PARALLEL FAMILIES OF \( \mathbf{1} \) 
HYPERSURFACES IN RIEMANNIAN SPACE \( V_n \). (1)

by John De Cicco and Robert V. Anderson

1. ELEMENTARY CONCEPTS CONCERNING A POSITIONAL FIELD OF FORCE \( \phi \). (2)

Consider a field of force \( \phi \), whose force vector acting at any point \( x = (x^i) \), of a Riemannian space \( V_n \), is \( \phi = \phi^i = \phi_i \). The force vector \( \phi \), is considered to be a vector function of position only, and
to be of class two over a certain region of \( V_n \). Thus the force vector \( \phi \) is a continuous function and possesses continuous partial derivates with respect to the \( x^i \) of first and second orders through out the given region of \( V_n \). Such a field of force \( \phi \), is termed a positional field of force \( \phi \) in the Riemannian space \( V_n \).

The positional field of force \( \phi \) may be given by either the covariant components \( \phi_i \), or the contravariant component \( \phi^i = g^{ij} \phi_j \).

Here the \( g^{ij} = g^{ji}(x) \), are the contravariant components of the fundamental metric tensor of the given Riemannian space \( V_n \). It is assumed that \( |\phi| = |c g^{ij} \phi_j|^{1/2} > 0 \), \( c = \pm 1 \). The trivial case when \( \phi \) is identically zero, which leads to the \( n^{2n-2} \) geodesics of \( V_n \), is omitted from consideration.

If a particle of constant mass \( m > 0 \), is constrained to move along a path \( C \) of the given region of \( V_n \), then its speed \( v \) is governed by the law

\[
T = \frac{mv^2}{2} = \int_{x_0}^{x} \phi^i dx^i = \frac{mv^2}{2} \tag{1.1}
\]

where the initial speed at the point \( x_0 \), is \( v_0 > 0 \). The integral appearing in (1.1), is termed the work \( W \) performed in moving from \( x_0 \) to \( x \) along the curve \( C \). The quantity \( T = 1/2.mv^2 \), is called the kinetic energy of the particle.

If the work \( W \) is independent of the path \( C \), then the positional field of force \( \phi \) is said to be conservative. In this case there exists a potential function \( V = V(x) \), such that

\[
\phi_i = -\frac{\partial V}{\partial x^i} = -\text{grad}(V) \tag{1.2}
\]

For a conservative field of force \( \phi \), the relation (1.1), becomes the energy equation, namely
(1.3) \[ T + V = \frac{mv^2}{2} + V(x) = E \]

where \( E \) is the total constant energy.

For any positional field of force \( \phi \), conservative or not, the simple family of \( n-1 \) Faraday lines of force \( C \), is composed of the \( n-1 \) integral solutions \( C \) of the system of \( n \) first order ordinary differential equations

\[
\frac{dx^i}{ds} = \frac{\delta^i_j + \phi_i}{|\phi|} \frac{\phi^i}{|\phi|},
\]

where \( s \) denotes the arc length along any Faraday line of force \( C \).

THEOREM 1.1. If \( \phi \) is a conservative field of force in \( V_n \), let \( L \) be a curve such that at each point \( P \) of \( L \), there exists a Faraday line of force \( C \), tangent to \( L \). Assume that these \( n-1 \) Faraday lines of force are all transversal to some curve \( L^* \). Let \( AD \) denote the line of force at the initial point \( A \) on \( L \) and terminal point \( D \) on \( L^* \). Similarly \( BE \) denotes the line of force at \( B \) on \( L \), terminating at \( E \) on \( L^* \). Then

\[ \int_{LAB} |\phi| ds = \int_{CAD} |\phi| ds - \int_{CBE} |\phi| ds. \]

Another way of writing this is in terms of the characteristic function \( r \) of \( V_n \). Thus

\[ \int_{LAB} r(x;\frac{\partial V}{\partial x}) ds = \int_{CAD} r(x;\frac{\partial V}{\partial x}) ds - \int_{CBE} r(x;\frac{\partial V}{\partial x}) ds. \]

It is noted that if \( p_i \) is an absolute covariant vector, then

\[ r(x;p) = |\epsilon g^{ij} p_i p_j|^{1/2}, \quad \epsilon = +1. \]

If the characteristic function

\[ r(x;\frac{\partial V}{\partial x}) = +1, \]

then (1.6), becomes an analogue of the Jacobi string condition for \( V_n \). Also \( r(x;\frac{\partial V}{\partial x}) = 1 \), is then the Hamilton Jacobi partial differential equation of first order.

2. The Lamé Differential Parameters of First and Second Orders

If \( u(x) \) and \( v(x) \) are two absolute scalar functions of at least class \( C^1 \), then their gradients \( \frac{\partial u}{\partial x^i} \) and \( \frac{\partial v}{\partial x^i} \), are two absolute covariant vectors orthogonal to the two \( (n-1) \) dimensional hypersurfaces \( u(x) = \) constant and \( v(x) = \) constant. The inner product between these two gradients is called the Lamé differential parameter of order one. It is written as
(2.1) \[ \Delta_1(u,v) = (\text{grad } u, \text{grad } v) = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}. \]

In particular if \( u(x) = v(x) \), this becomes

(2.2) \[ \Delta_1(u) = \Delta_1(u,u) = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}. \]

Now since in a conservative field of force there is a potential function \( V = V(x) \), such that \( \phi_i = \frac{\partial V}{\partial x^i} \), it is seen that

(2.3) \[ |\phi| = |\text{grad } V| = \left| \epsilon \Delta_1(V) \right|^{1/2} > 0. \]

If \( \phi_i \) is the covariant form of the force vector \( \phi \), then the divergence of \( \phi \) is defined by the absolute scalar

(2.4) \[ \theta = \text{div}(\phi) = g^{ik} \phi_{i,k} = g^{ik} \left( \frac{\partial \phi_i}{\partial x^k} - \Gamma^j_{ik} \phi_j \right). \]

A field of force \( \phi \) is said to be solenoidal if and only if the divergence of the force vector \( \phi \) is zero throughout the region of \( V \) under consideration.

If \( g = |g_{ij}| \), the determinant of the \( g_{ij} \), then another form of (2.4), is

(2.5) \[ \theta = \text{div}(\phi) = \frac{1}{\sqrt{|g|}} \left[ \frac{\partial}{\partial x^i} \sqrt{|g|} \phi_i \right]. \]

It is noted that the form of the divergence given in (2.5), is that most convenient for applications.

The Lamé differential parameter of the second order is the Laplacean. It is defined by

(2.6) \[ \Delta_2(u) = \phi^2 u = g^{ik} \left( \frac{\partial u}{\partial x^i} \right)_k = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} \frac{\partial u}{\partial x^j} \right), \]

where \( u = u(x) \) is an absolute scalar function of position.

A positional field of force is said to be Laplacean if and only if it is both conservative and solenoidal. Thus a conservative field of force \( \phi \), with a potential function \( V(x) \) is Laplacean if and only if \( V(x) \) is a harmonic function. That is \( \nabla^2 V = 0 \).

3. CONSERVATIVE FIELDS OF FORCE \( \phi \) FOR WHICH THE EQUIPOTENTIAL HYPERSONSURFACES FORM A PARALLEL FAMILY. Let \( \phi \) be a conservative field
of force with potential function \( V = V(x) \). The family \( V(x) = \text{constant} \) represents a set of \( =^1 \) hypersurfaces, called equipotential hypersurfaces, each of which is of deficiency one. It is assumed that \( \Delta_1(V) \neq 0 \). Then a unit vector orthogonal to these hypersurfaces is

\[
(3.1) \quad \xi_i = \frac{1}{\sqrt{|\Delta_1(V)|}} \frac{\partial V}{\partial x^i}.
\]

The contravariant form of this vector is

\[
(3.2) \quad \xi^i = \frac{1}{\sqrt{|\Delta_1(V)|}} g^{ij} \frac{\partial V}{\partial x^j}.
\]

Therefore the orthogonal trajectories \( C \) of this set of \( =^1 \) equipotential hypersurfaces \( V(x) = \text{constant} \) are the \( =^{n-1} \) Faraday lines of force given as the integral solutions \( C \) of the system of \( n \) first order ordinary differential equations \( \frac{dx^i}{ds} = \xi^i \).

The \( =^{n-1} \) curves \( C \) and the \( =^1 \) hypersurfaces \( V = \text{constant} \) are said to form a normal family. Also a set of \( =^1 \) hypersurfaces \( V(x) = \text{constant} \) is said to be parallel if and only if it cuts orthogonally a set of \( =^{n-1} \) geodesics \( C \) of \( V_n \).

THEOREM 3.1. The set of \( =^1 \) equipotential hypersurfaces \( V(x) = \text{constant} \) of a conservative field of force \( \Phi \), is a parallel family if and only if the Lamé parameter \( \Delta_1(V) \) is a function of the potential function \( V = V(x) \) alone. Thus

\[
(3.3) \quad \Delta_1(V) = g^{ij} \frac{\partial V}{\partial x^i} \frac{\partial V}{\partial x^j} = eF(V),
\]

where \( e = \pm 1 \), according as \( \Delta_1(V) \geq 0 \), or \( \Delta_1(V) < 0 \).

For, in this case the \( =^{n-1} \) orthogonal trajectories are the \( =^{n-1} \) geodesics \( C \) of \( V_n \). Hence parallel displacements of the unit tangent vectors along these curves \( C \) are also tangent to the respective curves \( C \). Therefore

\[
(3.4) \quad \left[ \frac{dx^i}{ds} \right] \frac{dx^k}{ds} _' \frac{\partial}{\partial x^i} \left[ \Delta_1^{-1/2} g^{ij} \frac{\partial V}{\partial x^j} \right] \frac{\partial V}{\partial x^a} = 0.
\]

where \( \Delta_1 = \Delta_1(V) \), and the symbol \( ( \cdot ) _' \) means the covariant derivative of the quantity inside the parenthesis with respect to the \( x^k \).

Thus
Since \( |g^{ij}| \neq 0 \), it is seen that

\[
(3.6) \quad \frac{1}{2} \frac{\partial^2}{\partial x^k \partial x^j} g^{ka} \frac{\partial V}{\partial x^a} + \Delta_1 \frac{\partial}{\partial x^j} \frac{\partial V}{\partial x^a}, k, \frac{\partial V}{\partial x^a} = 0
\]

However it is easily seen that

\[
(3.7) \quad g^{ka} \frac{\partial V}{\partial x^a} \left( \frac{\partial V}{\partial x^a} \right), k, \frac{\partial V}{\partial x^a} = \frac{1}{2} \frac{\partial^2}{\partial x^j} \frac{\partial V}{\partial x^a}
\]

Substitute this result into (3.6). Then interchange the dummy indices \( k \) and \( a \) and subtract. It is evident that

\[
(3.8) \quad \frac{\partial V}{\partial x^j} g^{ka} \left[ \frac{\partial^2}{\partial x^k \partial x^a} + \frac{\partial^2}{\partial x^a \partial x^k} \right] = 0
\]

Since at least one \( \frac{\partial V}{\partial x^j} \neq 0 \), and since \( |g^{ka}| \neq 0 \), these conditions become

\[
(3.9) \quad \frac{\partial^2}{\partial x^j} g^{ka} \left[ \frac{\partial^2}{\partial x^k \partial x^a} + \frac{\partial^2}{\partial x^a \partial x^k} \right] = 0
\]

Hence \( \Delta_1(V) \) and \( V \) are functionally dependent. As these steps are reversible the proof of Theorem 3.1, is complete.

Now recall that since \( V(x) \) is an absolute scalar function it is known that \( r(x; \frac{\partial V}{\partial x}) = |\epsilon_1(V)|^{1/2} \geq 0 \).

**THEOREM 3.2.** If \( V(x) = \text{constant} \), with \( \Delta_1(V) \neq 0 \), form a set of \(-1\) equipotential hypersurfaces of a conservative field of force \( \psi \), each of deficiency one, which form a parallel family, then exists a non-constant function \( x = x(V) \) which obeys the Hamilton-Jacobi partial differential equation of first order, namely

\[
(3.10) \quad r(x; \frac{\partial x}{\partial x}) = |\epsilon_1(x)|^{1/2} = |\epsilon g^{ij} \frac{\partial x}{\partial x^i} \frac{\partial x}{\partial x^j}|^{1/2} = +1
\]

Conversely if \( x(x) = \text{constant} \) obeys the Hamilton-Jacobi equation then it represents a parallel family.

For in Theorem 3.1 set \( x(V) = \int \frac{dV}{\sqrt{F(V)}} \), where \( F(V) \) is a non-constant function of \( V = V(x) \), and note that \( V(x) = \text{constant} \) is parallel if and only if
Thus, Theorem 3.2 is proved.

4. SOME CONDITIONS FOR PARALLEL FAMILIES OF \(=1\) HYPERSURFACES IN RIEMANNIAN SPACE \(V_n\). If \(\lambda_i\) is an absolute covariant vector in \(V_n\), its covariant first order partial derivative is given by

\[
\lambda_{i,j} = \frac{\partial \lambda_i}{\partial x^j} - \Gamma^a_{ij} \lambda_a
\]

In particular if \(V = V(x)\), is an absolute scalar function

\[
V_{,i} = \frac{\partial V}{\partial x^i}, \quad V_{,ij} = \frac{\partial^2 V}{\partial x^i \partial x^j} - \Gamma^a_{ij} \frac{\partial V}{\partial x^a}
\]

Thus the ordinary partial derivative of the first order Lamé differential parameter \(\Delta_1 = \Delta_1(V)\), with respect to \(x^a\) is

\[
\frac{\partial \Delta_1}{\partial x^a} = 2 g_{ij} \frac{\partial V}{\partial x^i} V_{,j} - g_{ij} \frac{\partial^2 V}{\partial x^i \partial x^j} - \Gamma^a_{ij} \frac{\partial V}{\partial x^a} \frac{\partial V}{\partial x^a}
\]

Let \(T_{a\beta}\), denote the absolute covariant tensor of second order defined by

\[
T_{a\beta} = \frac{1}{2} \left[ \frac{\partial V}{\partial x^a} \frac{\partial \Delta_1}{\partial x^\beta} - \frac{\partial V}{\partial x^a} \frac{\partial \Delta_1}{\partial x^a} - \Gamma^\gamma_{a\beta} \frac{\partial V}{\partial x^\gamma} \frac{\partial V}{\partial x^a} \right]
\]

Now supposing that \(\Delta_1(V) > 0\), and since \(g = |g_{ij}| \neq 0\), the following result is evident.

**THEOREM 4.1.** Let the potential function \(V = V(x)\), of a conservative field of force \(\phi\), be of at least class two in a given open region of Riemannian space \(V_n\), with \(\Delta_1(V)\) positive. Then the family of \(=1\) hypersurfaces, each of deficiency one, defined as the equipotential hypersurfaces \(V(x) = \text{constant}\) of the field of force \(\phi\), is a parallel family if and only if

\[
T_{a\beta} = g^{ij} V_{,i} \left[ V_{,\beta} V_{,j} - V_{,a} V_{,j\beta} \right] = 0
\]

Moreover, each component of this tensor, when \(a \neq \beta\), \(a, \beta\) fixed, namely

\[
V_{,\beta} V_{,j} - V_{,a} V_{,j\beta}
\]

is orthogonal to \(V(x) = \text{constant}\).
Now the condition (4.5) can be written in the form 

\[ V_{,\beta} g^{ij} V_{,i} V_{,j\alpha} = V_{,\alpha} g^{ij} V_{,i} V_{,j\beta}. \]  

Let \( \rho \) be defined so that 

\[ g^{ij} V_{,i} V_{,j\alpha} = \rho V_{,\alpha} , \quad g^{ij} V_{,i} V_{,j\beta} = \rho V_{,\beta}. \]  

The second set of equations in (4.8) follows from the first since at least one \( V_{,\alpha} \) is not zero.

**Theorem 4.2.** A family of equipotential hypersurfaces \( V(x) = \text{constant} \) of a conservative field of force \( \phi \), with potential function \( V = V(x) \), and \( \Delta_1(V) > 0 \), is a parallel family if and only if 

\[ \frac{1}{2} \frac{\partial \Delta_1}{\partial x^\alpha} = g^{ij} V_{,i} V_{,j\alpha} = \rho V_{,\alpha} , \]  

where \( \alpha = 1, 2, \ldots, n \), and \( \rho \) is an absolute scalar function. Thus if \( \Delta_1 > 0 \), is not a constant the two families \( \Delta = \text{constant} \) and \( V(x) = \text{constant} \) are identical.

It is noted that if \( \Delta_1 = C > 0 \), is identically constant, then by a suitable homothetic map \( T \), the conservative field of force \( \phi \), with potential function \( V = V(x) \), is such that the force vector 

\[ \phi = \sqrt{g_{ij}} \frac{\partial V}{\partial x^i} , \]  

has unit magnitude.

5. **Cartograms and Parallel Families.** Let \( V_n \) and \( \tilde{V}_n \) be two Riemannian spaces whose metrics are given by the two definite quadratic differential forms: 

\[ ds^2 = g_{ij} dx^i dx^j, \quad \text{and} \quad ds^2 = \tilde{g}_{ij} dx^i dx^j, \]  

where each of the \( 2n^2 \) functions \( g_{ij}(x) \), \( \tilde{g}_{ij}(x) \), is of at least class three in an \( n \) dimensional region of points of \( V_n \), or \( \tilde{V}_n \). Two points, one of \( V_n \) and the other of \( \tilde{V}_n \), are said to correspond if and only if they are determined by the same curvilinear coordinates \( x = (x^1, x^2, \ldots, x^n) \). This establishes a point to point transformation between \( V_n \) and \( \tilde{V}_n \). Such a transformation between \( V_n \) and \( \tilde{V}_n \), is called a cartogram \( T \).

If \( \rho \) is a positive scalar point function of at least class three in the given region of \( V_n \) such that \( ds = \rho ds \), then the cartogram \( T \) is said to be conformal and the two Riemannian spaces \( V_n \) and \( \tilde{V}_n \) are said to be conformally equivalent. Hence \( \tilde{g}_{ij} = \rho g_{ij} \), and \( \tilde{g}_{ij}^n = = \frac{1}{\rho^2} g_{ij} \). In this case \( \tilde{V}_n \) is called a conformal image of the space \( V_n \).
In particular if the positive scalar point function \( \rho \) is a constant then \( V_n \) is called a homothetic image of \( V_n \) and \( T \) is called a homothetic transformation \( T \). If \( \rho = +1 \), \( V_n \) is called an isometric image of \( V_n \) and \( T \) is an isometric correspondence. In this case \( V_n \) is said to be applicable to \( V_n \).

Now let \( V(x) = \text{constant} \) represent a parallel family of equipotential hypersurfaces of a conservative field of force \( \phi \) in \( V_n \). The condition for this is

\[
(5.1) \quad g^{ij} V_{,i} V_{,j} = \rho V^, \alpha ,
\]

for \( \alpha = 1, 2, \ldots, n \), and \( \rho \) a scalar function.

Suppose that this parallel family in \( V_n \) corresponds to a parallel family of \( \phi \) hypersurfaces in \( \tilde{V}_n \). Thus the condition (5.1) must hold in both \( V_n \) and \( \tilde{V}_n \). Hence for some scalar functions \( \rho(x) \) and \( \tilde{\rho}(x) \)

\[
(5.2) \quad g^{ij} \frac{\partial^2 V}{\partial x^i \partial x^j} = \rho \frac{\partial V}{\partial x^\alpha} , \quad g^{ij} \frac{\partial^2 V}{\partial x^i \partial x^j} = \tilde{\rho} \frac{\partial \tilde{V}}{\partial x^\alpha}.
\]

Hence

\[
(5.3) \quad (\rho g^{ij} - \tilde{\rho} g^{ij}) \frac{\partial V}{\partial x^i} \frac{\partial^2 V}{\partial x^j} - \rho \frac{\partial^2 V}{\partial x^\alpha} \frac{\partial \tilde{V}}{\partial x^\alpha} = 0.
\]

Now \( \rho \) and \( \tilde{\rho} \) are independent of the partial derivatives of \( V(x) \).

Clearly (5.3) is an identity for all such partial derivatives. Hence necessarily

\[
(5.4) \quad \rho g^{ij} - \rho \tilde{g}^{ij} = 0 , \quad \rho \tilde{g}^{ij}\tilde{\gamma}^\lambda_{ja} - \rho \tilde{g}^{ij}\tilde{\gamma}_j^\alpha = 0.
\]

Set \( u^2 = \frac{\rho}{\tilde{\rho}} > 0 \). Then from the first of equations (5.4) it is seen that \( \tilde{g}^{ij} = \frac{1}{u^2} g^{ij} \). Thus we have established that \( V_n \) and \( \tilde{V}_n \) are necessarily related by a conformal cartogram \( T \).

Now if \( T \) is a conformal cartogram the Christoffel symbols of the second kind correspond by the equations

\[
(5.5) \quad \tilde{\gamma}_j^\lambda = \gamma^\lambda_{ja} + \frac{1}{u} \left[ \delta^\lambda_{ja} \frac{\partial u}{\partial x^a} + \delta^\lambda_{ja} \frac{\partial u}{\partial x^j} - \delta_{ja} \tilde{\gamma}_{\alpha}^{\lambda} \frac{\partial u}{\partial x^\alpha} \right].
\]

However the second of equations (5.4) yields
Upon comparing (5.5) and (5.6), it is found that

\[ \frac{1}{u^2} g^{ij} \frac{\partial}{\partial x^i} u \frac{\partial}{\partial x^j} = 0. \]

Since \( u > 0 \) and \( g = |g_{ij}| \neq 0 \), these reduce to

\[ \frac{\lambda}{u} \frac{\partial u}{\partial x^i} + \frac{\partial}{\partial x^i} g_{ij} \frac{\partial u}{\partial x^j} = 0. \]

Multiply through by \( g^{ij} \). Upon simplification (5.8) reduces to

\[ \frac{\partial u}{\partial x^j} = 0. \]

Again since \( 1/g = |g^{ij}| \neq 0 \), it is seen that \( \frac{\partial u}{\partial x^j} = 0 \), for \( j=1,2,..,n \).

**Theorem 5.1.** Let \( V_n \) and \( \tilde{V}_n \) be two Riemannian spaces for which the fundamental quadratic differential forms are

\[ ds^2 = g_{ij} dx^i dx^j \]

and \( d\tilde{s}^2 \) respectively. Suppose that \( V(x) = \text{constant} \) is parallel family of \( m \) hypersurfaces in \( V_n \), where \( V(x) \) is the potential function of a conservative field of force \( \phi \) in \( V_n \). Then every such family will correspond to a parallel family of \( m \) hypersurfaces in \( \tilde{V}_n \), by a cartogram \( T \), if and only if \( T \) is a homothetic map.

For, in this case the scale function \( u(x) \) obeys \( \frac{\partial u}{\partial x^i} = 0 \), for \( i=1,2,..,n \). Since the scale function is always positive, it is seen that \( u = u(x) = c > 0 \), where \( c \) is a constant. Thus the proof of Theorem 5.1 is complete.

**References**
