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A GEOMETRIC OBSERVATION ABOUT LINEAR PARTIAL DIFFERENTIAL OPERATORS

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We consider a linear partial differential operator on an open $U \subset R^p$ of the form

$$(1.1) \qquad A\phi = \sum_{|\alpha| \leq k} a_\alpha \frac{\partial^\alpha \phi}{\partial U^\alpha} \qquad (\phi \in C_o^\infty(\bar{U}))$$

where the a_α are complex-valued functions defined on U an the $\frac{\partial^\alpha}{\partial U^\alpha}$ are the usual partial derivatives of order α, where $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $|\alpha| = \sum_i \alpha_i$. In this paper we make two observations:

1) The symbol of A, nowadays usually considered as a function on the cotangent bundle, can be considered, instead, as a function on a subset of $G_p(R^{p+2})$, where $G_p(R^{p+2})$ is the bundle of p-planes at points of R^{p+2}. The symbol can be defined naturally here, by geometric considerations, without using coordinates (tho it does use the decomposition of R^{p+2} as $R^p \times R^2$). If one then puts a coordinate system on $G_p(R^{p+2})$, the coordinate expression of this function becomes the usual expression for the symbol.

2) The most general definition of a partial differential equation (*equation* , as distinct from operator) seems to be as a subset of a higher order Grassman bundle. We indicate, again geometrically, how an operator of the form (1.1) gives rise to a partial differential equation in this sense. Furthermore, it gives rise to a sequence of such equations, of orders 1 to k; the j'th equation being of order j. The equation of order k is (1.1) considered as a differential equation; the equation of order 1 is the characteristic equation (the zeros of the symbol).

We wish to consider the A of (1.1) as a k-th order vector field on U, i.e. as a map which assigns to each $x \in U$ a k-th order complex tangent vector to R^p at x. Then we wish to show how such a k-th order vector field gives rise to functions on certain subsets of $G_p^{1,c}(U \times R^1), \ldots, G_p^{k,c}(U \times R^1)$, where $G_p^{\ell,c}(U \times R^1)$ is the set of complex ℓ-th order p-planes at points of $U \times R^1$. This is to be done separately at each point of U so what we wish to show is that a k-th order complex tangent vector at $x \in R^p$ gives rise to some complex valued functions defined on certain ℓ-th order p-spaces in R^{p+1}.

Since R^p is a real manifold and we are referring to complex tangent vectors and p-spaces we briefly discuss complex tangent vectors to a real manifold. Let M be a p-dimensional C^∞ real manifold and $m \in M$. We now define M_m^c, the complex tangent space to M at m. We could define M_m^c to be just the complexification $M_m \otimes C$ of the usual real tangent space M_m. However we prefer to define M_m^c directly, as follows. Let R_m be the complex local ring of M at m, i.e. the elements of R_m are the germs of complex valued C^∞ functions at m. Let I_m be the maximal ideal of R_m, so I_m consists of the germs that vanish at m. For each non-negative integer k we define the complex linear space $M_m^{k,c}$ to be the dual space of R_m/I_m^{k+1}. We call elements of $M_m^{k,c}$ k-th order *tangent vectors* at m and call $M_m^{k,c}$ the k-th order *tangent space* at m. Alternatively a k-th order tangent vector may be considered as a linear function on R_m that vanishes on I_m^{k+1}. Because every $f \in R_m$ can be uniquely expressed as $f = f_o + f_1$ where f_o is constant and $f_1 \in I_m$ it is easily seen that every $t \in M_m^{k,c}$ can be uniquely expressed as $t = t_o + t_1$ where $t_o f = cf_o$ ($c \in C$, independent of f) and t_1 is zero on constants. It is clear that $M_m^{k,c} \subset M_m^{k+1,c}$ and that the usual real k-th order tangent space M_m^k consists of those $t \in M_m^{k,c}$ such that t is real whenever f is real valued. And $M_m^{k,c} =$ $= M_m^k \oplus iM_m^k$. Also, a linear partial differential operator defined on $U \subseteq R^p$ is essentially the same thing as a map which assigns to each $x \in U$ an element of $(R^p)_x^{k,c}$. For this reason we call it a k-th order complex vector field on U.

Now we define the Grassman manifold ${}^{k,c}G_p(M)$ of complex k-th order p-spaces over the real d-dimensional manifold M. This is done essentially as for the real case but since we need below the explicit relation with the real case we give some details here. We shall write R_m^o, I_m^o for the local ring and maximal ideal formed from the *real* valued C^∞ functions at m. We define a p-*ideal* in

R_m to be an ideal I in R_m for which there exists a set of generators $f_{p+1},\dots,f_d$ such that $df_{p+1},\dots,df_d$ at m are linearly independent over C. Then we define a z-th *order* p-*ideal* at m to be any ideal in R_m of the form $I + I_m^{z+1}$, where I is any p-ideal in R_m. We define a z-th *order complex* p-*space* at m as any dual space of an $I_m/(I + I_m^{z+1})$ where I is any p-ideal at m. We now define ${}^{k,c}G_p(M)$ to be the set of all z-th order complex p-spaces, made into a real C^∞ manifold, and a bundle over M, with the following differentiable structure. Let $x_1,\dots,x_d$ be any coordinate system of M with a cubic domain Q. Let N be the submanifold of M defined by

$$N = [m \in Q \mid x_{p+1}(m) = \dots = x_d(m) = 0]$$

and let ρ be the associated projection of N into Q. Let ${}^zQ$ = all z-th order complex p-spaces (m,P) such that $m \in Q$ and P = $= (I_m/(I + I_m^{z+1}))^*$, where P is any p-ideal having a set of generators of the form

$$f_{p+1} = x_{p+1} - h_{p+1} \cdot \rho,\dots, f_d = x_d - h_d \cdot \rho$$

where the $h_{p+1},\dots,h_d$ are C^∞ functions on N. We then define the collection of functions w_i^o , $'w_r^\alpha$, $''w_r^\alpha$ for $\alpha = (\alpha_1,\dots,\alpha_p)$, $|\alpha| \leqslant z$, $p+1 \leqslant r \leqslant d$, by

$$w_i^o(m,P) = x_i(m) \quad , \quad w_r^\alpha(m,P) = \frac{\partial^\alpha h_r}{\partial x^\alpha}(m)$$

$$'w_r^\alpha(m,P) = \text{Re } w_r^\alpha(m,P) \ ,$$

$$''w_r^\alpha = \text{Im } w_r^\alpha$$

The set of all such $\{w_i^o , 'w_r , ''w_r\}$ make ${}^{k,c}G_p(M)$ into a real C^∞ manifold, and a bundle over M.

We note that the usual ${}^kG_p(M)$ is a submanifold of ${}^{k,c}G_p(M)$ consisting of all the (m,P) such that the corresponding p-ideal has a set of real-valued generators. These are the elements (in the domain of such a coordinate system) for which all $''w_r^\alpha = 0$. We call elements of ${}^kG_p(M)$ *real* z-th order p-spaces.

We now define, for each $y \in R$, an application Φ_y, of a subset of ${}^{k,c}G_p(R^{p+1}) \to {}^{k}G_p(R^{p+2})$. This is the application that carries each $x \in R^{p+1} \to (x,y) \in R^{p+2}$ and carries the complex p-plane whose coordinates (relative to the w_i^o, ${}'w_{p+1}^\alpha$, ${}''w_{p+1}^\alpha$, - obtained from the usual coordinate system of R^{p+1}) to the element of $G_p(R^{p+2})$ whose coordinates are these same numbers, i.e.

$$w_i^o \cdot \Phi_y = w_i^o \quad , \quad w_{p+1}^\alpha \cdot \Phi_y = {}'w_{p+1}^\alpha \quad , \quad w_{p+2}^\alpha \cdot \Phi_y = {}''w_{p+1}^\alpha$$

The geometric construction in going from a linear partial differential operator to a partial differential equation, or to the symbol of the operator, is just the projection of a k-th order vector into a plane P, skew to both factors, followed by a projection into the second factor. Then in case the second factor is R^1 the resulting vector, which is a function of P, can be described by its coefficients of orders 0 to k (there is one coefficient for each of these orders when the second factor is R^1), so we have k+1 functions of P (depending on the initial vector field), and these include the symbol and the partial differential equation. In the case where the second factor is of dimension greater than 1, we obtain vector-valued functions of P. We now describe this geometric process more precisely.

Let M and N be real C^∞ manifolds of dimensions p and q, and let d = p+q. Let m be a point of M and n a point of N. We shall speak of vectors *tangent* to M or to N at (m,n), meaning vectors tangent to the submanifolds M × (n) or (m) × N ; by tangent vector we shall always mean complex tangent vectors. Let ${}^{k,c}G_p(m,n)$ be the set of k-th order complex p-spaces of M × N at (m,n); let ${}^{k,c}M_{(m,n)}$ be the space of k-th order tangent vectors to M (really to M × (n) at (m,n), and define ${}^{k,c}N_{(m,n)}$ similarly. We have natural projections, that we denote by ρ and σ of ${}^{k,c}(M{\times}N)_{(m,n)}$ into ${}^{k,c}M_{(m,n)}$ and of ${}^{k,c}(M{\times}N)_{(m,n)}$ into ${}^{k,c}N_{(m,n)}$ ρ and σ are the k-th order differentials of the natural projections of M × N into M × (n) and (m) × N (and we could generalize to the case where M and N are only "k-th order factors" of some manifold Q). From ρ we define a subset E of ${}^{k,c}G_p(m,n)$ by: E consists of all elements of ${}^{k,c}G_p(m,n)$ on which ρ is nonsingular. We then define a map V of $E \times {}^{k,c}M_{(m,n)}$ into ${}^{k,c}N_{(m,n)}$ by: if $P \in E$ and $t \in {}^{k,c}M_{(m,n)}$ and if t' is the unique element of P such that $\rho t' = t$ then

$$V(P,t) = \sigma t' \quad .$$

We now write a formula for this V, in terms of coordinate systems, or equivalently, we wish coordinate expressions for V, in case $M = R^p$ and $N = R^q$, in terms of the usual coordinates of Euclidean space. We hence assume now that $M = R^p$, $N = R^q$ and let $u_1,\ldots,u_s$ be the usual coordinate system of R^s. Then, as usual, V(P,t) can be expressed as

$$V(P,t) = \sum_{|\beta|\leq k} v_\beta(P,t) \frac{\partial^\beta}{\partial U^\beta}$$

where the v_β are complex valued functions which we wish to find explicitly in terms of the coordinates of P and of t. Since V(P,t) is clearly linear in t the main thing will be to compute the functions $v_{\beta\alpha}$ defined by

$$v_{\beta\alpha}(P) = v_\beta \quad (P , \frac{\partial^\alpha}{\partial U^\alpha})$$

where $\alpha = (\alpha_1,\ldots,\alpha_p)$. $\beta = (\beta_1,\ldots,\beta_q)$, and $|\alpha| \leqslant k$, $|\beta| \leqslant k$. And we now wish to express these $v_{\beta\alpha}$ in terms of the above coordinates $\{w_i^o , {}'w_r^\alpha , {}''w_r^\alpha\}$ of P; we shall write $w_r^\gamma = {}'w_r^\gamma + i{}''w_r^\gamma$ and express the $v_{\beta\alpha}$ in terms of the w_r^γ .

First, for each $\alpha = (\alpha_1,\ldots,\alpha_p)$ and $\beta = (\beta_{p+1},\ldots,\beta_d)$ with $|\alpha| \geqslant |\beta|$ we define $P_{\beta\alpha}$ to be the unique polynomial such that, for all functions $g_{p+1},\ldots,g_d$ in C^∞ at m ,

$$\frac{\partial^\alpha}{\partial U^\alpha}(m)\,[(g_{p+1} - g_{p+1}(m))^{\beta_{p+1}} \ldots (g_d - g_d(m))^{\beta_d}]$$

$$= P_{\beta\alpha}\left(\ldots, \frac{\partial^\eta g_r}{\partial U^\eta}(m),\ldots\right) .$$

Hence if N = the number of $\eta = (\eta_1,\ldots,\eta_p)$ such that $0 \leq \eta \leq \alpha$ then $P_{\beta\alpha}$ is a polynomial in (d-p)N variables. It is clear that this defines unique polynomials $P_{\beta\alpha}$. It will be important that the variables here are labelled with the subscripts (η,r) as abov

The formula we desire to prove for the $v_{\beta\alpha}$ is:

$$(1.2) \qquad v_{\beta\alpha}(P) = P_{\beta\alpha}(\ldots,w_r^\eta(P),\ldots)/\beta!$$

and we now make the calculations to prove (1.2). We first seek the coordinates of that $t'_\alpha \in P$ such that $\rho t'_\alpha = \frac{\partial^\alpha}{\partial U^\alpha}$. That t'_α will have the form

$$(a) \qquad t'_\alpha = \sum_\gamma a_{\gamma\alpha} \frac{\partial^\gamma}{\partial U^\gamma}(m)$$

thru all $\gamma = (\gamma_1,\ldots,\gamma_d)$ with $|\gamma| \leq k$. Because

$$\sigma t'_\alpha = \sum_{\gamma_1=\ldots=\gamma_r=0} a_{\gamma\alpha} \frac{\partial^\gamma}{\partial U^\gamma}(m)$$

we have

$$(b) \qquad v_{\beta\alpha}(P) = a_{(0,\beta),\alpha} \quad .$$

Hence we are interested in the $a_{\gamma\alpha}$ for which $\gamma_1 = \ldots. = \gamma_p = 0$. But for the moment we consider general γ. We write

$$(u - u(m,n))^\gamma = \prod_i (u_i - u_i(m,n))^{\gamma_i} \quad .$$

From (a) we have the usual formula:

$$\begin{aligned} \gamma!\, a_{\gamma\alpha} &= t'_\alpha (U - U(m,n))^\gamma \\ &= (g_* \frac{\partial^\alpha}{U^\alpha}(m))(U - U(m,n))^\gamma \\ &= \frac{\partial^\alpha}{\partial U^\alpha}(m)((U - U(m,n))^\gamma \cdot g) \end{aligned}$$

$$= \frac{\partial^\alpha}{\partial U^\alpha}(m)\ [(\prod_{i=1}^{p}(U_i - U_i(m,n))^{\gamma_i})(\prod_{j=p+1}^{d}(g - g_j(m)^{\gamma_j})\ ]$$

We write $\gamma = \gamma' + \gamma''$ where $\gamma' = (\gamma_1,\ldots,\gamma_p,0,\ldots)$ and $\gamma'' = (0,\ldots,0,\gamma_{p+1},\ldots,\gamma_d)$. So the above becomes

$$\gamma!\ a_{\gamma\alpha} = \frac{\partial^\alpha}{\partial U^\alpha}(m)[(U-U(m))^{\gamma'}(g-g(m))^{\gamma''}]$$

(c) $$= \sum_{0\le\eta\le\alpha} \frac{\alpha!}{\eta!(\alpha-\eta)!}\ (\frac{\partial^\eta}{\partial U^\eta}(m)(U-U(m))^{\gamma'})(\frac{\partial^{\alpha-\eta}}{\partial U^{\alpha-\eta}}(m)(g-g(m))^{\gamma''})$$

We recall the fact, easily proved by induction,

(d) $$\frac{\partial^\eta}{\partial U^\eta}(m)(U-U(m))^\alpha = \eta!\ \delta_{\alpha\eta} \qquad \text{if} \quad 0 \leqslant \eta \leqslant \alpha$$

(where $\alpha = (\alpha_1,\ldots,\alpha_p)$, $\eta = (\eta_1,\ldots,\eta_p)$); then (c) and (d) give

(e) $$\gamma!\ a_{\gamma\alpha} = \frac{\alpha!}{(\alpha-\gamma')!}\ \frac{\partial^{\alpha-\gamma'}}{\partial U^{\alpha-\gamma'}}(m)((g-g(m))^{\gamma''})$$

If we take $\gamma = (0,\beta)$, so $\gamma' = 0$, $\gamma'' = \gamma$, this gives (1.2).

Now we specialize to the case where $q = 1$ ($d=p+1$), so the $v_{\beta\alpha}$, $P_{\beta\alpha}$, $a_{\beta\alpha}$ becomes $v_{o,\alpha},\ldots,v_{k,\alpha}$, $P_{o,\alpha},\ldots,P_{k,\alpha}$, $a_{o,\alpha},\ldots,a_{k,\alpha}$. In this case we give the explicit expression for the $P_{j,\alpha}$ by iterating the Leibnitz product rule. We have

$$\frac{\partial^\alpha h^j}{\partial U^\alpha} = \sum \frac{\alpha!}{(\alpha-\eta^1)!\ldots(\eta^{j-2}-\eta^{j-1})!\eta^{j-1}!}\ \frac{\partial^{\alpha-\eta^1}h}{\partial U^{\alpha-\eta^1}}\cdots\frac{\partial^{\eta^{j-2}-\eta^{j-1}}h}{\partial U^{\eta^{j-2}-\eta^{j-1}}}\frac{\partial^{\eta^{j-1}}}{\partial U^{\eta^{j-1}}}$$

where this sum is taken over all choices of $\eta^1,\ldots,\eta^{j-1}$ such that $\alpha \geqslant \eta^1 \geqslant \eta^2 \geqslant \ldots \geqslant \eta^{j-1} \geqslant 0$. Applied at m to $h = g-g(m)$, all undifferentiated terms vanish so

(1.3) $P_{j,\alpha}(\ldots,w^{\eta},\ldots) =$

$$= \sum \frac{\alpha!}{(\alpha-\eta^1)!\ldots(\eta^{j-2}-\eta^{j-1})!\,\eta^{j-1}!}\, w^{\alpha-\eta^1}\ldots w^{\eta^{j-2}-\eta^{j-1}} w^{\eta^{j-1}}$$

where this sum is taken over all $\eta^1,\ldots,\eta^{j-1}$ such that, $\alpha > \eta^1 > \eta^2 > \ldots > \eta^{j-1} > 0$. We note, in particular, that all non-zero terms of $P_{j,\alpha}$ are products of at least j w's, and contain no w^{η} with $|\eta| > k-j+1$. Hence

$$v_{j,\alpha} = \frac{1}{j!} \sum \frac{\alpha!}{(\alpha-\eta^1)!(\eta^1-\eta^2)!\ldots(\eta^{j-2}-\eta^{j-1})!\;\eta^{j-1}!} \cdot$$

$$\cdot\, w_{p+1}^{\alpha-\eta^1}\, w_{p+1}^{\eta^1-\eta^2}\ldots w_{p+1}^{\eta^{j-2}-\eta^{j-1}}\, w_{p+1}^{\eta^{j-1}}$$

where this sum is taken over all $\eta^1,\ldots,\eta^{j-1}$ such that

$\alpha > \eta^1 > \ldots > \eta^{j-1} > 0$. If j=k and this is $\neq 0$ then we must have $|\alpha| = k$, and all these products of w_{p+1}'s are the same, all being equal to

$$(w_{p+1}^{\delta_1})^{\alpha_1}\ldots(w_{p+1}^{\delta_p})^{\alpha_p}.$$

Furthermore, when $j = k = |\alpha|$, $k!/\alpha!$ is the number of such sequences $\eta^1,\ldots,\eta^{k-1}$ (as is shown by an easy induction) so

$$v_{k,\alpha} = (w_{p+1}^{\delta_1})^{\alpha_1} \ldots (w_{p+1}^{\delta_p})^{\alpha_p} \qquad \text{if} \qquad |\alpha| = k$$

Hence

$$(1.4) \qquad v_k(P,\textstyle\sum_{|\alpha|\leq k} a_\alpha \frac{\partial}{\partial U^\alpha}) = \sum_{|\alpha|=k} a_\alpha\, (w_{p+1}^{\delta_1})^{\alpha_1}\ldots(w_{p+1}^{\delta_p})^{\alpha_p}$$

which is the usual formula for the symbol of (1.1).

As given by (1.2) and (1.3) the $v_{j,\alpha}$ are functions on ${}^{k,c}G_p(M)$ but since all non-zero terms in (1.3) contain only w^{η} with

$|\eta| \leqslant k-j+1$ we see that $v_{j,\alpha}$ is the lift of a function defined on ${}^{k-j+1,c}G_p(M)$, hence the same is true of $v(P,\sum_{|\alpha|\leq k} a_\alpha \frac{\partial}{\partial U^\alpha})$ and this function defines the j-th partial differential equation associated with A, to which we referred in the introduction. In particular, the k-th equation, given by (1.4), is defined on ${}^{1,c}G_p(M)$.

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