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SOME LOCAL RESULTS ON THE INDUCED $\bar{\partial}$-OPERATOR

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§1. INTRODUCTION.

In a recent note [5], I proved an approximation theorem of the following type: if M is a C^∞ submanifold of C^n satisfying certain convexity conditions at a point $x \in M$, then any solution of the induced Cauchy-Riemann equations on M near x, can be uniformly approximated on M by holomorphic functions defined on open sets containing M. This theorem was proved under some restrictions on the dimension of M; to be precise, it was assumed that CR codim M $\leqslant 2$ (see §2 for definitions). The main object of the present paper is to prove a result of the same type (Theorem 4.3), without any restriction on the dimension of M, and under convexity conditions that are rather more general than those in [5]. I have also added a paragraph (§8) on some of the consequences of the approximation theorem. For example, under the right convexity conditions, M will present the phenomenon of Hans Lewy [4], that is, a solution of the induced Cauchy-Riemann equations on M will be locally extendible to a holomorphic function on some open set of C^n.

In §2 we give the basic definitions of CR (Cauchy-Riemann) manifolds and CR functions (C^∞ solutions of the induced Cauchy-Riemann equations on M). The convexity condition on M is given in §3, and in §4 we state the approximation theorem.

The next three paragraphs consist of a proof of this result.

The method of proof is quite similar to the methods that were used in [6], with one basic difference: in [6] we used the L^2 estimates for the $\bar{\partial}$ operator on a Stein domain, whereas here we will be also concerned with the support of the solution of the $\bar{\partial}$ equation, in the sense that we will have to keep the support of successive solutions away from M. It is in that process that the convexity of M is needed.

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§2. CAUCHY-RIEMANN MANIFOLDS.

As we will be concerned with local results, we consider M^r to be a C^∞ real submanifold of C^n of real dimension r. Let $T_x(M)$ be the tangent space to $M = M^r$ at $x \in M$; it is a r-dimensional real linear subspace of C^n, and we denote by $H_x(M)$ the maximal complex linear subspace of $T_x(M)$, and by $h_x(M)$ its complex dimension. We have

$$(2.1) \qquad \max(0,r-n) \leqslant h_x(M) \leqslant [r/2] \qquad \text{for} \quad x \in M .$$

DEFINITION 2.1. *We say that* M *is a* CR *submanifold of* C^n *if and only if* $h_x(M)$ *is constant on each connected component of* M. *If* M *is a connected* CR *submanifold,* $h = h_x(M)$ *is called the* CR *dimension of* M. *If* $h = 0$, M *is said to be totally real.*

If M is a CR submanifold of C^n and CR dim(M) > 0, we consider on M the induced $\bar\partial$ operator, which we write $\bar\partial_M$ (see [2], [6]). If a C^∞ function f satisfies the equations $\bar\partial_M f = 0$ near a point $x \in M$, we say that f is a CR function at x.

LEMMA 2.2. *Let* M *be a* CR *submanifold of* C^n *of* CR *dimension* m. *If* $x \in M$ *and* $\rho_1,\ldots,\rho_\ell$ *are real valued* C^∞ *functions defined on a neighborhood* N *of* x *in* C^n *such that*

(i) $\quad d\rho_1 \wedge \ldots \wedge d\rho_\ell \neq 0 \quad$ *on* $\quad M \cap N$

(ii) $\quad M \cap N = \{x \in N \; ; \; \rho_i(x) = 0 \; , \; i = 1,\ldots,\ell\}$,

then there are $k = n - m =$ CR codim (M) *functions among the* ρ, *say* $\rho_1,\ldots,\rho_k$, *such that*

(2.2) $$\bar\partial\rho_1 \wedge \cdots \wedge \bar\partial\rho_k \neq 0$$

on some neighborhood of x *in* M,*and* $f \in C^\infty(M)$ *is a* CR *function at* x *if and only if for any extension* $\tilde f \in C^\infty(N)$, *with* $\tilde f|M \cap N = f$, *we have*

$$\bar\partial\tilde f \wedge \bar\partial\rho_1 \wedge \cdots \wedge \bar\partial\rho_k \equiv 0$$

on some neighborhood of x *in* M.

The proof is elementary and we omit it.

§3. THE NOTION OF CONVEXITY.

Let V be an open subset of M, where M is a CR submanifold of CR dimension m in C^n. We say that $f \in C^\infty(V)$ is a CR function on V if it is CR at every point $x \in V$. Now let $x \in M$ and let $\{\rho_1,\dots,\rho_\ell\}$ be as in Lemma 2.2. Let U be an open neighborhood of x in C^n, where the ρ's are defined and $d\rho_1 \wedge \cdots \wedge d\rho_\ell \neq 0$ on U. For real numbers $t_1,\dots,t_\ell$ we write

$$V(t_1,\dots,t_\ell) = \{x \in U \; ; \; \rho_i(x) = t_i \; , \; i = 1,\dots,\ell\} \quad ,$$

so that

$$V(0,\dots,0) = V = U \cap M \; .$$

We will restrict our attention to systems $\{\rho_1,\dots,\rho_\ell\}$ for which the family of submanifolds $V(t_1,\dots,t_\ell)$ is locally CR trivial. More precisely, we make the

DEFINITION 3.1. *Let* $\{\rho_1,\dots,\rho_\ell\}$ *be functions defining* M *near* $x \in M$ *as in Lemma 2.2. We say that the system* $\{\rho_1,\dots,\rho_\ell\}$ *is* <u>*CR regular*</u> *for* M *at* x *if and only if there is an open neighborhood* U *of* x *in* C^n *such that* $\rho_1,\dots,\rho_\ell$ *are in* $C^\infty(U)$ *and* $d\rho_1 \wedge \cdots \wedge d\rho_\ell \neq 0$ *in* U, *and an open neighborhood* W *of the origin in* R^ℓ, *and a* C^∞ *diffeomorphism*

$$\Phi : W \times V \to U \quad .$$

such that

(i) $\Phi(0) = \Phi|\{0\} \times V$ = Identity on V

(ii) $\Phi(t_1,\dots,t_\ell) = \Phi|\{(t_1,\dots,t_\ell)\} \times V$ *is an isomorphism of the* CR *structures of the* CR *submanifolds* V *and* $V(t_1,\dots,t_\ell)$, *for every* $(t_1,\dots,t_\ell) \in \overset{\circ}{W}$.

Here the notion of CR isomorphism is the natural one, and in (ii) we are of course assuming that $V(t_1,\dots,t_\ell)$ is a CR submanifold for $(t_1,\dots,t_\ell) \in W$.

We observe first of all that CR regularity at x is invariant under holomorphic changes of coordinates. Secondly, for any CR submanifold M, and any point $x \in M$, there are CR regular systems for M at x. For there is a neighborhood U of x in C^n and real co ordinates $\{x_1,\dots,x_{2n}\}$ in U such that

$$M \cap U = \{x \in U \; ; \; x_i - y_i(x_{\ell+1},\dots,x_{2n}) = 0 \; , \; i=1,\dots,\ell\} \; , \tag{3.1}$$

where $y_i \in C^\infty(U)$. Since translations are holomorphic diffeomorphisms, hence CR isomorphisms, the system $\rho_i = x_i - y_i(x_{\ell+1},\dots,x_{2n})$, $i = 1,\dots,\ell$ is indeed CR regular at x.

With respect to a CR regular system, a CR function satisfies a stronger condition than the one in Lemma 2.2. We have:

LEMMA 3.2. *Let* $\{\rho_1,\dots,\rho_\ell\}$ *be a* CR *regular system for* M *at* x, *and let* $\rho_1,\dots,\rho_k$ (k=CR codim M) *satisfy* $\bar\partial\rho_1 \wedge \dots \wedge \bar\partial\rho_k \neq 0$ *at* x. *Then for every sufficiently small open neighborhood* V *of* x *in* M, *there is an open set* U *in* C^n *with* $U \cap M = V$ *and such that for any* CR *function* f *on* V *we can find an extension* $\tilde f \in C^\infty(U)$ *of* f *to* U *satisfying*

$$\bar\partial\tilde f \wedge \bar\partial\rho_1 \wedge \dots \wedge \bar\partial\rho_k \equiv 0 \quad on \quad U \; . \tag{3.2}$$

Proof. We may take V small enough so that the mapping Φ of Definition 3.1 is defined on it, and a small enough neighborhood of the origin in R^l, $W' \subset W$, so that on $\Phi(W' \times V)$ we still have $\bar\partial\rho_1 \wedge \dots \wedge \bar\partial\rho_k \neq 0$.

Then we define $\tilde f$ on $U = \Phi(W' \times V)$ as follows:

$$\tilde f | V(t_1,\dots,t_\ell) = f \cdot \Phi^{-1}(t_1,\dots,t_\ell) \quad \text{for} \quad (t_1,\dots,t_\ell) \in W'$$

Then $\tilde f$ is a CR function on each CR submanifold $V(t_1,\dots,t_\ell)$; $(t_1,\dots,t_\ell) \in W'$, and (3.2) is a consequence of Lemma 2.2.

The notion of convexity that we will need is the following:

DEFINITION 3.3. *A* CR *submanifold* M *is said to be strongly pseudoconvex at* $x \in M$ *if there is a* CR *regular system for* M *at* x, $\{\rho_1,\dots,\rho_\ell\}$ ($\ell = \text{codim}_R M$) *such that the functions* $\rho_1,\dots,\rho_\ell$ *are strictly plurisubharmonic at* x.

It is not hard to see that if there is a real C^∞ hypersurface containing M which is strongly pseudoconvex at $x \in M$, then M is strongly pseudoconvex at x in the sense of Definition 3.3.

§4. UNIFORM APPROXIMATION WITH HOLOMORPHIC FUNCTIONS.

We first intruduce some notation. For a compact set K, C(K) will stand for the uniform algebra of complex valued continuous functions on K. If K is a compact subset of C^n, we write $\Gamma(K,\mathcal{O})$ for the algebra of sections over K of $\mathcal{O}$, the sheaf of germs of holomorphic functions in C^n, and A(K) for the closure of the natural image of $\Gamma(K,\mathcal{O})$ in C(K). Similarly, if M is a CR submanifold of C^n and K is a compact subset of M, $\mathcal{O}_M$ denotes the sheaf of germs of CR functions on M, and $A_M(K)$ is the closure of $\Gamma(K,\mathcal{O}_M)$ in C(K).

The following result was proved in [6]:

THEOREM 4.1. *Let* K *be a compact subset of* M, *where* M *is a totally real submanifold of* C^n *(i.e.:* CR dim M = 0*). Then* A(K) = C(K).

In the case where M is an hypersurface, which implies that M is a CR submanifold of CR codimension 1, we have a local theorem that was also proved in [6]:

THEOREM 4.2. *Let* M *be a* C^∞ *hypersurface, and let* $x \in M$. *Then there is a fundamental system of compact neighborhoods* {K} *of* x *in* M *such that if* $K \in \{K\}$ *we have* $A(K) = A_M(K)$.

The main object of this paper is to prove an analogous result for higher CR codimension, but here we will need the extra assumption of convexity at x.

THEOREM 4.3. *Let* M *be a* CR *submanifold of* C^n *and let* $x \in M$. *Suppose* M *is strongly pseudoconvex at* x *in the sense of Definition 3.3. Then there is a fundamental system of compact neighborhoods* {K} *of* x *in* M *such that if* $K \in \{K\}$ *we have* $A(K) = A_M(K)$.

The following three paragraphs will be denoted to the proof of this theorem.

§5. EXTENDING CR FUNCTIONS FROM CR SUBMANIFOLDS.

Let I be a multi-index $(i_1,\dots,i_N)$, where the i's are chosen from among the integers $1,\dots,k$. We set $\|I\| = N$. Also if we have k numbers $\rho_1,\dots,\rho_k$, we set $\rho_I = \rho_{i_1}\cdot\rho_{i_2}\cdots\rho_{i_N}$, and

$$\sum_{\|I\|=N} \rho_I = \sum_{i_1,\dots,i_N=1}^{k} \rho_{i_1}\cdots\rho_{i_N}$$

If we are given a k-tuple of positive integers $\beta = (\beta_1,\dots,\beta_k)$, we set

$$|\beta| = \beta_1 + \dots + \beta_k$$

and

$$\rho^\beta = \rho_1^{\beta_1} \cdots \rho_k^{\beta_k} .$$

The following lemma is essentially proved in §3 of [6], but we state it here in a slightly different form convenient for our purposes, and then give a sketch of the proof for the sake of completeness.

LEMMA 5.1. *Let* $\{\rho_1,\dots,\rho_\ell\}$ *be a* CR *regular system for* M *at* x, *and let* $\rho_1,\dots,\rho_k$ (k = CR codim M) *satisfy* $\bar\partial\rho_1 \wedge \dots \wedge \bar\partial\rho_k \neq 0$ *at* x. *Then for every sufficiently small open neighborhood* V *of* x *in* M *there is an open set* U *in* C^n *with* $U \cap M = V$ *such that the following is true: for any* CR *function* f *on* V *and any positive integer* q, *there is a function* $u \in C^\infty(U)$ *satisfying:*

(i) $u|V = f$

(ii) $\bar\partial u \wedge \bar\partial\rho_1 \wedge \dots \wedge \bar\partial\rho_k \equiv 0$ *on* U

(iii) $\bar\partial u = \sum_{|\beta|=q} h_\beta\, \rho^\beta$ *on* U, *where* h_β *are* $C^\infty(0,1)$ *forms on* U, *and* $\rho^\beta = \rho_1^{\beta_1}\dots\rho_k^{\beta_k}$ *as above.*

Proof. We take V and U as in the proof of Lemma 3.2, and take $u_o = \tilde f$, which satisfies conditions (i) and (ii). Hence on U we have

$$\bar\partial u_o = \sum_{i=1}^{k} h_i\, \bar\partial\rho_i \quad , \quad h_i \in C^\infty(U) \ .$$

Applying $\bar\partial$ to this equation we get

$$\sum_{i=1}^{k} \bar\partial h_i \wedge \bar\partial\rho_i \equiv 0 \quad \text{on} \quad U \ ,$$

and therefore

$$\bar\partial h_i = \sum_{j=1}^{k} h_{ij}\, \bar\partial\rho_j \quad \text{on} \quad U \quad , \ i = 1,\dots,k \ ; \ h_{ij} \in C^\infty(U) \ .$$

We define inductively h_I for any multi-index of the type describe at the beginning of this §, and we observe that the h_I are symmet ric, i.e., remain unchanged when one permutes the indices of I

The desired function is given then by

$$u = u_o + \sum_{m=1}^{q} \frac{(-1)^m}{m!} \sum_{\|I\|=m} h_I \rho_I$$

Q.E.D.

Obseve that (iii) means that we can choose u so that the components of $\bar{\partial}u$ vanish on M to any prescribed order.

§6. THE $\bar{\partial}$ OPERATOR ON L^2 FORMS.

In this number we will be concerned with the $\bar{\partial}$ cohomology with restrictions on the support of the differential forms. In stating the result, we will not aim at full generality, but will restrict ourselves to what is needed in §7.

LEMMA 6.1. *Let* ϕ *be a* C^∞ *real valued strictly plurisubharmonic function on some open set of* C^n. *Let* z_o *be such that* $\phi(z_o) = 0$. *Then there is a fundamental system* $\{\Omega\}$ *of Stein open neighborhoods of* z_o *such that the following is true:*

For a real number t *set* $R(t) = \{x \in \Omega \; ; \; \phi(x) \leqslant t\}$. *Let* t_1 *and* t_2 *be two real numbers such that* $R(t_1)$ *and* $R(t_2)$ *are non-empty, with* $t_1 < t_2$. *Let* f *be a differential form of type* (0,q) *with* $1 \leqslant q < n$ *in* $L^2(R(t_2))$ *such that*

$\bar{\partial}f = 0$ *on* $R(t_2)$ *in the sense of distributions*

$\mathrm{supp}(f) \subset R(t_1)$.

Then there is a differential form of type (0,q-1) *in* $L^2(R(t_2))$ *satisfying:*

(1) $\bar{\partial}u = f$ *on* $R(t_2)$ *in the sense of distributions*

(2) $\mathrm{supp}(u) \subset R(t_1)$

(3) $\|u\| \leqslant C \; \|f\|$,

where the norms are the L^2 *norms on* $R(t_2)$*, and the constant* C *is independent of* f*, and of* t_1 *and* t_2.

The proof of this result is omitted here, but will be included in a forthcoming paper. It is akin to the arguments in §3 of [1].

We observe that as a corollary of the above Lemma, it is not difficult to verify that the same result holds if we replace ϕ by the supremum of a finite number of such functions.

§7. PROOF OF THEOREM 4.3.

Let $\ell = \operatorname{codim}_R M$, $k = \text{CR codim } M$, and let $\{\rho_1,\dots,\rho_\ell\}$ be a strictly plurisubharmonic CR regular system at $x \in M$. Assume that $\bar\partial\rho_1 \wedge \dots \wedge \bar\partial\rho_k \neq 0$ at x, and let Ω be a Stein open neighborhood of x in C^n such that $K = \overline{\Omega \cap M}$ is contained in a neighborhood V of x in M for which the assertions of Lemmas 3.2, 5.1 and 6.1 hold. Then if f is a CR function on some neighborhood V' of K in M, it has an extension u to a neighborhood U' of V' in C^n which satisfies (i), (ii) and (iii) of Lemma 5.1. The order of the vanishing of u on M, which we call τ, will be chosen at the end of the proof.

Consider now a family of functions $\psi_t(x)$ of a real variable x in $C^\infty(R)$, with $t \in R$, $t > 0$, such that:

(7.1)

(1) $\psi_t(x) = 1$ for $|x| \leqslant t$

(2) $\psi_t(x) = 0$ for $|x| \geqslant 2t$

(3) $0 \leqslant \psi_t(x) \leqslant 1$ for every x and every t

(4) $\left|\dfrac{d^m \psi_t}{dx^m}(x)\right| \leqslant C\, t^{-m}$ for some constant C.

Let

$$\psi_{t,i} = \psi_t(\rho_i) \qquad i = 1,\dots,\ell \quad .$$

We use the following notation: for a multi-index J with $\|J\| = N$, and differential forms $f_{j_1}, \ldots, f_{j_N}$, we write

$$\bigwedge_j f_j = \bigwedge_{j \in J} f_j = f_{j_1} \wedge \ldots \wedge f_{j_N} \; ; \; \bigwedge_J f_j = 1 \quad \text{if} \quad N = 0 .$$

Let now J_k be a multi-index of length $\|J\| = k$ whose indices are chosen from the numbers $\{1, \ldots, \ell\}$ and are all distinct. We claim that

$$(7.2) \qquad \Big(\prod_{i \notin J_k} \psi_{t,i} \Big) \cdot \Big(\bigwedge_{J_k} \bar\partial \psi_{t,j} \Big) \wedge \bar\partial u \equiv 0 \quad \text{on} \quad \Omega$$

for $t > 0$ sufficiently small. In fact, by choosing t small enough, we can make the expression in (7.2) vanish on $\Omega - \Omega \cap U$ and on $\Omega \cap U'$ it vanishes because $\bar\partial \psi_{t,j} = \psi_t'(\rho_j)\bar\partial\rho_j$, $\{\bar\partial\rho_1, \ldots, \bar\partial\rho_k\}$ is a maximal linearly independent subset of $\{\bar\partial\rho_1, \ldots, \bar\partial\rho_\ell\}$ in U', and u satisfies (ii) of Lemma 5.1 on U'.

For a multi-index J_{k-1} of length k-1 which as before, has distinct indices from among $\{1, \ldots, \ell\}$, we define the (0,k) form on Ω (for t small enough)

$$(7.3) \qquad \xi(t, J_{k-1}) = \prod_{\substack{i=1,\ldots,\ell \\ i \notin J_{k-1}}} \psi_{t,i} \cdot \bigwedge_{J_{k-1}} \bar\partial \psi_{t,j} . \bar\partial u \quad .$$

Equation (7.2) implies that the forms $\xi(t, J_{k-1})$ are $\bar\partial$-closed on Ω.

At this point we make the observation that we may assume that $k < n$, for if $k = n$, M is totally real, and the result is contain ed in Theorem 4.1.

Consider now the support of $\xi(t, J_{k-1})$. To this end we introduce the following sets: for a real number s and $i = 1, \ldots, \ell$, define

$$R_i(s) = \Omega \cap \{\rho_i \leqslant s\} \quad ,$$

and for a multi-index J as above, write

$$R_J(s) = \bigcap_{i\in J} R_i(x) \quad ; \quad R_J(s) = \Omega \quad \text{if} \quad \|J\| = 0.$$

Then from (7.3) we see that the support in Ω of $\xi(t,J_{k-1})$ is contained in

$$(7.4) \qquad R_{J_{k-1}}(-t) \cup (\Omega - R_{J_{k-1}}(t)) .$$

Suppose that $k > 1$. Because of our assumptions on the functions $\rho_1,\dots,\rho_\ell$, we can assume that Ω is so small to begin with, that all these functions are strictly plurisubharmonic on a neighborhood of $\bar{\Omega}$, and so will be the maximum of any number of them. By Lemma 6.1 we can find forms $\omega(t,J_{k-1})$ of type $(0,k-1)$ such that:

(7.5)

(1) $\omega(t,J_{k-1}) \in L^2(\mathring{R}(t))$, where $R(t) = \bigcap_{i=1}^{\ell} R_i(t)$

(2) $\bar{\partial}\omega(t,J_{k-1}) = \xi(t,J_{k-1})$ on $\mathring{R}(t)$

(3) $\operatorname{supp}\omega(t,J_{k-1}) \subset R_{J_{k-1}}(-t)$

(4) $\|\omega(t,J_{k-1})\| < C\,\|\xi(t,J_{k-1})\|$

(5) $\omega(t,J_{k-1})$ is anti-symmetric in the indices of J_{k-1}

Pick now a multi-index J_{k-2} of the type described above, with $\|J_{k-2}\| = k-2$.

We will denote by $i\,J_{k-2}$ the multi-index of length $k-1$ obtained by adjoining i at the beginning of J_{k-2}. Then the $(0,k-1)$ L^2-forms

$$(7.6) \qquad \xi(t,J_{k-2}) = \prod_{\substack{i=1,\dots,\ell \\ i\notin J_{k-2}}} \psi_{t,i} \cdot \bigwedge_{J_{k-2}} \bar{\partial}\psi_{t,j} \cdot \wedge\bar{\partial}u - \sum_{\substack{i=1,\dots,\ell \\ i\notin J_{k-2}}} \omega(t,iJ_{k-2})$$

are $\bar{\partial}$-closed on $\mathring{R}(t)$, by (2) of (7.5).

Also, the support of $\xi(t,J_{k-2})$ in $\mathring{R}(t)$ is contained in $R_{J_{k-2}}(-t)$, by (3) of (7.5). So, as before we can find forms $\omega(t,J_{k-2})$ of type (0,k-2) such that:

(7.5)'

(1) $\omega(t,J_{k-2}) \in L^2(\mathring{R}(t))$

(2) $\bar\partial\omega(t,J_{k-2}) = \xi(t,J_{k-2})$ on $\mathring{R}(t)$

(3) $\mathrm{supp}\,\omega(t,J_{k-2}) \subset R_{J_{k-2}}(-t)$

(4) $\|\omega(t,J_{k-2})\| < C\,\|\xi(t,J_{k-2})\|$

(5) $\omega(t,J_{k-2})$ is anti-symmetric in indices of J_{k-2}.

(Observe that (5) of (7.5)' is a consequence of (5) of (7.5)).

If k = 2 , we stop here. If instead $k > 2$, we proceed by taking any multi-index J_{k-3} of length k-3 of aforementioned type. Then by (7.6) we have:

$$\bar\partial\left[\prod_{\substack{i=1,\dots,\ell\\ i\notin J_{k-3}}}\psi_{t,i}\cdot\bigwedge_{J_{k-3}}\bar\partial\psi_{t,j}\wedge\bar\partial u\right] =$$

$$= \sum_{\substack{h=1,\dots,\ell\\ h\notin J_{k-3}}}\prod_{\substack{i=1,\dots,\ell\\ i\notin hJ_{k-3}}}\psi_{t,i}\cdot\bigwedge_{hJ_{k-3}}\bar\partial\psi_{t,j}\wedge\bar\partial u =$$

$$= \sum_{\substack{h=1,\dots,\ell\\ h\notin J_{k-3}}}\xi(t,hJ_{k-3}) + \sum_{\substack{h=1,\dots,\ell\\ h\notin J_{k-3}}}\sum_{\substack{i=1,\dots,\ell\\ i\notin h\,J_{k-3}}}\omega(t,ihJ_{k-3})\ .$$

The double sum vanishes because of (5) of (7.5). Therefore, the (0,k-2) forms

$$\xi(t,J_{k-3}) = \prod_{\substack{i=1,\dots,\ell\\ i\notin J_{k-3}}}\psi_{t,i}\cdot\bigwedge_{J_{k-3}}\bar\partial\psi_{t,j}\wedge\bar\partial u - \sum_{\substack{h=1,\dots,\ell\\ h\notin J_{k-3}}}\omega(t,hJ_{k-3})$$

are in $L^2(\mathring{R}(t))$ and $\bar\partial$-closed on $\mathring{R}(t)$, and their support is contained in $R_{J_{k-3}}(-t)$, by (7.5)'.

Continuining in this way, we finally arrive at a $\bar\partial$-closed (0,1) form in $L^2(\mathring{R}(t))$:

$$(7.7) \qquad \prod_{i=1}^{\ell} \psi_{t,i} \cdot \bar\partial u - \sigma(t) = \zeta_t$$

where $\sigma(t)$ vanishes identically on a neighborhood of $M \cap \Omega$. In view of (7.1), we can make the L^2-norm of the form in (7.3) arbitrarily small by taking t small, if we only choose τ, the order of the vanishing on $\bar\partial u$ on M, big enough. Because of conditions (4) in (7.5) and (7.5)', the same is true of the successive forms $\xi(t,J_{k-i})$ and $\omega(t,J_{k-i})$. Hence the L^2-norm of (7.7) can be made arbitrarily small in $\mathring{R}(t)$ by taking a small t. We then solve the equation

$$(7.8) \qquad \bar\partial v_t = \prod_{i=1}^{\ell} \psi_{t,i} \cdot \bar\partial u - \sigma(t) = \zeta_t \quad \text{on} \quad \mathring{R}(t)$$

with

$$(7.9) \qquad \|v_t\| \leqslant C \, \|\zeta_t\| \quad .$$

The function $u - v_t$ is as a result holomorphic on some neighborhood of $M \cap \Omega$ (where $\sigma(t) \equiv 0$ and $\psi_{t,i} \equiv 1$ for all i). The difference between this function and u itself, is $-v_t$ on $M \cap \Omega$, and it is an easy matter to see that the supremum of $|v_t|$ on any compact subset of $M \cap \Omega$ can be made arbitrarily small if we take t to be small.

This is a consequence of the fact that we can take τ as big as we please, of (7.8), (7.9) and Sobolev's Lemma. The details can be either found in [6], or be provided by the reader. The proof is complete.

§8. SOME APPLICATIONS.

a) Let M be a CR submanifold of C^n. Let H(M) be the vector bundle over M whose fiber over $x \in M$ is $H_x(M) \otimes_R C$. Let $\mathcal{L}(M)$ be the Lie algebra generated by the sections of H(M) over M, which we *assume* to have constant dimension, so that $\mathcal{L}(M)$ is the algebra of sections of a vector bundle V. We define

$$e = \text{fiber } \dim_C \frac{V}{H(M)} \quad . \tag{8.1}$$

On the other hand, given a compact subset K of C^n, and connected subset K' of C^n with $K \subset K'$, $K \neq K'$, we say that K is extendible to K' if the restriction map

$$\Gamma(K',\mathcal{O}) \longrightarrow \Gamma(K,\mathcal{O})$$

is onto.

Greenfield [2] has proved the following result:

THEOREM 8.1. *If* M *is a "generic"* CR *submanifold of* C^n *(i.e., if equality holds in the first inequality of* (2.1), *so that* $\max(0,r-n) = h_x(M)\ \forall x \in M$), *and if* $\dim_R M > n$, *then* M *is locally extendible to a set containing a differentiable manifold* N, *with* $\dim N = \dim M + e$. *If* $e = 0$, *then* M *is locally holomorphically convex.*

(For the definitions of local extendibility and holomorphic convexity, see [2]).

In [3] this theorem is proved for real analytic CR submanifolds of C^n which are not necessarily generic, and also for non-generic C^∞ CR submanifolds with CR codim = 1. In these cases, one also has that M is not locally extendible to a set of dimension greater than dim M + e. For the proof in the case of a non generic C^∞ CR submanifold of CR codimension 1, Greenfield makes use of Theorem 4.2, and points out that local approximation of CR functions by germs of holomorphic functions implies a theorem like Theorem 8.1. Making use of Theorem 4.3 we get the following

THEOREM 8.2. *Let* M *be a* CR *submanifold of* C^n *which is strongly pseudoconvex at* $x \in M$*, and suppose* $\dim_R M > n$. *Then if* $e > 0$ *,* M *is locally extendible at* x *to a set containing a differentiable manifold* N*, with* $\dim N = \dim M+e$ *, and is not locally extendible to a set of dimension bigger than* $\dim M+e$. *If* $e = 0$ *, then* M *is locally holomorphically convex at* x.

The proof of this theorem (assuming the result of Theorem 4.3) is essentially described in [3] , and we omit it here.

b) Under the assumptions of Theorem 8.2, with $e > 0$, we get a stronger extendibility result, of the type known as "H. Lewy's phenomenon". We say that the set $K \subset M$ is CR extendible to K' (with K and K' as in (a)), if the restriction map

$$A(K') \longrightarrow \Gamma(K,\mathcal{O}_M)$$

is onto. (Recall the definitions at the beginning of §4).

Then, as a consequence of Theorems 4.3 and 8.2 we get:

THEOREM 8.3. *Let* M *be a* CR *submanifold of* C^n *which is strongly pseudoconvex at* $x \in M$*, and suppose* $\dim_R M > n$. *Then if* $e > 0$ *, and* K *is a sufficiently small compact neighborhood of* x *in* M*,* K *is* CR *extendible to a set* K' *which contains a differentiable manifold of dimension* $\dim M+e$.

Proof. Let $f \in \Gamma(K,\mathcal{O}_M)$. By Theorem 4.3, if K is sufficiently small, we can find a sequence f_j , $j = 1,2,\ldots$ of elements in $\Gamma(K,\mathcal{O})$ which converges uniformly to f on K. By Theorem 8.2, each of these germs f_j can be extended to germs $\hat{f}_j$ in $\Gamma(K',\mathcal{O})$, for some set $K' \supset K$ which contains a differentiable manifold of dimension $\dim M+e$. Then we observe that the uniform convergence of $\{f_j\}$ on K implies the uniform convergence of $\{\hat{f}_j\}$ on K'. This is a simple consequence of the well-known fact that the extension of germs on K to germs on K' is unique. Hence $\{\hat{f}_j\}$ converges uniformly to some element $\hat{f} \in A(K')$, and it is obvious that $\hat{f}|K = f$.

Q.E.D.

When M is such that the extension of germs can be made to a set

K' containing an open set of C^n, we have the exact analogue of the phenomenon of H. Lewy. In particular, if M is generic, and e is as big as possible, that is, if V in (8.1) is the whole tangent bundle $T(M) \otimes C$, then we have dim M+e = 2n, and K' will contain an open set of C^n. In this situation, a CR function f defined in a small neighborhood K of x in M (M is strongly pseudoconvex at x), can be extended to an holomorphic function defined on some open set U of C^n, such that K lies on its boundary.

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