DISTRIBUTION OF POTENTIAL IN A SPHERICAL RING

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ABSTRACT. As an illustration of the use of Legendre transforms in mathematical physics, Churchill [1] has studied the distribution of potential interior to the unit sphere with boundary conditions of third type at the surface. In this paper the distribution of potential interior to a spherical ring with boundary conditions of third type at the surface is found, by means of the theory of integral transforms. The ring is defined by:

\[ 0 < r_1 < r < r_2, \ 0 < \theta_1 < \theta < \theta_2 < \pi, \ 0 < \phi < 2\pi \]

INTRODUCTION. Our purpose is to find the distribution of potential in a spherical ring, without sources in the interior and with symmetry with respect to the angle \( \phi \). In this case the Laplace's equation:

\[ \nabla^2 V(r,x) = \frac{\partial}{\partial r} \left[ r \frac{\partial V(r,x)}{\partial r} \right] + \frac{\partial}{\partial x} \left[ (1-x^2) \frac{\partial V(r,x)}{\partial x} \right] = 0 \]

where \( x = \cos \theta \)

must be solved with the following boundary conditions:

\[ \left. \begin{array}{l}
\alpha_1 \frac{\partial V(r,x)}{\partial x}(r,x) + \alpha_2 V(r,x) \bigg|_{x=x_1} = f_1(r) \\
\beta_1 \frac{\partial V(r,x)}{\partial x}(r,x) + \beta_2 V(r,x) \bigg|_{x=x_2} = f_2(r)
\end{array} \right\} \]

where \(-1 < x_2 < x_1 < 1\)
An integral transform which satisfies the boundary condition (2) can be obtained.

The general solution of the equation:

\[ \frac{d}{dx} \left( (1-x^2) \frac{d}{dx} \right) y(x) + \lambda y(x) = 0 \]

is the function:

(4) \[ y(x;\lambda) = C u(x;\lambda) + D v(x;\lambda) \]

where:

\[ u(x;\lambda) = \sum_{n=0}^{\infty} A_{2n}(\lambda) x^{2n} \]
\[ v(x;\lambda) = \sum_{n=0}^{\infty} A_{2n+1}(\lambda) x^{2n+1} \]

being:

\[ A_{2n}(\lambda) = \frac{1}{(2n)!} \prod_{k=1}^{n} \frac{[(2k-2)(2k-1)-\lambda]}{[(2k-2)(2k-1)-\lambda]} \]
\[ A_{2n+1}(\lambda) = \frac{1}{(2n+1)!} \prod_{k=1}^{n} \frac{[(2k-1)2k-\lambda]}{[(2k-1)2k-\lambda]} \]

Using the convention that for \( n = 0 \) is: \( A_{2n}(\lambda) = A_{2n+1}(\lambda) = 1 \)

The constants \( C \) and \( D \) can be determined applying to the solution (4) the boundary conditions:

\[ \alpha_1 y'(x_1) + \alpha_2 y(x_1) = 0 \]
\[ \beta_1 y'(x_2) + \beta_2 y(x_2) = 0 \]

where the primes denote differentiation with respect to \( x \). So that:
\[ C_i = \alpha_1 v'(x_i;\lambda_i) + \alpha_2 v(x_i;\lambda_i) + \beta_1 v'(x_2;\lambda_i) + \beta_2 v(x_2;\lambda_i) \]

\[ D_i = -[\alpha_1 u'(x_i;\lambda_i) + \alpha_2 u(x_2;\lambda_i) + \beta_1 u'(x_2;\lambda_i) + \beta_2 u(x_2;\lambda_i)] \]

being \( \lambda_i \) the positive roots of the transcendental equation:

\[ \Delta = \begin{vmatrix} \alpha_1 u'(x_1;\lambda) + \alpha_2 u(x_1;\lambda) & \alpha_1 v'(x_1;\lambda) + \alpha_2 v(x_1;\lambda) \\ \beta_1 u'(x_2;\lambda) + \beta_2 u(x_2;\lambda) & \beta_1 v'(x_2;\lambda) + \beta_2 v(x_2;\lambda) \end{vmatrix} = 0 \]

We now define the finite integral transform:

\[ T(F(x)) = \bar{F}(\lambda_i) = \int_{x_1}^{x_2} F(x) S(x;\lambda_i) \, dx \]

whose kernel is:

\[ S(x;\lambda_i) = C_i u(x;\lambda_i) + D_i v(x;\lambda_i) \]

Because of the orthogonality of the functions \( S(x;\lambda_i) \) in the closed interval \([x_1, x_2]\) we have the inversion theorem:

\[ F(x) = \sum_{i} \frac{\bar{F}(\lambda_i)}{r_i} S(x;\lambda_i) \]

where:

\[ r_i = \int_{x_1}^{x_2} \left[ C_1^2 Z(x, 2m, 2n, 1) + 2C_1 D_1 Z(x, 2m, 2n+1, 2) + D_1^2 Z(x, 2m+1, 2n+1, 3) \right] \, dx \]

being:

\[ Z(x, am+b, cn+d, s) = \sum_{m,n} \frac{A_{am+b}(\lambda_i) A_{cn+d}(\lambda_i)}{am+cn+s} x^{am+cn+s} \]

Integrating by parts, we find that:

\[ T\left( \frac{d}{dx} \left( (1-x^2) \frac{d}{dx} \right) F(x) \right) = \frac{1-x^2}{\beta_1} S(x_2;\lambda_i)[\beta_1 F'(x_2) + \beta_2 F(x_2)] - \frac{1-x^2}{\alpha_1} S(x_1;\lambda_i)[\alpha_1 F'(x_1) + \alpha_2 F(x_1)] - \lambda_i T(F(x)) \]
SOLUTION OF THE PROBLEM.

By means of (5), and taking into account (7), equation (1) is transformed into:

\[
(8) \quad r^2 \frac{d^2 \bar{V}}{dr^2}(r; \lambda_1) + 2r \frac{d \bar{V}}{dr}(r; \lambda_1) - \lambda_1 \bar{V}(r; \lambda_1) = \bar{h}(r; \lambda_1)
\]

where:

\[
\bar{h}(r; \lambda_1) = \frac{1-x_1^2}{\alpha_1} S(x_1; \lambda_1) f_1(r) - \frac{1-x_2^2}{\alpha_1} S(x_2; \lambda_1) f_2(r)
\]

The solution of (8) with the boundary conditions (3) is:

\[
\bar{V}(r; \lambda_1) = H(r; \lambda_1) + k_1(\lambda_1) r^{-\frac{1}{2} + \sqrt{\lambda_1 + \frac{1}{4}}} + k_2(\lambda_1) r^{-\frac{1}{2} - \sqrt{\lambda_1 + \frac{1}{4}}}
\]

being:

\[
H(r; \lambda_1) = \frac{1}{2\sqrt{\lambda_1 + \frac{1}{4}}} \left[ \int r^{-\frac{1}{2} + \sqrt{\lambda_1 + \frac{1}{4}}} \bar{h}(r; \lambda_1) r^{-\frac{1}{2} - \sqrt{\lambda_1 + \frac{1}{4}}} \, dr - \int r^{-\frac{1}{2} - \sqrt{\lambda_1 + \frac{1}{4}}} \bar{h}(r; \lambda_1) r^{-\frac{1}{2} + \sqrt{\lambda_1 + \frac{1}{4}}} \, dr \right]
\]

\[
k_1(\lambda_1) = \frac{Q(\gamma_1, \gamma_2, r_1) P(\delta_1, \delta_2, r_2, -) - Q(\delta_1, \delta_2, r_2, -) P(\gamma_1, \gamma_2, r_1, -)}{P(\gamma_1, \gamma_2, r_1, +) P(\delta_1, \delta_2, r_2, -) - P(\gamma_1, \gamma_2, r_1, -) P(\delta_1, \delta_2, r_2, +)}
\]

\[
k_2(\lambda_1) = \frac{Q(\delta_1, \delta_2, r_2) P(\gamma_1, \gamma_2, r_1, +) - Q(\gamma_1, \gamma_2, r_1, +) P(\delta_1, \delta_2, r_2, +)}{P(\gamma_1, \gamma_2, r_1, +) P(\delta_1, \delta_2, r_2, -) - P(\gamma_1, \gamma_2, r_1, -) P(\delta_1, \delta_2, r_2, +)}
\]

with:

\[
P(p, q, r_j, \pm) = p \left( \frac{1}{2} \pm \sqrt{\lambda_1 + \frac{1}{4}} \right) r_j^{\frac{3}{2}} + q r_j
\]

\[
Q(p, q, r_j) = \tilde{g}_j(\lambda_1) - p H'(r_j; \lambda_1) - q H(r_j, \lambda_1)
\]
where the primes denote differentiation with respect to $r$.

Finally, using the inversion theorem (6), we obtain the solution:

$$V(r, x) = \sum_i \frac{S(x; \lambda_i)}{r_i} \left\{ \frac{H(r, \lambda_i)}{r} \left( k_1(\lambda_i) r^{\frac{1}{2} + \sqrt{\lambda_i}} + k_2(\lambda_i) r^{\frac{1}{2} - \sqrt{\lambda_i}} \right) \right\}$$

BIBLIOGRAPHY
