Let $\Delta$ denote the open unit disc in the complex plane $\mathbb{C}$, and let $H^\infty(\Delta)$ denote the algebra of bounded analytic functions on $\Delta$. We wish to prove the following theorem, which was proved in the case that $E$ is open by A. Stray [5].

**THEOREM 1.** Let $f \in H^\infty(\Delta)$, and let $E$ be a subset of $\mathbb{C}$ such that $f$ extends continuously to each point of $E$. Then there is a sequence $f_n \in H^\infty(\Delta)$ such that each $f_n$ extends to be analytic on some neighborhood of $E$, and $f_n$ converges uniformly to $f$ on $\Delta$.

For $E$ a subset of $\mathbb{C}$, let $H^\infty_E$ denote the subalgebra of $H^\infty(\Delta)$ of functions which extend continuously to each point of $E$. The theorem asserts that the functions in $H^\infty(\Delta)$ which extend analytically to a neighborhood of $E$ are dense in $H^\infty_E$. Combining the theorem with Carleson's corona theorem, we obtain the following corollary, which is due to Détraz [2].

**COROLLARY.** The open unit disc $\Delta$ is dense in the maximal ideal space of $H^\infty_E$.

**Proof of the main theorem.** We proceed now directly to the proofs. The symbols $C_0$, $C_1$, ..., will all denote universal constants. All norms will be suprema norms.

**LEMMA 1.** Let $Q$ be a closed subset of $\mathbb{C}$, let $W$ be an open subset of $\mathbb{C}$ at a positive distance from $Q$, and let $\epsilon > 0$. Let $f$ be a bounded Borel function on $C$, such that $f$ is analytic on $\Delta$. Suppose there is a continuous function $u$ in a neighborhood of $Q$ such that

$$|f(z) - u(z)| < \epsilon$$

for all $z \in \Delta$ which are near $Q$. Then there is a bounded Borel function $h$ such that
(i) $h$ is analytic on an open set containing $\Delta \cup Q$.

(ii) $h$ extends analytically across any arc on $\partial \Delta$ across which $f$ extends analytically.

(iii) $f-h$ is analytic on $W$ and satisfies $|f-h| < \epsilon$ there.

(iv) $|f(z)-h(z)| < C_1d$ for all $z \in \Delta$.

**Proof.** For $\delta > 0$, the open $\delta$-neighborhood of $Q$ will be denoted by $Q(\delta)$. By hypothesis, we can choose $\delta_0 > 0$ so small that $Q(\delta_0)$ does not meet $W$, that $u$ is defined on $Q(\delta_0)$, and that $|f(z)-u(z)| < d$ for $z \in Q(\delta_0) \cap \Delta$. Since $u$ is uniformly continuous in a neighborhood of $Q$, we can shrink $\delta_0$ so that also $|u(z)-u(\zeta)| < d$ for all $z, \zeta \in Q(\delta_0)$ satisfying $|z-\zeta| < 2\delta_0$.

Let $\Gamma$ be the union of the arcs on $\partial \Delta$ across which $f$ extends analytically. There is then an open set $U$ containing $\Gamma$ such that $|f(z)-u(z)| < d$ for all $z \in Q(\delta_0) \cap U$. Let $F$ be the function which coincides with $u$ on $Q(\delta_0) \setminus (\Delta \cup U)$, and which coincides with $f$ elsewhere. Then $F$ is a bounded Borel function which satisfies

$$|F(z)-F(\zeta)| < 3d \quad \text{whenever } z, \zeta \in Q(\delta_0), \, |z-\zeta| < 2\delta_0.$$  

Since $F$ coincides with $f$ on $\Delta$, on $W$, and in a neighborhood of $\Gamma$, it will suffice to obtain the conclusions of the lemma, with $f$ replaced by $F$.

Now we are in position to use Vitushkin's scheme for approximation, as developed for instance in Chapter VIII of [3], or in [6]. Because we are working on the unit circle, we can employ the version of this technique matching only one coefficient of the appropriate Laurent expansions (cf. [6], V.4). The details are as follows.

For a fixed $\delta$ satisfying $0 < \delta < \delta_0$, choose discs

$\Lambda_k = \{|z-z_k| < \delta\}$, $z_k \in Q$, which cover $Q$, and choose functions $g_k$ supported on $\Lambda_k$ such that $0 < g_k < 1$, $\sum g_k = 1$ in a neighborhood of $Q$, $|\frac{\partial g_k}{\partial z_k}| < 4/\delta$, and no point $z$ is contained in more than $C_2$ of the discs $\Lambda_k$. If

$$G_k(\zeta) = \frac{1}{\pi} \int \int \frac{F(z) - F(\zeta)}{z-z_k} \frac{\partial g_k}{\partial z_k} \, dx \, dy, \quad \zeta \in C,$$  

then $G_k$ is analytic in $\Delta \cup Q$. The main point is that $|G_k(\zeta)| < C_2t\delta$ for $\zeta \in C$.
then $G_k$ is a bounded Borel function, $G_k$ is analytic wherever $F$ is analytic, $G_k$ is analytic off $A_k$, and $G_k(\infty) = 0$. Moreover, $F - \sum G_k$ is analytic on the interior of the set on which $\sum g_k$ assumes the value 1. In particular, $F - \sum G_k$ is analytic in a neighborhood of $Q$. The condition (*) can be used to estimate $G_k$, yielding the bound

$$|G_k| \leq C_j d.$$ 

Suppose the expansion of $G_k$ near $\infty$ is given by

$$G_k(z) = \frac{a_1}{z - z_k} + \ldots .$$

By Schwarz's lemma we have

$$|a_1| \leq |G_k| \delta .$$

Now the analytic capacity of the connected open set $A_k \setminus \delta$ is at least one fourth its diameter. Hence we can find a continuous function $H_k$ on $\mathbb{C}$ such that $H_k$ is analytic off a compact subset of $A_k \setminus \delta$,

$$|H_k| \leq 5 |G_k| .$$

$$H_k(z) = \frac{a_1}{z - z_k} + \ldots .$$

Now $|G_k - H_k| \leq C_4 d$ so that

$$|G_k(z) - H_k(z)| \leq C_4 d \delta^2 / |z - z_k|^2$$

for $z \in \Delta_k$. As $H_k$ has been defined so that $G_k - H_k$ has a double zero at $\infty$, the estimate (***) persists for all $z \in \mathbb{C}$.

Now we define

$$h = F - \sum (G_k - H_k) .$$

Since $F - \sum G_k$ is analytic wherever $F$ is analytic, and each $H_k$ is
analytic in a neighborhood of $\tilde{a}$, the function $h$ is analytic on $\Delta$ and extends analytically across $\Gamma$. Moreover, $h$ is analytic in a neighborhood of $Q$, so that (i) and (ii) are valid. Since $F - h$ is analytic off $Q(\delta)$, $F - h$ is analytic on $W$. To complete the proof, it suffices now to obtain the estimates in (iii) and (iv).

To verify (iii), fix $z \in W$ and consider $F(z) - h(z) = \sum [G_k(z) - H_k(z)]$. Since no point lies in more than $C_2$ discs $\Delta_k$ and each $\Delta_k$ meets $b\delta$, there is a grand total of at most $2\pi C_2/\delta$ discs $\Delta_k$. Thus by (**)

$$(***) \quad \sum |G_k(z) - H_k(z)| < 2\pi C_2 C_4 d \frac{\delta}{\text{dist}(W, \cup \Delta_k)}^2.$$  

Taking $\delta$ much smaller than $\text{dist}(W, Q(\delta_0))$, we get $|F - h| < \epsilon$ on $W$.

To verify (iv), we first observe that $F - h = \sum [G_k - H_k]$ is analytic off $Q(\delta)$, so that it suffices to obtain the estimate

$$\sum |G_k(z) - H_k(z)| < C_1 d$$  

for $z \in Q(\delta)$. So fix a point $z \in Q(\delta)$. Let $M(m)$ be the number of discs $\Delta_k$ whose centers satisfy $m\delta \leq |z - z_k| < (m + 1)\delta$. Since no point $z$ is contained in more than $C_2$ discs, there will be a constant $C_2$ such that

$$M(m) \leq C_5 \quad \text{if} \quad 0 \leq m \leq 1/\delta,$$

providing $\delta$ is sufficiently small. (Here we use the geometry of the unit circle, and the fact that $z$ is close to the unit circle) Using the estimate $|G_k(z) - H_k(z)| < C_4 d$ for the at most $C_2$ indices $k$ for which $|z - z_k| < \delta$, the estimate (**) for those $k$ for which $m\delta \leq |z - z_k| < (m + 1)\delta$ and $1 \leq k < 1/\delta$, and the same estimate used to obtain (***) for those $k$ for which $|z - z_k| \geq 1$, we find that

$$\sum |G_k(z) - H_k(z)| \leq C_2 C_4 d + \sum_{k=1}^{1/\delta} M(k) C_4 d/k^2 + 2\pi C_2 C_4 d \delta \leq C_1 d.$$  

That completes the proof.
LEMMA 2. Let \( f \in H^w(\Delta) \), and let \( E \) be a subset of \( b\Delta \). Suppose there is an open set \( U \) containing \( E \), and a function \( u \) defined and continuous on \( U \), such that \( |f(z) - u(z)| < \delta \) for all \( z \in U \cap \Delta \). Then there is \( h \in H^w(\Delta) \) such that \( h \) extends to be analytic in a neighborhood of \( E \), and

\[
\sup_{z \in E} |f(z) - h(z)| \leq C_0 \delta.
\]

Proof. By replacing \( E \) by \( U \cap b\Delta \), we can assume that \( E \) is relatively open in \( b\Delta \). Then we can write \( E = (\cup Q_n) \cup (\cup R_n) \), where \( Q_1, Q_2, \ldots \) are pairwise disjoint closed intervals, \( R_1, R_2, \ldots \) are pairwise disjoint closed intervals, each \( Q_n \) joins the endpoints of two of the \( R_k \)'s, and each \( R_n \) joins the endpoints of two of the \( Q_k \)'s. Then we can choose \( \delta_n > 0 \) so that the \( \delta_n \)-neighborhoods of the \( Q_n \)'s are pairwise disjoint.

Starting with \( \phi_0 = f \), we construct by induction a sequence of Borel functions \( \phi_n \) such that

1. \( \phi_n \) is analytic on \( \Delta \), and \( \phi_n \) is analytic on a neighborhood of \( Q_n \).

2. \( \phi_n - \phi_{n-1} \) is analytic off the \( \delta_n \)-neighborhood of \( Q_n \) and satisfies \( |\phi_n - \phi_{n-1}| < d/2^n \) there.

3. \( \|\phi_n - \phi_{n-1}\| < 2C_1d \).

Indeed, having chosen \( \phi_{n-1} \), we note that on the part of \( \Delta \) near \( Q_n \) we have

\[
|\phi_{n-1} - u| < |\phi_{n-1} - \phi_{n-2}| + \ldots + |\phi_1 - f| + |f - u| < d/2^{n-1} + \ldots + d/2 + d < 2d,
\]

so that Lemma 1 will provide the desired function \( \phi_n \).

For each \( z \), \( |\phi_j(z) - \phi_{j-1}(z)| < d/2^j \) for all but at most one index \( j \), while always \( |\phi_j - \phi_{j-1}| < 2C_1d \). Hence the \( \phi_j \) converge point-
wise to a function $\psi$ satisfying

$$|\psi(z) - f(z)| \leq \sum |\psi_j(z) - \psi_{j-1}(z)| \leq (2C_1 + 1)d.$$ 

The convergence is uniform on any compact set at a positive distance from $\lim Q_n = bE$, so that $\psi$ is analytic on $\Delta$. Since $\psi_j - \psi_{j-1}$ is analytic on the $\delta_n$-neighborhood of $Q_n$ for $j \neq n$, while $\psi_n - \psi_{n-1}$ is analytic in a neighborhood of $Q_n$, $\psi - f$ will also be analytic in a neighborhood of each $Q_n$.

Now we perform essentially the same construction on the $R_n$'s, being careful to retain analyticity across the $Q_n$'s. Choose $\varepsilon_n > 0$ so that the $\varepsilon_n$-neighborhoods of the $R_n$'s are disjoint. Starting with $\psi_0 = \psi$, construct by induction a sequence $\psi_n$ such that

(i) $\psi_n$ is analytic on a neighborhood of $\Delta \cup R_n$.
(ii) $\psi_n$ is analytic across the arcs of $b\Delta$ across which $\psi_{n-1}$ is analytic.
(iii) $\psi_n - \psi_{n-1}$ is analytic off the $\varepsilon_n$-neighborhood of $R_n$ and satisfies $|\psi_n - \psi_{n-1}| < \varepsilon_1^2$ there.
(iv) $||\psi_n - \psi_{n-1}|| < C_7d$.

This is again possible by Lemma 1. As before we see that the $\psi_n$ converge to a function $h$, uniformly on sets at a positive distance from $bE$, such that $h \in H^\omega(\Delta)$, $h$ extends analytically across each $Q_n$ and across each $R_n$, and $|h - \psi| < (C_7 + 1)d$. Then $h$ is analytic across $E$, and $|h - f| < (C_7 + 2C_1 + 2)d$, so that $h$ is the required function.

**Corollary.** Let $f \in H^\omega(\Delta)$, let $E$ be a subset of $b\Delta$, and let $d > 0$. Suppose that for each $z \in E$, the diameter of the cluster set of $f$ at $z$ is less than $d$. Then there is $h \in H^\omega(\Delta)$ such that $h$ extends to be analytic in a neighborhood of $E$, and

$$\sup_{z \in \Delta} |f(z) - h(z)| < C_0d.$$
Proof. As the diameter of the cluster set of $f$ at $z \in \partial\Delta$ is an upper semicontinuous function of $z$ we can replace $\Delta$ by a larger open set. It is now easy to construct a continuous function satisfying the hypotheses of Lemma 2.

Proof of Theorem 1. If $f$ extends continuously to each point of $\Delta$, then we can take the $d$ of the preceding corollary to be arbitrarily small. The resulting $h$'s will approximate $f$ uniformly on $\Delta$, and they will be analytic on $\Delta$.

Proof of the Corollary to Theorem 1. To show that $\Delta$ is dense in the maximal ideal space of $H^w_{\Delta}$, one must show that if $f_1, \ldots, f_n \in H^w_{\Delta}$ satisfy $|f_1| + \cdots + |f_n| > \delta > 0$ on $\Delta$, then there are $g_1, \ldots, g_n \in H^w_{\Delta}$ satisfying $\sum f_j g_j = 1$. In fact, it suffices to show this for $f_1, \ldots, f_n$ lying in any dense subalgebra of $H^w_{\Delta}$, so that by Theorem 1 we can assume that $f_1, \ldots, f_n$ extend analytically to a neighborhood of $\Delta$. Then there is a simply connected open set $U \supseteq \Delta \cup \Delta$ such that $f_1, \ldots, f_n$ are bounded on $U$ and satisfy $|f_1| + \cdots + |f_n| > \delta/2$ there. By Carleson's theorem, applied to $U$, there are bounded analytic functions $g_1, \ldots, g_n$ on $U$ satisfying $\sum f_j g_j = 1$. Since the $g_j$'s belong to $H^w_{\Delta}$, they are the required functions.

CONCLUDING REMARKS. For a subset $\Delta$ of $\partial\Delta$, let $L^w_{\Delta}$ denote the uniform closure of the functions in $L^w(\partial\Delta)$ which extend continuously to an open set containing $\Delta$. Then $L^w_{\Delta}$ consists of the functions in $L^w$ which are constant on each "fiber" of the maximal ideal space of $L^w$ lying over points of $\Delta$. If we identify functions in $H^w(\Delta)$ with their radial boundary values, we can regard $H^w(\Delta)$ as a subalgebra of $L^w(\partial\Delta)$. Under this identification, $H^w_{\Delta}$ becomes a subalgebra of $L^w_{\Delta}$. In fact, $H^w_{\Delta} = H^w \cap L^w_{\Delta}$, and $H^w_{\Delta}$ is a logmodular subalgebra of $L^w_{\Delta}$ (cf. Détraz [2]).

For $f \in L^w(\partial\Delta)$, we define as usual the distance from $f$ to $L^w_{\Delta}$ by
and we define $d(f, H_E^w)$ similarly. Lemma 2 can be restated as follows.

**Theorem 2.** There is a universal constant $C_0$ such that for all $E \subseteq b\Delta$ and all $f \in H^w(\Lambda)$,

$$d(f, L_E^w) \leq d(f, H_E^w) \leq C_0 d(f, L_E^w).$$

We hope to study the smallest possible constant $C_0$ in another paper.

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**References**


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