THE FUNCTORS $K^n$ FOR THE RING OF A CURVE

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We freely use the notations and terminology of [KV].

In this note we compute the groups $K^n(A)$, $n > 0$, for an affine curve $C$, where $A$ is the affine ring of $C$.

As it has been proved by Bass [1], these groups are zero for $C$ non singular, hence the nonzero values depend on the singularities of $C$.

We prove essentially two results:

1) $K^1(A) = 0$ for $i \geq 2$ (dimension theorem?)

2) If $C$ has singular points $P_\alpha$ and $s_\alpha$ is the number of branches of $C$ passing through $P_\alpha$, then $K^1(A) = \mathbb{Z}(s_\alpha - 1)$, where $\mathbb{Z}$ denotes the additive group of integers.

Let $k$ be a field, $A$ an affine one dimensional integral $k$-algebra and $m_1, \ldots, m_h$ those maximal ideals in $A$ such that the local rings $A_{m_i}$ are not regular. Let $\bar{A}$ be the integral closure of $A$ in its quotient field and $c = \text{Ann}(\bar{A}/A)$ the conductor of $\bar{A}$ in $A$.

**Lemma 1.** If $B$ is an affine finite dimensional integral $k$-algebra and $m$ a maximal ideal, then $B_m$ is not integrally closed if and only if $m$ contains the conductor of $B$ in $B$.

**Proof.** See [4], Ch. II, § 2.2d.

Since we are dealing with one dimensional $k$-algebras, the lemma says that the maximal ideals $m$ such that $A_m$ is not regular are ex
actly those containing the conductor $c$. If $m_i$ is one of such ideals, $m_i \overline{A}$ is an ideal in $\overline{A}$ and we may call $B_i$ the (finite) set of maximal ideals in $\overline{A}$ containing $m_i \overline{A}$. Hence $B = \bigcup B_i$ is the set of maximal ideals in $\overline{A}$ containing $c$. In fact, if $I$ is a maximal ide-}

al in $\overline{A}$, $I \supseteq c$, then $J = I \cap A \supseteq c$ and $J$ is maximal in $A$. Let us call $r_i = \bigcap_{M \in B_i} M$, $r = \bigcap_{M \in B} M$.

Remark that the radical of $\overline{A_m} = \overline{A} \otimes_A A_m$ is $r_i \overline{A_m}$, and the con-

ductor of $A_m$ in $\overline{A_m}$ is $c \otimes A_m$.

Let now $r = \bigcap r_i \subseteq \overline{A}$. Then $r \supseteq c$ and there exists an exponent
$s_i$ such that $(r_i \otimes A_{m_i})^{s_i} \subseteq c \otimes A_{m_i}$. If $s = \max(s_i)$, then $r^s \subseteq c$, 
hence $r/c$ is a nilpotent ideal in $\overline{A}/c$. Since $\bigcap m_i / c = r$, we also have that $\bigcap m_i / c$ is a nilpotent ideal in $A/c$.

**LEMMA 2.** Let $B$ be a ring with identity and $J \subseteq B$ a nilpotent i-
deal. Then $K^i(J) = 0$ for every $i \geq 0$.

**Proof.** $J^*$ (i.e., the ring obtained from $J$ by adding an identity) has a nilpotent ideal $J$ and $J^*/J = k$ is a regular ring. By ap-
plying [1], Th. 10.1, Ch. XII we obtain $K^i(J^*) = 0$ for every $i > 0$ and besides, $K^0(J^*) \cong K^0(J^*/J) \cong K^0(k) = \mathbb{Z}$. Hence $K^0(J) = 0$.

**LEMMA 3.** If $c$ is contained in exactly $p$ different maximal ideals
of $\overline{A}$, then $K^i(r) = 2^{p-1}$, $K^i(r) = 0$ for every $i \geq 2$.

**Proof.** Let $M_1, \ldots, M_p$ be the different maximal ideals of $\overline{A}$ which contain $c$ and $J_q = \bigcap_{1}^{q} M_i$, $1 \leq q \leq p$. We shall prove by induction on $q$ that $K^i(J_q) = 2^{q-1}$, $K^i(J_q) = 0$ for $i \geq 2$. This implies the lemma since $J_p = r$.

If $q = 1$, we have the exact sequence

$$0 \longrightarrow M_1 \longrightarrow \overline{A} \longrightarrow \overline{A/M_1} \longrightarrow 0$$
which gives us

\[ K^0(M_1) \rightarrow K^0(\overline{A}) \xrightarrow{\alpha} K^0(\overline{A}/M_1) \rightarrow K^1(M_1) \rightarrow K^1(\overline{A}) \rightarrow 0 \rightarrow \]
\[ \cdots \rightarrow 0 \rightarrow K^i(M_1) \rightarrow K^i(\overline{A}) \rightarrow 0 \rightarrow \cdots \]

since \( \overline{A}/M_1 \) is a field.

Besides, \( K^0(\overline{A}/M_1) = \mathbb{Z} \) and \( \alpha \) is the rank map, i.e., for a finitely generated \( \overline{A} \)-projective module \( P \), \( \alpha(P) = \text{rk} P \), hence \( \alpha \) is surjective and, since \( \overline{A} \) is regular, \( K^i(\overline{A}) = 0 \) for \( i > 0 \), so we obtain \( K^i(M_1) = 0 \) for \( i > 0 \).

Assume then to have \( K^1(J_q) = \mathbb{Z} q^{-1} \), \( K^i(J_q) = 0 \) for \( i > 1 \), \( 1 \leq q < p \), and consider the exact sequence

\[ 0 \rightarrow J_{q+1} \rightarrow J_q \xrightarrow{\beta} k' \rightarrow 0 \]

where \( k' = \overline{A}/M_{q+1} \) and \( \beta \) is surjective since \( J_{q+1} \neq J_q \).

We have then

\[ K^0(J_{q+1}) \rightarrow K^0(J_q) \xrightarrow{\alpha} K^0(k') \rightarrow K^1(J_{q+1}) \rightarrow K^1(J_q) \rightarrow \]
\[ 0 \rightarrow \cdots \rightarrow 0 \rightarrow K^i(J_{q+1}) \rightarrow K^i(J_q) \rightarrow 0 \rightarrow \cdots \]

\((i > 1) \) since \( K^i(k') = 0 \) for \( i > 0 \) because \( k' \) is a field.

So, we have \( K^i(J_{q+1}) = 0 \) for \( i > 1 \) and to finish the proof we must show that \( \alpha \) is the zero map.

Since \( \overline{A} \) is a \( k \)-algebra, by taking \( J_q^* \) by adding \( k \) to \( J_q \), we have \( J_q \subset J_q^* \subset \overline{A} \) and the natural map \( \overline{A} \rightarrow \overline{A}/M_{q+1} = k' \) induces the commutative diagram

\[
\begin{array}{ccc}
J_q & \xrightarrow{i} & J^*_q \\
\downarrow{\beta} & & \downarrow{\bar{\beta}} \\
k & & k'
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow{\gamma} & & \\
k' & & \\
\end{array}
\]
and, obviously, $\bar{\beta}$ induces the rank map $K^0(J_+^q) \to K^0(k')$.

Since $K^0(J_+^q) = Ker[K^0(J_+^q) \to K^0(k)]$ and $K^0(\beta) = K^0(\gamma) K^0(\delta) K^0(i)$, $
\alpha = K^0(\beta) = 0$ is the zero map.

**Lemma 4.** Let $B$ be a commutative ring with connected spectrum (i.e., without non trivial idempotents), $M_1, \ldots, M_p$ maximal ideals in $B$, $\alpha : K^0(B) \to K^0(B/\cap M_i) = \bigoplus_1^p K^0(B/M_i) \cong \mathbb{Z}^p$, $\alpha = K^0(B)$, $\beta$ the canonical map $B \to B/\cap M_i$. Then $\Im \alpha$ is the diagonal $\Delta$ of $\mathbb{Z}^p$. If $P$ is a finitely generated projective $B$-module and $[P]$ its class in $K^0(B)$, then $\alpha[P] = (h, h, \ldots, h)$, $h = \text{rk } P$.

**Proof.** If $P$ is a finitely generated projective module it is obvious that the image of $[P]$ in $K^0(B/\cap M_i)$ is $\text{rk } P \otimes_B B/M_i$, but, as it is well known, $\text{rk } P \otimes_B B/M_i = \text{rk } P \otimes_B B_{M_i}$ where $B_{M_i}$ is the local ring at $M_i$. Since Spec$B$ is connected, $\text{rk } P \otimes_B B_{M_i}$ is independent of $M_i$ and equal to $\text{rk } P$, hence the lemma is proved.

**Theorem.** If $h$ is the number of maximal ideals of $A$ containing $c$ and $p$ the number of maximal ideals of $A$ containing $c$, then

$K^1(A) = \mathbb{Z}^{p-h}$, $K^i(A) = 0$ for every $i > 1$.

**Proof.** From the exact sequences

$0 \to c \to r \to r/c \to 0$

and

$0 \to c \to \cap M_i \to \cap M_i/c \to 0$

since $r/c$ and $\cap M_i/c$ are nilpotent, we obtain $K^i(c) = K^i(r) = K^i(\cap M_i)$ for all $i > 0$, and

$0 \to \cap M_i \to A \to A/\cap M_i \to 0$

gives, using that $K^i(A/\cap M_i) = \bigoplus K^i(A/m_i) = 0$ if $i > 0$, that

$K^i(A) = K^i(\cap M_i)$ for $i > 2$, so
\( K^i(A) = K^i(r) = 0 \) for \( i > 2 \), by lemma 3.

To compute \( K^1(A) \) we consider the commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & \cap m_i & \rightarrow & A & \rightarrow & A/\cap m_i & \rightarrow & 0 \\
& & \downarrow & \searrow & i & \downarrow & \& \\
0 & \rightarrow & r & \rightarrow & A & \rightarrow & A/r & \rightarrow & 0
\end{array}
\]

induced by the inclusion \( i : A \rightarrow A \). Hence

\[
\begin{array}{ccccccc}
K^0(\cap m_i) & \rightarrow & K^0(A) & \varphi_0 & K^0(A/\cap m_i) & \gamma & K^1(\cap m_i) & \rightarrow & K^1(A) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K^0(r) & \rightarrow & K^0(\bar{A}) & \varphi_0 & K^0(\bar{A}/r) & \delta & K^1(r) & \rightarrow & 0
\end{array}
\]

is exact and commutative.

If \( \{N_{i,j}\} \) is the set of maximal ideals of \( \bar{A} \) containing \( \bar{A}.m_i \), the images of \( K^0(A/m_i) \rightarrow \bigoplus_j K^0(A/N_{i,j}) \), \( \varphi_0 \) and \( \psi_0 \) are the diagonals of the codomains \( \mathbb{Z}^n \), hence they are direct summands of such codomains.

Since \( \text{Im} \varphi_0 = \text{Im} \psi_0 = \mathbb{Z} \) we have

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z}^h & \gamma & \mathbb{Z}^{p-1} & \rightarrow & K^1(A) & \rightarrow & 0 \\
& \uparrow & & \uparrow \alpha_0 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z}^p & \delta & \mathbb{Z}^{p-1} & \rightarrow & \mathbb{Z} & \rightarrow & 0
\end{array}
\]
Hence $\text{Im } \gamma$ is a direct summand in $\mathbb{Z}^{p-1}$, so $K^1(A) = \text{Coker } \gamma = \mathbb{Z}^{p-h}$.

REFERENCES


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Recibido en abril de 1971.