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SOME COMMENTS ON THE SPECTRAL THEOREM

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The spectral theorem for self-adjoint operators in a Hilbert space (with real, complex, or quaternionic scalars) generalizes the classical theorems on the canonical reduction of quadratic or hermitian forms and their matrices. Usually two steps are needed, the first passing from finite-dimensional spaces to bounded operators in general spaces, the second from bounded operators to unbounded. There has always been a certain interest (see [1], [2], [3], [4], [5]) in carrying out this generalization by "pure" Hilbert space methods - that is to say, by using only intrinsic algebraic and geometric properties of abstract Hilbert space without recourse to special theorems drawn from classical analysis. For bounded operators the spectral theorem was treated in this spirit by F. Riesz [1] and by Lengyel and Stone [2], for unbounded operators by Y. Y. Tseng [3]. The present paper, while closely related to Tseng's, expounds a variant of his approach that may appear somewhat simpler and may shed some additional light on the techniques required.

All methods for treating the case of an unbounded self-adjoint operator A involve the discussion of certain related bounded operators. Most of them also use the spectral theorem for the bounded case, either explicitly or implicitly. Here we shall assume the bounded case, as treated in [2], and apply it to one of the operators appearing in the characteristic matrix of A (see [6]) in such a way as to settle the unbounded case. We shall not assume any knowledge of [6], but shall develop on the spot the essential properties of the elements of the characteristic matrix for A.

As commutativity of operators is continually stressed in our arguments, we must recall that a bounded linear operator D commutes with the self-adjoint operator A if and only if it maps the domain of A into itself and AD is an extension of DA. The set of all operators D commuting with A is called the *commutant* of A, while the set of all bounded linear operators commuting with every member of the commutant is called the *second commutant* of A. Clearly, if D is in the commutant of A, then so is its adjoint D*: for, if x and

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y are vectors in the domain of A, we have $(Ax)(D^*y) = (DAx)y = (ADx)y = x(D^*Ay)$; and then, since A is self-adjoint, D^*y must be in the domain of A and satisfy the relation $AD^*y = D^*Ay$. Similarly, the second commutant contains both D and D^* if it contains either.

In order to state the spectral theorem and present its proof it will be convenient to introduce the following

DEFINITION 1. *A projection* P *splits a self-adjoint operator* A *at* λ, $-\infty < \lambda < +\infty$, *if and only if*

(1) P *commutes with* A.

(2) *if* x *is a vector in the domain of* A *and in the range of* P, *then* $(Ax)x \leq \lambda \|x\|^2$.

(3) *if* x *is a vector in the domain of* A *and in the range of* I-P, *then* $\lambda \|x\|^2 \leq (Ax)x$ *with equality holding if and only if* x = 0.

Here we note that the ranges of P and I-P are mutually orthogonal subspaces and that every vector x is the sum of components Px and (I-P)x in these two subspaces respectively, in just one way. The commutativity required in (1) shows that x is in the domain of A if and only if its two components are. Commutativity shows further that A acts on each of these subspaces as a self-adjoint operator therein and that the behavior of A is completely determined by what it does there, in accordance with the equations $Ax = APx + A(I-P)x = PAx + (I-P)Ax$ where $APx = PAx$ and $A(I-P)x = (I-P)Ax$. The concept of splitting demands in addition a certain quantitative behavior (semi-boundedness) in each of these subspaces, as described by (2) and (3) respectively.

We shall now state the spectral theorem, in two parts.

THEOREM 1a. (Spectral Theorem, Analytic Part.) *If* A *is a self-adjoint operator, then there exists for each real* λ, $-\infty < \lambda < +\infty$, *a unique projection* E_λ *splitting* A *at* λ. *The projections* E_λ *necessarily have the following properties:*

(1) E_λ *is in the second commutant of* A, *as well as in the commutant.*

(2) $E_\lambda E_\mu = E_\nu$ *where* $\nu = \min(\lambda,\mu)$.

(3) $\lim_{\varepsilon \to 0} E_{\lambda+\varepsilon}x = E_\lambda x$ *strongly when* $\varepsilon > 0$.

(4) $\lim_{\lambda \to -\infty} E_\lambda x = 0$ *strongly.*

(5) $\lim_{\lambda \to +\infty} E_\lambda x = x$ *strongly.*

We recall that a family of projections satisfying (2) to (5) above is called a *spectral family* or a *canonical resolution of the identity*. With this terminology we state the second part of the spectral theorem as follows.

THEOREM 1b. (Spectral Theorem, Synthetic Part). *If* E_λ, $-\infty < \lambda < +\infty$, *is a spectral family of projections, then there exists a unique self-adjoint operator* A *such that* E_λ *splits* A *at* λ.

Here we shall prove only Theorem 1a. The proof of Theorem 1b, as is well-known, depends on the construction of A as a limit of Riemann-Stieltjes sums

$\sum_{k=0}^{n-1} \lambda_{k+1}^1 (E_{\lambda_{k+1}} - E_{\lambda_k})$, $\lambda_k \leqslant \lambda_{k+1}^1 \leqslant \lambda_{k+1}$, and the verification of the splitting property for E_λ.

The proof of Theorem 1a depends in the last analysis on the following specialization.

THEOREM 2. (Splitting Theorem). *If* A *is a self-adjoint operator, there exists a projection* E *that splits* A *at* 0 *and is in the second commutant of* A.

Indeed, we shall begin by proving

THEOREM 3. *The Splitting Theorem implies the Spectral Theorem, Analytic Part.*

The proof will be presented as a series of lemmas and theorems.

LEMMA 1. *A projection* P *splits* A *at* λ *if and only if it splits* $A-\lambda I$ *at* 0.

Proof. This proof will be left to the reader.

COROLLARY 1. *The Splitting Theorem implies that there exists a* *projection* E_λ *in the second commutant of* A *such that* E_λ *splits* A *at* λ, $-\infty < \lambda < +\infty$.

LEMMA 2. *If* P_μ *and* Q_λ *are commuting projections that split* A *at* μ *and* λ *respectively, then* $\mu \leqslant \lambda$ *implies* $P_\mu Q_\lambda = P_\mu$ *and* $\lambda = \mu$ *implies* $P_\mu = Q_\lambda$.

Proof. Since P_μ and $I-Q_\lambda$ are commuting projections, the intersection of their ranges is a subspace with $R = R_{\lambda\mu} = P_\mu(I-Q_\lambda)$ as its projection. Thus for arbitrary x the vector $y = Rx$ is in the ranges of P_μ and $I-Q_\lambda$. Since P_μ and Q_λ split A at μ and at λ respectively, we have $\lambda\|y\|^2 \leqslant (Ay)y \leqslant \mu\|y\|^2$ with equality on the left if and only if $y = 0$. Thus $\mu \leqslant \lambda$ implies $\lambda\|y\|^2 = (Ay)y$ and hence $y = 0$. It follows that $Rx = 0$ or $P_\mu Q_\lambda = Q_\lambda P_\mu = P_\mu$. When $\lambda = \mu$, we can interchange P_μ and Q_λ, obtaining $Q_\lambda = P_\mu Q_\lambda = Q_\lambda P_\mu = P_\mu$.

COROLLARY 2.1. *The Splitting Theorem implies that, if* P_λ *splits* A *at* λ, *then* P *is unique and is in the second commutant of* A.

Proof. Let E_λ be the projection of which the existence is asserted by the Splitting Theorem; and let P_λ split A at λ. We verify that P_λ and E_λ commute. In fact, P_λ commutes with A and E_λ is in the second commutant of A, so that this is obvious. In Lemma 2 we can now take $\lambda = \mu$, $Q_\lambda = E_\lambda$ and conclude that $P_\lambda = E_\lambda$.

COROLLARY 2.2. *The Splitting Theorem implies all statements combined in the Spectral Theorem, Analytic Part, except those concerning properties* (3) , (4) , (5).

Proof. The existence of a splitting family is given by Corollary 1. Its uniqueness and its inclusion in the second commutant of A are guaranteed by Corollaries 1 and 2.1. Property (2) is then evident from Lemma 2.

THEOREM 4. *If* $\lambda \leqslant \mu$, *the projection* $F = F_{\lambda\mu} = E_\mu - E_\lambda = E_\mu(I-E_\lambda)$ *commutes with* A *and has range lying in the domain of* A. *For all* y *in the range of* F $\lambda\|y\|^2 \leqslant (Ay)y \leqslant \mu\|y\|^2$ *with equality on the left if and only if* $y = 0$.

Proof. Apart from the notation, all of this theorem except for the assertion that the range of F lies in the domain of A is proved in the discussion of Lemma 2: it is only necessary to take

$P_\mu = E_\mu$ and $Q_\lambda = E_\lambda$ there. Now if x is in the domain of A so is $y = Fx$. Since $\|y\|^2 = (Fx)x$ and $(Ay)y = (AFx)x$ we have $\lambda(Fx)x \leqslant$ $\leqslant (AFx)x \leqslant \mu(Fx)x$ or, equivalently, $0 \leqslant ((A-\lambda I)Fx)x \leqslant (\mu-\lambda)(Fx)x \leqslant$ $\leqslant (\mu-\lambda)\|x\|^2$. Thus the operator $H = F(A-\lambda I)$ has the same domain as A and satisfies the relations $(Hx)z = x(Hz)$ and $0 \leqslant (Hx)x \leqslant$ $\leqslant (\mu-\lambda)\|x\|^2$ for all x and z in the domain of A. A standard use of polarization shows that $\|Hx\| \leqslant (\mu-\lambda)\|x\|$ for all x in the domain of A. If y is an arbitrary vector in the range of F, there is a sequence z_n in the domain of A converging strongly to y. Thus $y_n = Fz_n$ and Hz_n are Cauchy sequences converging to $Fy = y$ and z^* respectively. Hence for all x in the domain of A we have

$$\begin{aligned}(Ax)y &= \lim_{n\to\infty} (Ax)y_n = \lim_{n\to\infty} (FAx)z_n \\ &= \lim_{n\to\infty} ((H+\lambda F)x)z_n = \lim_{n\to\infty} x(Hz_n) + \lim_{n\to\infty} x(\lambda y_n) \\ &= x(z^* + \lambda y) .\end{aligned}$$

Since A is self-adjoint, we conclude that y is in the domain of A (and $Ay = z^* + \lambda y$).

LEMMA 3. E_λ *has property* (3).

Proof. If $\nu \leqslant \mu$ we have $\|E_\mu x - E_\nu x\|^2 = ((E_\mu - E_\nu)x)x = (E_\mu x)x - (E_\nu x)x$, so that $(E_\mu x)x$ is a monotone increasing function of μ with real values between 0 and $\|x\|^2$. As a function of x it is quadratic. When $\lambda \leqslant \mu$ the function $q(x) = \lim_{\mu\to\lambda} ((E_\mu - E_\lambda)x)x$ exists and is also quadratic with real values between 0 and $\|x\|^2$. Hence there exists a bounded self-adjoint operator F such that $(Fx)y = \lim_{\mu\to\lambda}((E_\mu - E_\lambda)x)y$. Thus if $\lambda < \nu \leqslant \mu$ we have

$$\begin{aligned}((E_\mu - E_\lambda)Fx)y &= (Fx)((E_\mu - E_\lambda)y) = \lim_{\nu\to\lambda} ((E_\nu - E_\lambda)x)((E_\mu - E_\lambda)y) \\ &= \lim_{\nu\to\lambda} ((E_\nu - E_\lambda)x)y = (Fx)y .\end{aligned}$$

Hence $(E_\mu - E_\lambda)Fx = Fx$, so that Fx is in the range of $E_\mu - E_\lambda$. Thus by theorem 4 we see that Fx is in the domain of A with $\lambda\|Fx\|^2 \leqslant$ $\leqslant (AFx)x \leqslant \mu\|Fx\|^2$, where the equality holds on the left if and only if $Fx = 0$. If we let μ tend to λ in this double inequality we obtain $\lambda\|Fx\|^2 = (AFx)x$ and hence $Fx = 0$. It follows that

$$\lim_{\mu\to\lambda} \|E_\mu x - E_\lambda x\|^2 = \lim_{\mu\to\lambda} ((E_\mu - E_\lambda)x)x = (Fx)x = 0 .$$

Putting $\mu = \lambda+\varepsilon$, $\varepsilon > 0$, we conclude that $\lim_{\varepsilon\to 0} \|E_{\lambda+\varepsilon}x - E_\lambda x\| = 0$.

LEMMA 4. E_λ *has property* (4).

Proof. Let x be an arbitrary vector and ε an arbitrary positive real number. We can then take y in the domain of A so that $\|x-y\| \leqslant \frac{1}{2}\varepsilon$. Let $\lambda < -2\|Ay\|/\varepsilon$. Then $E_\lambda y$ is in the domain of A and $(AE_\lambda y)y \leqslant \lambda\|E_\lambda y\|^2$. Hence $|\lambda|\,\|E_\lambda y\|^2 \leqslant (-AE_\lambda y)y = (E_\lambda y)(-Ay) \leqslant$ $\leqslant \|E_\lambda y\|\,\|Ay\|$ and $\|E_\lambda y\| \leqslant \|Ay\|/|\lambda| \leqslant \frac{1}{2}\varepsilon$. Finally $\|E_\lambda x\| \leqslant \|E_\lambda y\| + \|E_\lambda(x-y)\| \leqslant \frac{1}{2}\varepsilon + \|x-y\| \leqslant \varepsilon$, as was to be proved.

LEMMA 5. E_λ *has property* (5).

Proof. The discussion is similar to that of Lemma 4. For given x and ε we choose y as before and λ so that $\lambda > 2\|Ay\|/\varepsilon$. We observe that $(I-E_\lambda)y$ is in the domain of A and that

$$\lambda\|(I-E_\lambda)y\|^2 \leqslant (A(I-E_\lambda)y)y \leqslant \|(I-E_\lambda)y\|\,\|Ay\| \quad .$$

Since $\|x-y\| \leqslant \frac{1}{2}\varepsilon$ and $\|(I-E_\lambda)y\| \leqslant \|Ay\|/\lambda \leqslant \frac{1}{2}\varepsilon$, we conclude that $\|x-E_\lambda x\| \leqslant \varepsilon$.

We have thus established Theorem 3 and reduced the proof of Theorem 1a to the proof of Theorem 2, the Splitting Theorem. For the latter we need to introduce the bounded self-adjoint operators B and C that occur in the characteristic matrix

$$\begin{Vmatrix} B & C \\ C & I-B \end{Vmatrix}$$

of the self-adjoint operator A (see [6]). It is then easily shown that the projection E supplied for C by the Splitting Theorem serves also as the desired splitting projection for A. Thus the Splitting Theorem for bounded self-adjoint operators is seen to imply the theorem for all self-adjoint operators. With this motivation we turn to the discussion of the operators B and C.

Following von Neumann [7], we study the graphs of the relations $y = Ax$, $-Ay = x$ in the Hilbert space of ordered vector-pairs (x,y) with the scalar product $(x_1,y_1)(x_2,y_2) = x_1x_2 + y_1y_2$. The *graph of* A or the *graph of the equation* $y = Ax$ is the set $M_A = \{(x,y)\ ;\ y = Ax\}$. Similarly the *inverse graph of the operator* $-A$ or the *graph of the equation* $-Aw = z$ is the set $N_A = \{(z,w)\ ;\ -Aw = z\}$. The orthogonality of elements (x,y) and (z,w) chosen from these sets is expressed by the equation $xz + yw = x(-Aw) + (Ax)w = 0$ or $(Ax)w = x(Aw)$. The fact that (z,w) is orthogonal to every element of M_A is expressed by the statement that $xz + yw = xz + (Ax)w = 0$ for all x in the domain of A; and the latter statement is valid for self-adjoint A if and only if w is in the domain of A and $Aw = -z$, that is, if and only if (z,w) is in N_A. Thus when A is self-adjoint, N_A is the orthogonal complement M_A^* of M_A. Similarly, M_A is the orthogonal complement of N_A. Thus M_A and N_A are both closed linear subsets, or subspaces, of the Hilbert space of vector pairs. We denote by $P = P_A$ the projection of the latter on M_A, the graph of A. Now the operators B and C are defined as the composite mappings $x \to z$ and $x \to w$, respectively, read off from the diagram

$$x \to (x,0) \xrightarrow{P} (z,w) \begin{array}{l} \nearrow z \\ \searrow w \end{array}$$

Since each arrow in the diagram represents a bounded linear mapping from source to target, B and C are bounded linear mappings or operators with the original Hilbert space as source and target. Since P is the projection on M_A , the projection on N_A is $I-P$. The equation $(x,0) = P(x,0) + (I-P)(x,0)$ shows that $P(x,0) = (Bx,Cx)$ is in M_A and that $(I-P)(x,0) = (x-Bx,-Cx)$ is in V_A Thus Bx is in the domain of A and $Cx = ABx$, while $-Cx$ is in the domain of $-A$ and $(-A)(-Cx) = x - Bx$ or $Bx + ACx = x$. It follows that Bx is in the domain of A^2 (which is the same as that of $I+A^2$) and $A^2(Bx) = A(AB)x = ACx$, $(I+A^2)Bx = Bx + ACx = x$. We have thus proved

LEMMA 6. B *and* C *have ranges contained in the domains of* $I + A^2$ *and* A *respectively. The operators* A, B, C *satisfy the identities*

$$(1)\quad C = AB \quad ; \quad (2)\quad B + AC = I \quad ; \quad (3)\quad (I + A^2)B = I \ .$$

LEMMA 7. B *is a self-adjoint operator with self-adjoint inverse* $I + A^2$.

Proof. By Lemma 6 (3) we see that $Bx = 0$ implies $x = 0$. Hence B has an inverse, of which $I+A^2$ must be an extension. Now if y is an arbitrary vector in the domain of $I+A^2$, we put $z=y-B(I+A^2)y$, noting that $(I+A^2)z = 0$. Thus $\|z\|^2 + \|Az\|^2 = ((I+A^2)z)z = 0$, $z = 0$, $y = B(I+A^2)y$, and y is in the range of B. Hence the range of B is the domain of $I+A^2$ and the two operators are inverses of one another. Now $(Bx)y = (Bx)((I+A^2)By) = ((I+A^2)Bx)By = x(By)$ for all x and y because A is self-adjoint. To show that $I+A^2$ is, like B, self-adjoint, let y and y^* be such that $((I+A^2)x)y = xy^*$ for all x in the domain of $I+A^2$. Here we can put $x = Bz$, obtaining $zy = (Bz)y^*$ for all z. It follows that $By^* = y$ because B is self-adjoint. Hence y is in the domain of $I+A^2$ and $(I+A^2)y = y^*$. Accordingly, $I+A^2$ is self-adjoint.

We turn now to some commutativity properties of A, B, C.

LEMMA 8. *B and C commute with A and with each other. Consequently C is self-adjoint, as are A and B.*

Proof. If x is in the domain of A, the equation $ACx = Bx - x$ shows that ACx is also in the domain of A and that $A^2Cx = ABx - Ax = Cx - Ax$. Thus $Ax = (I+A^2)Cx$ and $BAx = Cx = ABx$. Thus B commutes with A. It now follows that $Cx = BAx$ is in the domain of A and that $ACx = A^2Bx = ABAx = CAx$ because B commutes with A. Hence C commutes with A. Now for all z we have $BCz = BABz = ABBz = CBz$ because B commutes with A. Hence B and C commute. Finally we observe that $C = AB$ implies that C^* is an extension of $B^*A^* = BA \subset AB$. Thus C and C^* coincide on the domain of A and must be identical by continuity, since the domain of A is everywhere dense. Thus C is self-adjoint.

LEMMA 9. *A bounded linear operator D commutes with A if and only if it commutes with both B and C.*

Proof. If D commutes with A we have $DCz = DABz = ADBz$, $Dz = DBz + DACz = DBz + ADCz = DBz + A^2DBz = (I+A^2)DBz$. Hence $BDz = DBz$; and B and D commute. We then have from the first equation $DCz = ADBz = ABDz = CDz$, so that C and D also commute. On the other hand, if D commutes with B and C and x is in the domain of A, we have $CDx = DCx = DABx = DBAx = BDAx$ and hence $Dx = B(Dx) + (AC)(Dx) = BDx + ABDAx$. Thus Dx is in the domain of A. We now have $BADx = CDx = DCx = BDAx$ and hence $ADx = DAx$.

We are now ready to prove our principal result.

THEOREM 5. *The projection* E *supplied by the Splitting Theorem for the bounded self-adjoint operator* C *serves as the operator required in order to validate the Splitting Theorem for* A.

Proof. We have to show that E splits A at 0 and is in the second commutant of A. E is in the second commutant of C and therefore commutes with B and C, both of which commute with C by Lemma 8. Hence E commutes with A, by Lemma 9. If D commutes with A, it also commutes with C, by Lemma 9. Hence it commutes with E, because E is in the second commutant of C. Thus E is seen to be in the second commutant of A. To show that E splits A at 0, we take x in the domain of A and note that $Ax = (I+A^2)BAx = (I+A^2)ABx = (I+A^2)Cx$ by Lemmas 7 and 8. Now if x is in the range of E so is Ax because $EAx = AEx = Ax$. We therefore have $(Ax)x = ((I+A^2)Cx)x = (Cx)x + (ACx)(Ax) = (Cx)x + (CAx)(Ax) \leqslant 0$, because E splits C at 0. Similarly, when x is in the range of I-E we see that Ax is in the range of I-E. We then have $(Ax)x = (Cx)x + (CAx)(Ax) \geqslant 0$ with equality if and only if $(Cx)x = 0$ and hence if and only if $x = 0$.

The proof of the Spectral Theorem, Analytic Part, is thus completed by reference to the paper of Lengyel and Stone [2], where it is shown by "pure" methods that the Splitting Theorem holds for every bounded self-adjoint operator.

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