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NOTE ON MEAN CONVERGENCE OF EIGENFUNCTION EXPANSIONS

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ABSTRACT. In this paper the mean convergence of series of eigenfunctions of the equation

$$(x^{2\alpha}y')' + \lambda y = 0$$

is studied, completing results of Generozov, (Math. Review #527, (1970)).

I. INTRODUCTION.

1. The problem that we shall consider in this paper is a particular case of the following general question: let $\{\phi_n(x)\}$ be an orthornormal (L^2-) complete system over an interval (x_o,x_1), $-\infty < x_o < x_1 < \infty$, with respect to a measure $\rho(x)dx$. Determine the greatest interval $a < p < b$ or $A \leqslant p \leqslant B$, for which this system is a basis in L^p (= the space of p-integrable functions over (x_o,x_1) with respect to the measure ρdx).

We shall always suppose that $p \in (1,\infty)$ and that $\phi_n \in L^p \cap L^{p^*}$. Then it is easy to see that if $\{\phi_n\}$ is a basis in L^p, the the series $\sum_1^\infty a_n\phi_n$ that converges in norm to $f \in L^p$ is its Fourier series. That is,

$$a_n = a_n(f) = \int_{x_o}^{x_1} f(x)\phi_n(x)\rho dx \quad .$$

It is known (cfr. [6], p. 268) that if for every $f \in L^p$, the Fou-

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rier series of f converges weakly to f, then for every $g \in L^{p^*}$, $1/p + 1/p^* = 1$, the Fourier series of g converges in L^{p^*} to g. Therefore, if $\{\phi_n\}$ is a basis in L^p, it is also a basis in L^{p^*}. So a and b are conjugate exponents. If we know that the system is complete in L^2 and $\int_{x_0}^{x_1} \rho dx < \infty$, the to prove that it is a basis in L^p, it is enough to prove:

$$(1) \qquad \|\sum_1^N a_n(f)\phi_n\|_p \leq C_p \|f\|_p \quad , \quad \forall f \in L^p \quad ,$$

with C_p independent of N and f.

In fact, by the L^2-completeness,

$$(2) \qquad \|\sum_1^N a_n(g)\phi_n - g\|_p \to 0 \quad \text{for} \quad g \in L^p \cap L^2 \ , \ p < 2 \ .$$

Then by (1), (2) holds for all $g \in L^p$. So $\{\phi_n\}$ is a basis in L^p and also in L^{p^*} by the previous argument.

If $\int_{x_0}^{x_1} \rho dx = \infty$, then instead of the L^2-completeness, it is sufficient to ask that (2) holds for $g \in E^p$ = a dense subset of L^p. Then again (1) implies that $\{\phi_n\}$ is a basis in L^p.

The most celebrated particular case is one due to M. Riesz: the system $\{\cos nx, \sin nx, n=0,1,2,\ldots\}$ is a basis in $L^p(0,2\pi)$ for $1 < p < \infty$ (cfr. [18], ch. 7, §3). This is false for $p = 1$.

2. Particular orthonormal systems appear as solutions of Sturm-Liouville problems. The simplest case is that of the equation

$$(3) \qquad y'' + \lambda y = 0$$

with the boundary conditions

$$(4) \qquad \begin{aligned} &y(0)\cos\alpha + y'(0)\sin\alpha = 0 \quad , \\ &y(2\pi)\cos\beta + y'(2\pi)\sin\beta = 0 \quad , \quad -\pi/2 < \alpha , \beta \leq \pi/2 . \end{aligned}$$

For the following particular values of α and β the orthonormal systems are:

case	$\alpha = 0$	, $\beta = 0$	, $\{\sin(nx/2)\}$	, $n = 1,2,\ldots$;
case	$\alpha = 0$	, $\beta = \pi/2$	, $\{\sin((2n+1)x/4\}$	, $n = 0,1,2,\ldots$;
case	$\alpha = \pi/2$	, $\beta = \pi/2$	, $\{\cos(nx/2)\}$	, $n = 0,1,\ldots$;
case	$\alpha = \pi/2$	, $\beta = 0$	, $\{\cos((2n+1)x/4\}$	, $n = 0,1,\ldots$;

It is easy to see that these four systems are bases in L^p for $1 < p < \infty$. For example, we shall prove this for the second system. Let us consider the family of periodic functions of period 2π, odd with respect to π and even with respect to $\pi/2$. The Fourier series of such a function is a series in $\sin(2n+1)x$, $n = 0,1,\ldots$. By the theorem of M. Riesz, it converges to f in $L^p(0,2\pi)$ if $f \in L^p$. On the other hand, such a function can be defined arbitrarily on $[0,\pi/2)$. So, the system just mentioned, restricted to $[0,\pi/2)$, is a basis in $L^p(0,\pi/2)$. This means that $\{\sin((2n+1)x/4\}$; $n = 0,1,\ldots$ is a basis in $L^p(0,2\pi)$. Q.E.D.

A more general theorem holds:

THEOREM 1. *The system of eigenfunctions of the equation*

$$y'' - q(x)y + \lambda y = 0 \quad , \quad 0 \leqslant x \leqslant 2\pi \quad , \tag{5}$$

with the boundary conditions (4) *is a basis in* L^p, $1 < p < \infty$, *if* $q(x)$ *is continuous in* $[0,2\pi]$, ([19]).

In fact, this is a consequence of the preceding result and the following theorem (cfr. [15], ch. I, th 1.9): let $f \in L^1(0,2\pi)$ and call $S_n(f)$ the nth partial sum of its Fourier series with respect to the system of eigenfunctions of the boundary problem (5), (4). Call $\{T_n(f)\}$ the partial sums of the Fourier series of f with respect to the eigenfunctions of equation (3) and boundary conditions (4) but with $|\text{sgn } \alpha|\pi/2$ instead of α and $|\text{sgn } \beta|\pi/2$ instead of β. Then

$$|S_n(f) - T_n(f)| = o(1) \quad \text{uniformly in } x \in [0,2\pi] . \tag{6}$$

Now theorem 1 follows easily. Q.E.D.

3. Other cases of the problem have been treated by several authors. Pollard in [12], [13], [14], considers the cases of series of Legendre, Jacobi, Laguerre and Hermite and his results are complemented by others. In [2] series of Laguerre and Hermite are considered and in [10] an open question left by Pollard is settled. [1] gives a review of the problem and Wing in [17] extends some results of Pollard and investigates Jacobi series. Muckenhoupt in [9] completes results of Pollard and Wing. The intervals where the mean convergence is considered for the polynomials of Jacobi, Laguerre and Hermite are $(-1,1)$, $(0,\infty)$ and $(-\infty,\infty)$, respectively.

Essentially, these are the only orthogonal polynomials generated by a Sturm-Liouville problem: Merlo, [8], proves that if an orthonormal system has the form $\{P_n(x)\sqrt{\rho(x)}$; $n = 0,1,\ldots$ with P_n a polynomial of degree n and is a solution of a problem of Sturm-Liouville, then except for a change of scale the $P_n(x)$ are the polynomials of Jacobi, Laguerre or Hermite. [7] contains some results in this direction which are used by Merlo.

Eigenfunctions of the Bessel equation

$$x^2w'' + x\,w' + (\lambda x^2 - \nu^2)w = 0$$

give place to Fourier-Bessel expansions whose mean convergence has been treated in [17] for $\nu \geqslant -1/2$ and in [3] when $-1 < \nu < -1/2$.

Generozov in [4] studies this for expansions in eigenfunctions of the equation

$$(x^{2\alpha}w')' + \lambda w = 0 \quad , \quad 0 \leqslant \alpha < 1, 0 \leqslant x \leqslant 1, \; w(1)=0 \; , \tag{7}$$

$$w(0)=0 \;\; \text{if} \;\; 0 \leqslant \alpha < 1/2 \; ; \; w \text{ bounded if } \; 1/2 \leqslant \alpha < 1 \; .$$

In this paper we shall continue the study of this equation.

II. THE EQUATION $(x^{2\alpha}w')' + \lambda w = 0$.

4. Let us consider an equation of the form

$$\frac{d}{dx}\left(P\,\frac{dX}{dx}\right) + (\lambda\rho - Q)X = 0 \; , \; 0 < x < 1 \quad , \; \rho \geqslant 0 \; , \; P > 0 \; . \tag{8}$$

If $(\rho/P)^{1/2}$ is integrable on (0,1), then the change of variables

$$t = t(x) = \int_o^x (\rho/P)^{1/2} dx$$

and the change of function

$$u(t) = (\rho P)^{1/4} X = (\rho(x(t)) P(x(t)))^{1/4} X(x(t)) ,$$

transforms equation (8) into

(9) $$u''(t) + (\lambda - \bar{q}(t))u = 0 , \quad 0 < t < T = \int_o^1 (\rho/P)^{1/2} dx ,$$

where $\bar{q}(t) = \frac{Q}{\rho} - \frac{3}{16}(\frac{\rho'}{\rho})^2 - \frac{3}{16}(\frac{P'}{P})^2 + \frac{1}{8}\frac{P'\rho'}{P\rho} + \frac{\rho''}{4\rho} + \frac{P''}{4P}$, $(' = \frac{d}{dt})$

5. If instead of the given transformation we use $\tau = t/T$, the function $v(\tau) = u(T\tau)$, verifies

$$v''(\tau) + T^2(\lambda - \bar{q})v = 0 , \quad 0 < \tau < 1 ,$$

or, what is the same thing

(10) $$v''(\tau) + (\Lambda - q)v = 0 , \quad 0 < \tau < 1 ,$$

where $\Lambda = T^2\lambda$, $q = T^2\bar{q}$.

6. Let X_n be the eigenfunction corresponding to λ_n in equation (8) and to certain boundary conditions, say

(11) $X(1) = X(0) = 0$ or boundedness of X near a singular endpoint.

Then $u_n(t) = (\rho P)^{1/4} X_n(x(t))$ is an eigenfunction for equation (9) and the boundary conditions corresponding to (11). Let $f(t)$ and $F(x)$ be related as $u(t)$ and X: $f = (\rho P)^{1/4} F$. Then

(11') $$\int_o^T f^2 dt = \int_o^1 F^2 \rho\, dx .$$

Also the Fourier coefficients of f with respect to $\{u_n\}$ are the same than that of F with respect to $\{X_n\}$ and ρ. That is

$$f \sim \sum_1^\infty a_n u_n \quad , \quad F \sim \sum_1^\infty a_n X_n \text{ with } a_n = \frac{\int_o^T f(t)u_n(t)dt}{\int_o^T u_n^2 dt} = \frac{\int_o^1 F\rho X_n dx}{\int_o^1 X_n^2 \rho\, dx}$$

It is easy to see then that

$$(12) \qquad \int_o^T |\sum_1^N a_n u_n - f|^2 dt \underset{N\to\infty}{\to} 0 \text{ iff } \int_o^1 |\sum_1^N a_n X_n - F|^2 \rho\, dx \underset{N\to\infty}{\to} 0 \quad ,$$

and in general, for $1 < q < \infty$,

$$(13) \qquad \int_o^T |\sum_1^N a_n u_n - f|^q dt \to 0 \text{ iff } \int_o^1 |\sum_1^N a_n X_n - F|^q \rho^{q/4+1/2}\, P^{q/4-1/2}\, dx \to 0$$

7. Analogous results hold if we consider the equation (10) instead of (9): the eigenfunction associated to (10) and $\Lambda_n = T^2\lambda_n$ is now $w_n(t) = \sqrt{T}\, (\rho P)^{1/4} X_n(x(t))$. Here $t = T^{-1}\int_o^x (\rho/P)^{1/2} dx$.

Let f and F be related in the same way as w and X: $f = T^{1/2}(\rho P)^{1/4}F$. Then

$$(14) \qquad \int_o^1 f^2 dt = \int_o^1 F^2 \rho\, dx \quad ,$$

and the Fourier coefficients of F with respect to $\{X_n\}$ and ρ and that of f with respect of $\{w_n\}$ are the same. Consequently ,

$$(15) \qquad \int_o^1 |\sum_1^N a_n w_n - f|^2 dt \to 0 \text{ iff } \int_o^1 |\sum_1^N a_n X_n - F|^2 \rho\, dx \to 0 \quad ,$$

Since $\int_o^1 |\sum_1^N a_n w_n - f|^q dt = \int_o^1 |\sum_1^N a_n X_n - F|^q \rho^{q/4+1/2}\, P^{q/4-1/2}\, T^{q/2-1}\, dx$,

$$(16) \qquad \int_o^1 |\sum_1^N a_n w_n - f|^q dt \to 0 \text{ iff } \int_o^1 |\sum_1^N a_n X_n - F|^q \rho^{q/4+1/2}\, P^{q/4-1/2} dx \to 0.$$

The preceding considerations are implicitly used in the proofs of part III.

8. An interesting particular case we have if $\rho P = 1$. In that case (13) reads:

$$(17) \qquad \int_0^T |\sum_1^N a_n u_n - f|^q dt \to 0 \quad \text{iff} \quad \int_0^1 |\sum_1^N a_n X_n - F|^q \rho\, dx \to 0 .$$

That is $\{X_n\}$ is a basis in L^q with respect to the measure ρdx iff $\{u_n\}$ is a basis in L^q with respect to the Lebesgue measure. This last property can be often established by theorem 1.

The Tchebicheff polynomials constitute an important example of this case. They satisfy the equation

$$((1-x^2)^{1/2}y')' + (1-x^2)^{-1/2}\lambda y = 0 \quad , \quad -1 < x < 1 \quad ,$$

which is of the form (8) with $P = (1-x^2)^{1/2}$, $Q = 0$, $\rho P = 1$. The corresponding equation (9) is $u''+\lambda u = 0$, $-\pi < t < 0$. So in this case Theorem 1 and the preceding observation assure that the system of Tchebicheff polynomials is a basis in L^q with respect to the measure $(1-x^2)^{-1/2}dx$ for $1 < q < \infty$.

9. Let us consider now the equation (cfr. [4])

$$(18) \qquad (x^{2\alpha}y')' + \lambda y = 0 \quad , \quad 0 < x < 1 \quad ,$$

with one boundary condition $y(1) = 0$.

Let $\alpha < 1$. By the transformation of section 4, which in this case is $t = x^{1-\alpha}/(1-\alpha)$, $u(t) = x^{\alpha/2}y(x(t))$, we get

$$(19) \quad u''(t)-\bar{q}(t)u(t)-\lambda u=0 \quad , \quad 0 < t < 1/(1-\alpha) \quad , \quad u(1/(1-\alpha))=0$$

with

$$(20) \qquad \bar{q}(t) = (\nu^2 - \tfrac{1}{4})/t^2 \quad , \quad \nu = \left|\frac{1-2\alpha}{2(1-\alpha)}\right|$$

Two linearly independent solutions of this equation are

$$(21)\quad \begin{cases} \sqrt{t}\, J_\nu(t\sqrt{\lambda})\ ,\ \sqrt{t}\, J_{-\nu}(t\sqrt{\lambda}) & \text{if}\quad \nu \neq \text{integer}\ , \\ \sqrt{t}\, J_\nu(t\sqrt{\lambda})\ ,\ \sqrt{t}\ \ Y_\nu(t\sqrt{\lambda}) & \text{if}\quad \nu = \text{integer}\ . \end{cases}$$

If we add the boundary condition

$$(22)\quad \begin{cases} y(0) = 0 \quad \text{in case} \quad \alpha < 1/2\ , \\ y \text{ bounded in case} \quad 1/2 \leqslant \alpha < 1\ , \end{cases}$$

we obtain the solutions of (18)

$$(23)\qquad y_n(x) = C\ x^{(1-2\alpha)/2}\ J_\nu(x^{1-\alpha} s_n)\quad ,$$

where s_n are the positive solutions of $J_\nu(s_n) = 0$ and $\lambda_n = s_n^2(1-\alpha)^2 > 0$

If $\alpha \geqslant 1$, *then a solution of* (19) *that vanishes at* $x=1$ *does not belong to* L^2, *so no eigenfunction exists.*

In fact, if $\alpha=1$, then for different values of λ, the general solutions of (18) are:

$$\lambda > 1/4\quad ,\quad y = \frac{C_1}{\sqrt{x}} \sin(\sqrt{\lambda - 1/4}\ \lg x + C_2)\quad ,$$

$$\lambda = 1/4\quad ,\quad y = (C_1 \lg x + C_2)/\sqrt{x}\quad ,$$

$$\lambda < 1/4\ .\ ,\quad y = \frac{C_1}{\sqrt{x}} \sinh(\sqrt{1/4-\lambda}\,\lg x + C_2)\quad .$$

None belongs to L^2. If $\alpha > 1$ we can reduce equation (18) to (19) by the same transformation, only now $-\infty < t < 1/(1-\alpha) < 0$. We consider separately the cases $\lambda > 0$, $\lambda < 0$. In the first case the general solution of (19) is:

$$(24)\qquad u = \sqrt{-t}\{C_1 H_\nu^{(1)}(-t\sqrt{\lambda}) + C_2 H_\nu^{(2)}(-t\sqrt{\lambda})\}\quad .$$

Therefore, the general solution of (18) is

$$(25)\qquad y = x^{1/2-\alpha}\{b_1 H_\nu^{(1)}(x^{1-\alpha}\frac{\sqrt{\lambda}}{\alpha-1}) + b_2 H_\nu^{(2)}(x^{1-\alpha}\frac{\sqrt{\lambda}}{\alpha-1})\}\ .$$

Using the asymptotic formulae for $H_\nu^{(1)}$ and $H_\nu^{(2)}$, we obtain

$$(26)\qquad y(x) \sim x^{1/2-\alpha}(x^{(\alpha-1)/2}b_1'\ e^{ix^{1-\alpha}\sqrt{\lambda}/(\alpha-1)}+b_2'\ e^{-ix^{1-\alpha}\sqrt{\lambda}/(\alpha-1)})=$$

$$= x^{-\alpha/2}(b_1'\ e^{ix^{1-\alpha}\sqrt{\lambda}/(\alpha-1)}+b_2'\ e^{-ix^{1-\alpha}\sqrt{\lambda}/(\alpha-1)}) .$$

This function does not belong to L^2 unless $b_1' = b_2' = 0$.

If $\lambda < 0$ the general solution of (18) is

$$(27)\qquad y = x^{1/2-\alpha}\{B_1\ H_\nu^{(1)}(x^{1-\alpha}\ e^{i\pi/2}\sqrt{-\lambda}/(\alpha-1)) +$$

$$+ B_2\ H_\nu^{(2)}(x^{1-\alpha}e^{i\pi/2}\sqrt{-\lambda}/(\alpha-1))\} ,$$

again using the asymptotic formulae for $H_\nu^{(i)}$, we get,

$$(28)\qquad y \sim x^{-\alpha/2}(B_1'\ e^{-x^{1-\alpha}\sqrt{-\lambda}/(\alpha-1)} + B_2'\ \ e^{x^{1-\alpha}\sqrt{-\lambda}/(\alpha-1)})$$

Since $H_\nu^{(1)}(ix) \neq 0$ for real $x > 0$, (cfr. Watson [16], pp. 78 and 511), the solution that vanishes at $x = 1$ must correspond to some $b_2 \neq 0$. But then (28) does not belong to L^2. We have proved so that:

If equation (18) *admits a system of eigenfunctions which is a basis in* L^2, *then* $\alpha < 1$.

Generozov proved in [4] for $0 \leqslant \alpha < 1$ the following theorem (cfr. fig. 1).

THEOREM 2. *Let* $-\infty < \alpha < 1$. *The system* $\{y_n/\ \|y_n\|_2\ ;\ n = 1,2,\dots\}$ *(see* (23)*) of eigenfunctions of the equation*

$$(29)\qquad (x^{2\alpha}y')' + \lambda y = 0 ,$$

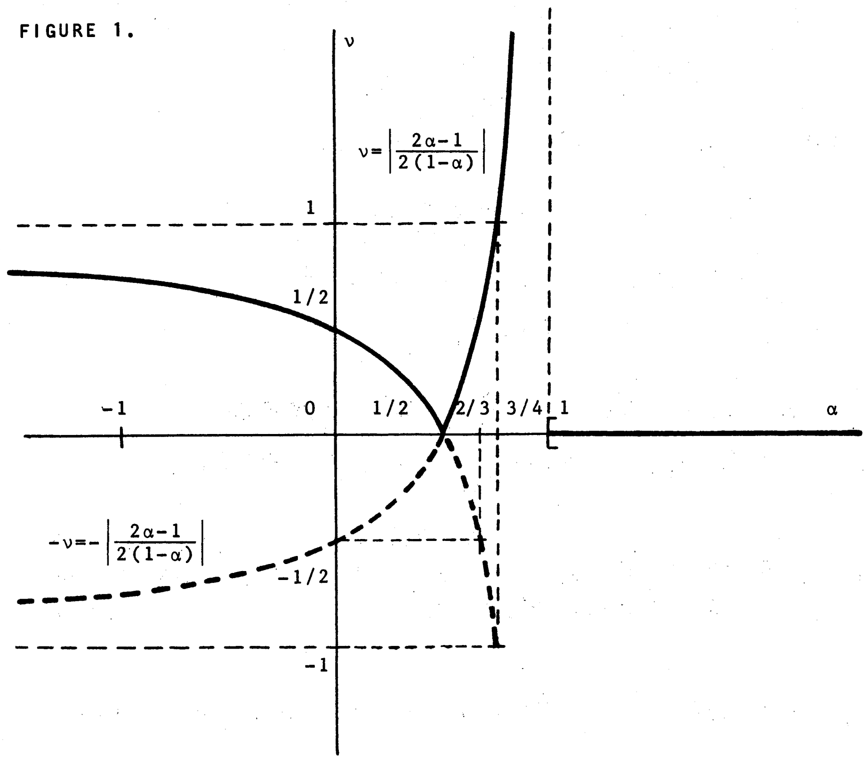
with boundary conditions,

$y(1) = 0$; $y(0) = 0$ *for* $-\infty < \alpha < 1/2$, y *bounded for* $1 > \alpha \geqslant 1/2$, *is a basis in* $L^p(0,1)$ *for*

$$(30)\qquad \frac{2}{2-(\alpha\vee 0)} < p < \frac{2}{\alpha\vee 0}$$

($\alpha \vee 0$ means $\sup(\alpha,0)$).

FIGURE 1.



10. We consider now, for $\alpha < 3/4$, the system of functions

$$(31) \qquad y_n(x) = x^{(1-2\alpha)/2} J_{-\nu}(x^{1-\alpha}\sqrt{\lambda_n}/(1-\alpha)) \ , \ \nu = \left|\frac{1-2\alpha}{2(1-\alpha)}\right|$$

which are solutions of equation (18) with $\lambda = \lambda_n$, determined by the condition $y_n(1) = 0$, (cfr. figure 1).

(For $1 > \alpha \geqslant 3/4$, (23) *gives the only solution of* (18) *wich belongs to* L^2*).*

These functions verify at the origin

(32) $$xy_n'(x) - cy_n(x) = O(x^{c+2(1-\alpha)}) .$$

where $c = \min(1-2\alpha,0)$. *These conditions uniquely determine, but for a constant factor, the solutions* (31).

(32) implies the orthogonality of these functions as we shall see below and might be considered as the boundary condition at the origin. So, *the functions* y_n *defined by* (31) *are also solutions of a a Sturm-Liouville problem.*

The orthogonality of $\{y_n\}$ follows from,

$$(\lambda_m-\lambda_n)\int_\varepsilon^1 y_n y_m dx = [(x^{2\alpha}y_n')y_m-(x^{2\alpha}y_m')y_n]_\varepsilon^1 =$$

$$= -\varepsilon^{2\alpha}y_n'(\varepsilon)y_m(\varepsilon) + \varepsilon^{2\alpha}y_m'(\varepsilon)y_n(\varepsilon) =$$

$$= -\varepsilon^{2\alpha-1}[(\varepsilon y_n'(\varepsilon)-cy_n(\varepsilon))y_m(\varepsilon) - (\varepsilon y_m'(\varepsilon)-cy_m(\varepsilon))y_n(\varepsilon)] =$$

$$= \text{by}(32) = \varepsilon^{2\alpha-1}O(\varepsilon^{c+2(1-\alpha)})O(\varepsilon^c) = O(\varepsilon^{2c+1}) = o(1) ,$$

(recall $\alpha < 3/4$).

The following theorem holds:

THEOREM 3. *Let* $\alpha < 3/4$. *The orthonormal system* $\{y_n/\|y_n\|_2\}$, y_n *defined by* (31), *is a basis in* $L^p(0,1)$ *if*

i) $$\frac{2}{2-\alpha} < p < \frac{2}{\alpha} \quad \textit{when} \quad 2/3 \geqslant \alpha \geqslant 0 ,$$

ii) $\frac{1}{2(1-\alpha)} < p < \frac{1}{2\alpha-1}$ *when* $3/4 > \alpha > 2/3$,

iii) $1 < p < \infty$, $\alpha < 0$

We consider first the case $p = 2$.

LEMMA 1. *The orthonormal systems of theorems 2 and 3 are complete in* L^2.

Proof. This follows from the equality

$$\int_0^1 f(x)\, x^{(1-2\alpha)/2} J_\nu(x^{1-\alpha} s_n)\, dx = \frac{1}{1-\alpha}\int_0^1 f(t^{1/(1-\alpha)})\, t^{\alpha/2(1-\alpha)} J_\nu(ts_n)\, t^{1/2} dt$$

and the fact that when $\nu > -1$, $\{J_\nu(ts_n)t^{1/2}\}$ is a complete system in L^2, (cfr. [3] and the references there mentioned).

III. PROOF OF THE MAIN RESULTS.

11. *Proof of theorem 2.* We follow the same line of proof as Generozov, [4]. Improved estimations will permit to cover a greater range of α. Let $-\infty < \alpha < 1$, and $\nu \geqslant -1/2$. Actually we shall prove that the system $y_n(x) = x^{(1-2\alpha)/2} J_\nu(x^{1-\alpha} s_n)$, s_n = positive zero of $J_\nu(x)$, is a basis in L^p for $\frac{2}{2-(\alpha \vee 0)} < p < \frac{2}{\alpha \vee 0}$.

Then, by taking $\nu = \left|\frac{1-2\alpha}{2(1-\alpha)}\right|$ we have theorem 2 and by taking $\nu = -\left|\frac{1-2\alpha}{2(1-\alpha)}\right|$ we have i) of theorem 3.

Since by lemma 1, the system $y_n(x)$ is complete in L^2, it is enough to prove for our purpose that the Dirichlet kernel

$$D_N(x,\xi) = \sum_{n=1}^{N} \frac{y_n(x)\, y_n(\xi)}{\|y_n\|_2^2}$$

verifies

(33) $$\int_0^1 \left| \int_0^1 D_N(x,\xi) F(x)dx \right|^p d\xi \leq C_p \int_0^1 |F(\xi)|^p d\xi \quad ,$$

for $2/(2-(\alpha V0)) < p < 2/(\alpha V0)$. Let us call

(34) $$y_n(t) = \sqrt{t}\, J_\nu(ts_n) = t^{\alpha/2(1-\alpha)} y_n(t^{1/(1-\alpha)}) \quad ,$$

(35) $$\bar{D}_N(t,\tau) = \sum_{n=1}^{N} \bar{y}_n(t)\, \bar{y}_n(\tau)/\|y_n\|_2^2 \quad .$$

Making the change of variables $x = t^{1/(1-\alpha)}$, $\xi = \tau^{1/(1-\alpha)}$ in (33) and observing that $\|\bar{y}_n\|_2^2 = (1-\alpha)\|y_n\|_2^2$, we get (cf. §4-7):

(36) $$\int_0^1 \left| \int_0^1 \bar{D}_N(t,\tau) \left(\frac{t}{\tau}\right)^{\alpha/2(1-\alpha)} F(t^{1/(1-\alpha)})dt \right|^p \tau^{\alpha/(1-\alpha)} d\tau \leq$$

$$\leq C'_p \int_0^1 |F(t^{1/(1-\alpha)})|^p\, t^{\alpha/(1-\alpha)} dt \quad .$$

Finally calling $f(t) = F(t^{1/(1-\alpha)})t^{\alpha/p(1-\alpha)}$, we see that (33) holds iff

(37) $$\int_0^1 \left| \int_0^1 \bar{D}_N(t,\tau) \left(\frac{t}{\tau}\right)^{(\frac{1}{2}-\frac{1}{p})\frac{\alpha}{1-\alpha}} f(t)dt \right|^p d\tau \leq C'_p \int_0^1 |f(t)|^p dt \quad .$$

But, (cfr. [17]),

(38) $$\bar{D}_N(t,\tau) = \frac{\sqrt{t\tau}\Lambda_N}{2} \frac{J_\nu(\Lambda_N t) J_{\nu+1}(\Lambda_N \tau)}{\tau - t} + \frac{\sqrt{t\tau}\Lambda_N}{2} \frac{J_{\nu+1}(\Lambda_N t) J_\nu(\Lambda_N \tau)}{t-\tau} +$$

$$+ \frac{O(1)}{\tau + t} + \frac{O(1)}{2-t-\tau} \quad ,$$

where $\Lambda_N = (N+\frac{\nu}{2}+\frac{1}{4})\pi$.

Since $\sqrt{t}\, J_\nu(t)$ is bounded for $\nu \geq -1/2$, the first two terms are of the form $h(\tau)k(t)/(\tau-t)$, where h and k are bounded independently of N. Then, to prove that (37) holds it is enough to prove that the kernels

$$\left(\frac{t}{\tau}\right)^\beta \frac{1}{\tau - t} \quad , \quad \left(\frac{t}{\tau}\right)^\beta \frac{1}{\tau + t} \quad , \quad \left(\frac{t}{\tau}\right)^\beta \frac{1}{2-\tau-t} \quad ,$$

with $\beta = (\frac{1}{2}-\frac{1}{p})\frac{\alpha}{1-\alpha}$ define continuous operators in $L^p(0,1)$.

The operator

$$(39) \qquad If(\tau) = \int_0^1 (\frac{t}{\tau})^\beta \frac{f(t)}{2-t-\tau}\, dt \quad ,$$

is continuous in $L^p(0,1)$ if (cf. [3] ,§6),

$$(40) \qquad \beta \vee 0 < 1/p < 1 \wedge (1+\beta)$$

In our case (40) reads

$$0 \vee \frac{\alpha}{1-\alpha}(\frac{1}{2}-\frac{1}{p}) < \frac{1}{p} < 1 + [\frac{\alpha}{1-\alpha}(\frac{1}{2}-\frac{1}{p}) \wedge 0]$$

or equivalently

$$(41) \qquad \frac{2}{\alpha \vee 0} > p > \frac{2}{2-(\alpha \vee 0)} \quad .$$

Also the operators

$$(42) \qquad \text{p.v.} \int_0^1 (\frac{t}{\tau})^\beta \frac{f(t)}{t \pm \tau}\, dt$$

are continuous in $L^p(0,1)$ (cf. [5] or [11]) for $-1/q < \beta < 1/p$, $1 < p < \infty$, $1/p+1/q = 1$, which is just (40) and then in our case (41). Q.E.D.

12. *Proof of theorem 3.* In the proof of theorem 3 we need the following auxiliary result.

LEMMA 2. *The operator*

$$(43) \qquad \text{p.v.}\int_{-\infty}^{+\infty} \frac{1+|\tau|^{-a}}{1+|t|^{-a}} |\frac{t}{\tau}|^\beta \frac{f(t)}{t-\tau}\, dt = \text{p.v.}\int_{-\infty}^{+\infty} \frac{1+|\tau|^{a}}{1+|t|^{a}} |\frac{\tau}{t}|^{-\beta-a} \frac{f(t)}{t-\tau}\, dt$$

is continuous in $L^p(-\infty,\infty)$ *if* $-1/p < -\beta$, $-a-\beta < 1/q$.

It is an easy consequence of the lemma in p. 308 of Muckenhoupt's paper, [9] .

(It can be proved using Theorem 3 of [3] with $r(x,t) = |x/t|^{1/pq}$).

To prove the theorem we first observe that we have already proved i) in §11. For the rest of the theorem we follow the proof of the orem we follow the proof of theorem 2 but taking $-1 < \nu < -1/2$, $\nu = -|1-2\alpha|/|2(1-\alpha)|$. Then again $\{y_n(x)\}$ is a basis in $L^p(0,1)$ if (37) holds. Now the estimation (38) of the Dirichlet kernel $\bar{D}_N(t,\tau)$ does not hold but instead we have (cf. [3] ,§4):

$$(44)\qquad \bar{D}_N(t,\tau) = K(A_N,t;\tau) + K(A_N,\tau,t) + H(A_N,t,\tau) + H(A_N,\tau,t) + $$

$$+ \frac{O(1)}{2-t-\tau} + O(1)(t\tau)^{\nu+1/2} \quad ,$$

where $K(A,\tau,t) = J_\nu(A\tau)J_{\nu+1}(tA)\ \dfrac{A(t\tau)^{1/2}}{2(t-\tau)}$,

$$H(A,\tau,t) = J_\nu(A\tau)J_{\nu+1}(tA)\ \frac{A(t\tau)^{1/2}}{2(t+\tau)} \quad .$$

To prove (37) for some p, it is then enough to see that each term in the right hand side of (44) multiplied by $|t/\tau|^\beta$, $\beta = (1/2 - 1/p)\alpha/(1-\alpha)$, defines a continuous operator in $L^p(0<\tau<1)$, whose norm is uniformly bounded in N.

We call these operators respectively K,K^*,H,H^*,I,J, and show that:

a) K and H are uniformly continuous in $L^p(0<\tau<1)$ for

$$\frac{1}{2(1-\alpha)} < p < \frac{2}{\alpha} \quad \text{when} \quad \frac{2}{3} < \alpha < \frac{3}{4} \quad ,$$

$$1 < p < \infty \quad \text{when} \quad \alpha < 0 \quad ,$$

b) K^* and H^* are uniformly continuous in L^p for

$$\frac{2}{2-\alpha} < p < \frac{1}{2\alpha-1} \quad \text{when} \quad \frac{2}{3} < \alpha < \frac{3}{4} \quad ,$$

$$1 < p < \infty \quad \text{when} \quad \alpha < 0 \quad ,$$

c) I is continuous in $L^p(0,1)$ for $\frac{2}{2-(\alpha\vee 0)} < p < \frac{2}{\alpha\vee 0}$, $\alpha < 1$,

d) J is continuous in L^p for

$$\frac{1}{2(1-\alpha)} < p < \frac{1}{2\alpha-1} \quad \text{when} \quad \frac{2}{3} < \alpha < \frac{3}{4} ,$$

$$1 < p < \infty \quad \text{when} \quad \alpha < 0 .$$

Then for $\alpha \leqslant 0$, there is no restriction on p and iii) follows. If $2/3 < \alpha < 3/4$, the interval $1/2(1-\alpha) < p < 1/(2\alpha-1)$ given by d), contained in the remaining intervals given by a), b) and c), and ii) follows.

c) we have already proved in §11, by considering operator (39).

d) The operator

(45) $$\int_0^1 \tau^{\nu+\frac{1}{2}-\beta}\, t^{\nu+\frac{1}{2}+\beta} f(t)\,dt$$

is continuous in L^p if $t^{\nu+\frac{1}{2}+\beta} \in L^q(0,1)$ and $t^{\nu+\frac{1}{2}-\beta} \in L^p(0,1)$.

That is, if

(46) $$\nu + \frac{1}{2} - \beta > -\frac{1}{p} \quad \text{and} \quad \nu + \frac{1}{2} + \beta > -\frac{1}{q} .$$

Replacing ν by $-|2\alpha-1|/|2(1-\alpha)|$ and β by $(1/2 - 1/p)\alpha/(1-\alpha)$, (46) is equivalent to

(47) $$\begin{cases} \frac{1}{2(1-\alpha)} < p < \frac{1}{2\alpha-1} & \text{if} \quad \frac{2}{3} < \alpha < \frac{3}{4} , \\ 1 < p < \infty & \text{if} \quad \alpha < 0 , \end{cases}$$

That proves d).

b) We consider the operators

(48) $$\text{p.v.}\int_0^\infty \frac{J_{\nu+1}(At)J_\nu(A\tau)}{t\pm\tau} (At.A\tau)^{1/2} \left(\frac{t}{\tau}\right)^{(\frac{1}{2}-\frac{1}{p})\frac{\alpha}{1-\alpha}} f(t)\,dt .$$

An easy change of variables shows that their norms as operators in $L^p(0,\infty)$ do not depend on A. So we take A = 1 and determine the p's for which (48) is continuous in $L^p(0,1)$.

Let us call

(49) $$a = -(\nu+\tfrac{1}{2}) \quad , \quad \beta = (1/2 - 1/p)\alpha/(1-\alpha) .$$

Since $0 < a < 1/2$, we have:

(50) $$\begin{cases} x^{1/2}J_\nu(x) = O(1)(1+x^{-a}) & , \quad 0 < x < \infty , \\ x^{1/2}J_{\nu+1}(x) = O(1)(1+x^{-a})^{-1}, & 0 < x < \infty . \end{cases}$$

This implies that the operators defined by (48) are continuous in $L^p(0,\infty)$ whenever the operator (43) with a and β given by (49) is continuous.
It is easily verified that it is true for any $p \in (1,\infty)$ if $\alpha < 0$. When $\alpha \in (2/3,3/4)$, the mentioned operators are continuous if $1/(2\alpha-1) > p > 2/(2-\alpha)$. This is the condition to which reduce those of lemma 2.

a) follows from b) by duality if we observe that K^* with $\beta = (\frac{1}{2}-\frac{1}{p})\frac{\alpha}{1-\alpha}$ as operator in L^p is the adjoint operator of K with $\beta = (\frac{1}{2}-\frac{1}{q})\frac{\alpha}{1-\alpha}$ as operator in L^q, (cf. [3], §6). Q.E.D.

ADDED IN PROOF. Theorem 1 still holds even when one only requires that $q(x) \in L^1$:

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