for which there is \(x_1 \in R\) satisfying \([x_1, x_2, \ldots, x_n] \in S^R\).

Clearly we have \(\mathcal{D} \subseteq S^R\). But by LEMMA 2, \(S^R = \mathcal{D}\). So \(D = S^R\).

Furthermore \([v_2, \ldots, v_n] \in \mathcal{D} = S^R = S\) according to the inductive hypothesis. Of course we need to know that \(S\) is finitely generated, but this follows from LEMMA 1 and the definition of \(S\).

We have then that \([0, v_2, \ldots, v_n] \in S^R = S\) and finally

\[ U = V + U' \subseteq S \]

This means that \(S = S\) and THEOREM 2 is proved.

**COROLLARY.** Let \(R\) be a left and right semipointed ring. Assume that \((c^a)_1\) and \((c^a)_r\) holds. Then \((a^o)_1 \iff (a^o)_r\).

**Proof:** Assume that \((a^0)_1\) holds. Then

\[ (a^o)_1 \iff (h^o)_1 \]
\[ \iff (h^o)_1 \]
\[ \iff (a^o)_r \]

by Theorem 2

by Theorem 1

The other implication follows in the same way.

3. VON NEUMANN RINGS.

In this section we give characterizations of von Neumann rings in terms of purity. We recall that a von Neumann ring is a ring \(R\) satisfying: for every \(a \in R\) there is \(x \in R\) such that \(a.x.a = a\).

We shall say that a ring is absolutely flat (resp. pure) if any right \(R\)-module is flat (resp. pure).

**THEOREM 3.** Let \(R\) be a ring. The following conditions are all equivalent:

a) \(R\) is absolutely pure
b) \(R\) is a von Neumann ring
c) \(R\) is absolutely flat
d) every cyclic right \(R\)-module is pure

**Proof:** a) \(\iff\) b) Let \(z \in R\). Then the right ideal \(z.R\) is pure in \(R\).

Since \(R\) has identity we can write \(z = 1.z\). By the purity there is \(x \in R\) such that \(z = (z.x).z\) as we wanted to prove.
b) $\implies$ c) is a well known result

c) $\implies$ d) and c) $\implies$ a) are clear

d) $\implies$ b)

Let $I$ be a right ideal of $R$, $a \in R$ and $\phi: \langle a \rangle \to R/I$ be a homomorphism of the right ideal $\langle a \rangle$ generated by $a$ into the cyclic module $R/I$. Let $S$ be an injective right module containing $R/I$.

There exists $s \in S$ satisfying

$$\phi(a) = s \cdot a$$

and since $R/I$ is pure in $S$, we can find $c \in R/I$ such that

$$\phi(a) = c \cdot a$$

This means that $\phi$ can be extended to a homomorphism of $R$ into $R/I$. Being $I$ and $a \in R$ arbitrary we can apply Th. 3 of (2) to conclude that $R$ is a von Neumann ring.

Proof of Theorem 3 is now complete.

**Remark 1.** Using the absolute purity of von Neumann rings, as shown in THEOREM 3, we can give an immediate answer to a question posed in (4), §25.1. Namely: Let $A$ be a right $R$-module, where $R$ is a von Neumann ring. Suppose that $A$ is generated by $n$ elements. Then every finitely generated submodule of $A$ is generated by $n$ elements. In fact, let $A'$ be a finitely generated submodule of $A$, $a_1, \ldots, a_n$ a set of generators of $A$ and $a'_1, \ldots, a'_m$ a set of generators of $A'$.

We have $r_{ji} \in R$ satisfying

$$a'_i = \sum_{j=1}^{n} a_j \cdot r_{ji}, \quad i=1, \ldots, m$$

Being $A'$ pure in $A$ there exist $x'_j \in A'$, $j=1, \ldots, n$ satisfying

$$a'_i = \sum_{j=1}^{n} x'_j \cdot r_{ji}$$

Clearly, $x'_j$ is a set of generators of $A'$.

Next we characterize those right semihereditary rings which are von Neumann rings.

**Lemma.** (Compare (3), Chap. I, §2, Exer. 18 a). Let $R$ be a right semihereditary ring and let $B$ be an injective right $R$-module containing $R$ such that $R$ is pure in $B$. Then any finitely generated submodule of a projective right $R$-module is a direct summand of it.

**Proof:** Let $P$ be a projective right $R$-module and let $M$ be a finitely generated submodule of it. Without loss of generality we can assume that $P$ is finitely generated and free. In fact, if $F$ is a free module of which $P$ is a submodule then we can write $F = F_1 \oplus F_2$, with $F_1$ free, finitely generated and containing $M$. If $M$ is a direct
summand of $F_1$, it is also a direct summand of $F$ and therefore of $P$.
Being $R$ right semihereditary, $M$ is a projective module. Let $a_1, \ldots, a_n$
be elements of $M$ and $\phi'_1, \ldots, \phi'_n$ mappings of $M$ into $R$, satisfying
$$a = \sum_{i=1}^{n} a_i \cdot \phi'_i(a)$$
for every $a \in M$.

Since $R$ is pure in $B$, by PROP. 1.3, the mappings $\phi'_i$ can be ex-
tended to mappings $\phi_1: P \rightarrow R$. Let $\phi: P \rightarrow M$ be the mapping defin-
ed by $$\phi: x \mapsto \sum_{i=1}^{n} a_i \cdot \phi_i(x)$$
Clearly $\phi$ defines a projection of $P$ onto $M$. $M$ is then a direct
summand of $P$.

REMARK 2. The previous Lemma permits to give an immediate answer
to a question posed in (4), §25.(1). Namely, let $R$ be a von Neu-
mann ring. Then if every torsion free $R$-module is projective, $R$
is a left self-injective ring. In fact, let $h(R)$ be an injective
hull of $R$. Then $h(R)$ is torsion free, therefore it is projective.
Let $I = \langle e \rangle$ be a principal non-zero left ideal of $R$, $e$ an idempo-
tent. By the previous Lemma $I$ is a direct summand of $h(R)$ and so
$I$ is injective. Since $I$ was arbitrary, we have also that $J = \langle 1 - e \rangle$
is injective. Therefore $R = I \oplus J$ is injective as we wanted to
prove.

THEOREM 4. Let $R$ be a ring. Then $R$ is a von Neumann ring if
and only if $R$ is right semihereditary and pure (in some injective
right $R$-module containing it).

Proof: Apply the previous Lemma to $P = R$ to get that every fini-
tely generated right ideal of $R$ is a direct summand of $R$. This is
enough to assure that $R$ be a von Neumann ring.

Base in the same Lemma we have

PROPOSITION 3.1. Let $R$ be a ring. Then $R$ is a von Neumann ring if
and only if $R$ is right semihereditary and satisfies condition $(a^n)_R$

Proof: To prove part "if" we proceed as in the proof of the Lemma
applied to the situation $P = R$ and using condition $(a^n)_R$ to extend
the mappings $\phi'_i$.

COROLLARY. Let $R$ be a left noetherian, left semihereditary ring sat
isifying condition \((a^n)_r\). \(R\) is then a semisimple \(d.c.c.\) ring.

Proof: According to a result by L.W. Small ([5], COROLLARY 3) the two first hypothesis imply that \(R\) is right semihereditary. Condition \((a^n)_r\) and the previous proposition prove our claim, since a left noetherian von Neumann ring is necessarily semisimple \(d.c.c.\).

REFERENCES