EIGENVECTORS AND CYCLIC VECTORS FOR BILATERAL WEIGHTEDhiftS

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1. INTRODUCTION

In what follows a class of bilateral weighted shifts operators on Banach spaces is defined. Let B be one of these operators; the following results are proven to be true:

1) There exist two vectors, f, g, such that the linear span of \( B_k^k f, B_k^k g : k = 0,1,2,\ldots \) is dense in the whole space; moreover, if B satisfies certain additional conditions, then the intersection of the (closed) invariant subspaces generated by

\( B_k^k f : k = 0,1,\ldots \) and \( B_k^k g : k = 0,1,\ldots \)

is equal to \( \{0\} \), whenever f and g are suitably chosen.

2) If B has an eigenvector, then it also has a cyclic vector;

3) If the adjoint \( B^* \) of B has an eigenvector, then B has no cyclic vectors;

4) If either B or \( B^* \) has an eigenvector, then B has no algebraically complementary invariant subspaces, no roots and no logarithm.

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1. INTRODUCTION

Let $C_0$ be the Banach space of all complex (two-sided) sequences \( \{c_n\} \) ([n \in \mathbb{Z}, the set of all integers] converging to zero in both directions, and with the sup. norm. $\ell_1$ is the subset of all those sequences in $C_0$ such that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ and this sum is the norm of that sequence in $\ell_1$.

Throughout this paper, $X$ will denote an intermediate space between $C_0$ and $\ell_1$; i.e., (see [1],[7])

\[ \ell_1 \subset X \subset C_0 \]

and, whenever $T$ is a (bounded linear) operator on $C_0$ and $\ell_1$, its restriction to $X$ is also bounded there. Further, we shall assume that the sequence \( \{e_m\} \) ([m \in \mathbb{Z}], where $e_m = \{\delta_{mn}\} \) ([n \in \mathbb{Z}) \) ([\delta_{mn} = 0, if m\#n; \delta_{mm} = 1] is a Schauder basis for $X$ in the sense that, if $g = \{c_n\} \in X$, then

\[ \lim_{k,k' \to \infty} \|g - \sum_{m=-k}^{k'} c_m e_m\| = 0 \]

Using (2), $g$ can be written as $g = \sum c_n e_n$.

It is well known ([7, Chap.1]) that an intermediate space $X$ can be re-normed with an equivalent norm so that, whenever an operator $T$ satisfies the two inequalities

\[ \|T\|_{C_0} < 1, \|T\|_{\ell_1} < 1, \text{ then } \|T\|_{X} < 1. \]

Assume that the $X$-norm satisfies (3) and let $W$ be a unitary operator on $C_0$ and $\ell_1$, simultaneously (i.e., $W$ is an isometric operator mapping each of those spaces onto itself). Then $W$ is also a unitary operator on $X$; moreover, $W^*(=\text{the adjoint of } W \text{ acting on the dual space } X^* \text{ of } X)$ is a unitary operator too.

Applying these results to the operators $W$ and $V$ defined (on $C_0$ and $\ell_1$) by

\[ W e_n = e_{\sigma(n)} \] \[ V e_n = \lambda_n e_n \]

where $\sigma$ is a "permutation" (i.e., $\sigma$ is a bijective map of $\mathbb{Z}$), and

\[ \lambda_n \in \mathbb{C} \text{ and } |\lambda_n| = 1 \]

where $\lambda_n$ is a complex number of modulus one, for all $n \in \mathbb{Z}$, we get
0 \neq \|c_n\| = \|e_m\|$, for all $n, m \in \mathbb{Z}$,
and
\[
\|\sum c_n e_n\| = \|\sum |c_n| e_n\|,
\]
for all X-norm $\|\cdot\|$.

Hence, without loss of generality we can assume (and we shall!) that the X-norm satisfies (3) and, moreover, $\|e_n\| = 1$, for all $n \in \mathbb{Z}$.

A bilateral weighted shift (B.W.S.) $B$ on $X$ is an operator defined by
\[
B \sum c_n e_n = \sum c_n w_n e_{n+1},
\]
where $w_n (n \in \mathbb{Z})$ is a bounded (two-sided) sequence of non-zero complex numbers. Since $B$ is clearly bounded and linear on $C_0$ and $\ell_1$ (for every bounded sequence $\{w_n\}$), then it is so on $X$.

2. THE SPECTRUM OF $B$ ACTING ON $X$.

**Lemma 1** ([3, thm. 6]). The annulus (or disc)
\[
D = \{z: R_2 \leq |z| \leq R_1\},
\]
where
\[
R_1 = \lim_{n \to \infty} \sup_{m \to \infty} \left(\prod_{j=m+1}^{m+n} |w_j|\right)^{1/n}
\]
and
\[
R_2 = \lim_{n \to \infty} \inf_{m \to \infty} \left(\prod_{j=m+1}^{m+n} |w_j|\right)^{1/n},
\]
is contained in $\sigma(B)$.

**Lemma 2** ([3, thm.10]). If $X = C_0$ or $X = \ell_1$, then
\[
\sigma(B) = D.
\]

**Corollary 3.** $\sigma(B) = D$, for every intermediate subspace $X$ satisfying (2).

**Proof.** Let $\lambda \notin D$, then (by Lemma 2) $(B - \lambda)$ and $(B - \lambda)^{-1}$ are bounded on $C_0$ and $\ell_1$; hence, they are bounded on every intermediate space $X$. 
It follows that
\[ \sigma(B) \subseteq D. \]
The converse inclusion is the Lemma 1. q.e.d.

The dual space of \( C_0(\ell_1, \text{resp.}) \) is equal to \( \ell_1(\ell_\infty, \text{resp.}) \), the Banach space of all bounded sequences under the sup. norm, \( \text{resp.} \). Thus, the dual space \( X^* \) of \( X \) (satisfying (1)) is intermediate between \( \ell_1 \) and \( \ell_\infty \):
\[
\ell_1 \subseteq X^* \subseteq \ell_\infty.
\]
Moreover, if \( e^*_m = \{ \delta_{mn} \} \) (though as an element of \( X^* \)), then \( (e^*_n, e^*_m) = \delta_{nm} \), and the set of (bounded linear) functionals \( \{ e^*_m \} (m \in \mathbb{Z}) \) is total in the sense that
\[
g \in X, \ (g, e^*_m) = 0, \forall m \in \mathbb{Z}, \text{ implies } g = 0.
\]
It is also clear that \( \| e^*_m \| = 1, \forall m \in \mathbb{Z} \).

The finite linear combinations of the \( e^*_m \)'s are not dense in \( X^* \) in general (e.g., if \( X = \ell_1 \)), \( X^* = \ell_\infty \); however, every element \( f \in X^* \) can be represented as
\[
f = \sum c_n e^*_n,
\]
where the sum is understood as a weak* limit.

Let \( B \) be a B.W.S. on \( X \); as in the case when \( X = \ell_2 \) ([5, prob. 76]); see also [3], [4]), it is not hard to check that the point spectrum of \( B \) and \( B^* \) is invariant under rotations about the origin.

Since kernel \( (B) = \{ 0 \} \), every eigenvalue of \( B \) has positive modulus; thus, if \( \lambda \in \sigma_p(B) \), then \( \lambda \neq 0 \) and \( \sigma_p(B) \) contains the circle of radius \( |\lambda| \). The eigenvalues of \( B \) are simple; to be more precise, the only eigenvectors of \( B \) with eigenvalue \( \lambda \) are the multiples of
\[
g_{\lambda} = e_0 + \sum_{n=1}^{\infty} \left( \prod_{j=0}^{n-1} w_j \right) \lambda^{-n} e_n + \sum_{n=1}^{\infty} \left( \prod_{j=1}^{n} w_j \right)^{-1} \lambda^n e_{-n}.
\]
The analogous results are true for \( B^* \) and every eigenvector of \( B^* \) with eigenvalue \( \lambda \) is a multiple of
\[
h_{\lambda} = e^*_0 + \sum_{n=1}^{\infty} \left( \prod_{j=0}^{n-1} w_j \right)^{-1} \lambda^n e^*_n + \sum_{n=1}^{\infty} \left( \prod_{j=1}^{n} w_j \right) \lambda^{-n} e^*_{-n} = \sum_{n=-\infty}^{\infty} a_n \lambda^n e^*_n,
\]
In the general case, \( \sigma_p(B) (\sigma_p(B^*)) \) is a (possibly empty) annulus about the origin containing both, one or none of the two circles that form the boundary; this annulus could degenerate to a (closed or open) pointed disc, or to a circle. For example, if \( \omega_n = \frac{1}{(n+1)/(n+2)}^2 \) \((n = 0, 1, 2, \ldots)\), \( \omega_{-n} = \frac{1}{(n+1)/n}^2 \) \((n = 1, 2, 3, \ldots)\), then \( \sigma_p(B) = \sigma(B) = \sigma(B^*) \) is the unit circle.

Assume that \( \sigma_p(B^*) \neq \emptyset \); without loss of generality we can assume that

\[
\sigma_p(B^*) \supset \mathbb{H} = \{ z : |z| = 1 \}.
\]

Let \( g = \sum \lambda_n \in X \) and define

\[
g \rightarrow g(\lambda) = (g, h_\lambda) = \sum \lambda_n \lambda^n
\]

(the series converges absolutely; to see this, use the results of seat.1).

For fixed \( g \), the function \( g(\lambda) \) is continuous on \( \sigma_p(B^*) \) and analytic on the interior of this set (to see this, use (2) and (6')); moreover

\[
|g(\lambda)| = |(g, h_\lambda)| \leq \|g\| \|h_\lambda\|^* \tag{9}
\]

and the convergence of a sequence in \( X \) implies the convergence (uniformly on compact subsets of \( \sigma_p(B^*) \)) of the corresponding functions (given by the map (8)).

Let \( L : X \rightarrow C(\mathbb{H}) \) (the Banach algebra of all continuous functions on \( \mathbb{H} \), under the sup. norm \( \| \cdot \|_\infty \)) be the linear map defined by (8) when \( \lambda \) is restricted to the unit circle. The following facts can be easily checked:

\[
\tag{10} \begin{array}{l}
i) \ L \ v_n = a_n^{-1} \lambda^n \ ; \ \text{therefore} \ L(X) \ is \ dense \ in \ C(\mathbb{H}); \\
ii) \ |L| = \|h_1\|^* = K^*. \ (use \ (9) \ and \ the \ results \ of \ seat.1) \\
iii) \ L \ is \ one-to-one \ (use \ (6'), \ (8) \ and \ the \ uniqueness \\
property \ of \ the \ Fourier \ series); \\
iv) \ (LBg)(\lambda) = \lambda g(\lambda), \ for \ all \ g \in X.
\end{array}
\]
3. EIGENVECTORS AND CYCLIC VECTORS.

THEOREM 4. If \( \alpha_p(B^*) \neq \phi \), then \( B \) has no cyclic vectors.

Proof. Without loss of generality we can assume that (7) holds. Assume that the statement is false; then there exists \( g \in X \) such that
\[
X = \bigvee_{k=0}^{\infty} B^k g \text{ (where the sign "\bigvee" means "the closed linear span of")}.
\]
Since (by (10), i)) \( L(X) \) is dense in \( C(\mathbb{P}) \), it follows from ((10),ii)) that the finite linear combinations
\[
\sum_{k=0}^{N} \lambda^k g(\lambda)
\]
are dense in \( C(\mathbb{P}) \); in other words, \( g(\lambda) \) is a cyclic vector for \( S = \text{"multiplication by } \lambda\" \) on \( C(\mathbb{P}) \). To prove the theorem we only have to observe that \( S \) cannot have a cyclic vector; in fact, if \( g(\lambda) \in C(\mathbb{P}) \) and \( M = \bigvee_{k=0}^{\infty} S^k g(\lambda) \), then,

a) If \( g(\lambda_0) = 0 \), for some \( \lambda_0 \in \mathbb{P} \), then \( M \) is contained in the maximal proper ideal \( \{ f \in C(\mathbb{P}) : f(\lambda_0) = 0 \} \); or else,

b) If \( g(\lambda) \) never vanishes on \( \mathbb{P} \), then
\[
M = \{ fg : f \text{ ranges over } A(\mathbb{P}) \} = gA(\mathbb{P}),
\]
where \( A(\mathbb{P}) \) is the closure in \( C(\mathbb{P}) \) of the analytic trigonometric polynomials; then \( A(\mathbb{P}) = g^{-1} M \).

In either case, \( M \not\subseteq C(\mathbb{P}) \).

THEOREM 5. If \( \alpha_p(B) \neq \phi \), then \( B \) has a cyclic vector.

Proof. Without loss of generality we can assume that
\[
(7') \quad \alpha_p(B) \supset \mathbb{P}
\]
Let \( \{ \lambda_n \}_{n=1}^{\infty} \) be a dense subset of \( \mathbb{P} \) such that every \( \lambda_n \) has a rational argument; then \( \lambda_h^* \) is a primitive root of the identity of order \( k \), where \( k = 0 \) if and only if \( h = n \). Fix \( h \) and \( n \); for large values of \( m \), we shall certainly have
\[
(11) \quad (1/m!) \sum_{j=1}^{m} (\lambda_h^* \lambda_n)^j = \delta_{hn};
\]
moreover, the absolute value of the first member of (11) is less than or equal to one for all values of \( m \).
Let $g_n = g_{\lambda_n}$ be the eigenvector of $B$ given by (6) and set

$$g = \sum_{n=1}^{\infty} c_n g_n,$$

where $(c_n)_{n=1}^{\infty}$ is any summable sequence of positive reals. By (11) we have

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} g_n = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} g_n,$$

where $h \leq M$, as $m \to \infty$. This expression shows that $g_h \in \bigvee_{n=1}^{\infty} B^k g$, for all values of $h = 1, 2, 3, \ldots$.

Let $f \in X^*$ be an element of the annihilator of $\bigvee_{n=1}^{\infty} B^k g$; by the previous result, $(g_h, f) = 0$, for all $h$.

Since $\sigma_p(B) = \Pi$ we can repeat the above construction defining a continuous linear map $L': X^* \to C(\Pi)$ by means of

$$(8') \quad h \mapsto (\lambda, h) = (g_{\lambda}, h), \ h \in X^*$$

(in fact, (2) and (6) show that the map $\lambda \mapsto g_{\lambda}$ is continuous).

This map is well defined and it enjoys all the properties (10) (with the obvious changes); in particular, we have:

$$(10') \iii \quad L' \text{ is one-to-one}$$

Applying this result to the continuous function $f(\lambda)$, which (since $(g_h, f) = 0, h = 1, 2, \ldots$) has a zero at every point of the dense subset $(\lambda_n)_{n=1}^{\infty}$ of $\Pi$, we conclude that $f = 0$ and therefore

$$X = \bigvee_{n=1}^{\infty} B^k g.$$

$q.e.d.$

The unequal behavior of $B$ according to $\sigma_p(B^*) \neq \phi$ or $\sigma_p(B) \neq \phi$ yields the following result:

**COROLLARY 6.** Let $B$ be a B.W.S. on $X$ and let $B^*$ be its adjoint (acting on $X^*$); then either $\sigma_p(B) = \phi$ or $\sigma_p(B^*) = \phi$.

If $X^* \subset C_0$, then the (unweighted) bilateral shift $U$ (defined by $U e_n = e_{n+1}$) provides an example of a B.W.S. for which both $\sigma_p(U)$
and \( \sigma_p(U^*) \) are empty sets. On the other hand, as it is well known (see, e.g., [6]) \( U \) and \( U^* \) acting on \( \ell_2 \) have (the same!) cyclic vectors. This example suggests the following question: is it true, for an arbitrary B.W.S. \( B \) acting on a space \( X \) such that \( X^* \) is separable, that either \( B \) has a cyclic vector, or \( B^* \) has a cyclic vector?.

If we consider the same operator \( U^* \) acting on \( \ell_\infty (= \ell^1) \), then \( \sigma_p(U^*) = \Pi \) and the eigenvectors of \( U^* \) with eigenvalue \( \lambda \) are the multiples of \( h_\lambda = \sum \lambda^n e_n^* \); a short analysis of these eigenvectors shows that

\[
\|h_\lambda - h_\lambda^{\prime}\| > \sqrt{3}, \text{ for all } \lambda \neq \lambda^{\prime}.
\]

I.e., the map \( \lambda \to h_\lambda \) (defined by \( (5') \)) is continuous at no point of \( \Pi! \).

4. THE MULTIPLICITY OF A B.W.S. INVARIANT SUBSPACES.

Let \( p^+(p^-, \text{ resp.}) \) denote the projection of \( X \) onto \( X^+ = \vee \{ e_n : n > 0 \} \), \( X^- = \vee \{ e_n : n < 0 \} \), resp. defined by

\[
p^+ \sum c_n e_n = \sum_{h=0}^{\infty} c_n e_n (p^+ = I - p^*),
\]

and let \( T^+, T^- \) be the operators defined on \( X^+, X^- \) respectively, by

\[
T^+ g = Bg \ (g \in X^+) \ , \ T^- g = p^-Bg \ (g \in X^-)
\]

It is clear that (up to an isometric isomorphism) \( X^* = p^* X^* \) and \( T^{*} = p^* B^* \) (on \( X^* \)); similarly we can write \( T^{-*} = B^* \) restricted to \( X^{-*} = p^{-} X^{-*} \).

Following R. Gellar ([4]), we shall define

\[
I^+ = \lim \inf_{n \to +\infty} (\prod_{j=0}^{n-1} w_j)^{1/n} , \quad Q^+ = \lim \sup_{n \to +\infty} (\prod_{j=0}^{n-1} w_j)^{1/n} \\
I^- = \lim \sup_{n \to +\infty} (\prod_{j=1}^{n} w_{-j})^{1/n} , \quad Q^- = \lim \inf_{n \to +\infty} (\prod_{j=1}^{n} w_{-j})^{1/n}
\]

We have \( R_2 = I^+ < Q^+ < R_1 \ , \ R_2 < Q^- < I^- < R_1 \).
Even in the case when $\sigma_p(B^*) = \emptyset$, to every $g \in X$ we can associate a formal Laurent series

\[(8'') \quad g(\lambda) = \sum c_n \lambda^n \]

(i.e., $g(\lambda)$ is defined in such a way that if $\lambda \in \sigma_p(B^*)$, then $g(\lambda)$ coincides with the expression (8)); the correspondence $g \rightarrow g(\lambda)$ is clearly injective and $(Bg)(\lambda) = \lambda g(\lambda)$.

**Theorem 7** ([4, thm. 11]). If $f \in X$ then

i) $f^+(\lambda) = (P^+ f)(\lambda)$ converges to an analytic function in the region $|z| < I^+$.  

ii) $f^-(\lambda) = (P^- f)(\lambda)$ converges to an analytic function in the region $|z| > I^-$.  

iii) If $I^- < I^+$, then convergence in $X$ implies uniform convergence of the associated analytic functions on compact subsets of the region $1 < |z| < I^+$. 

The region $I^- < |z| < I^+$ is precisely the interior of $\sigma_p(B^*)$. Similarly, the regions $|z| < I^+$ and $|z| > Q^-$ coincide with the interior of $\sigma_p(T^*)$ and $\sigma_p(T^-)$, respectively.

We recall that an operator $R$ is called unicellular if its lattice of invariant subspaces is linearly ordered by inclusion ([5]). It is not hard to conclude from the above definition that a B.W.S. $B$ on $X$ is unicellular if and only if its lattice of invariant subspaces consists of exactly the following elements:

\[(12) \quad \{0\}, X, X_m = \bigvee (e_n : n \geq m) = \bigvee_{k=0}^{\infty} B^k e_m \quad (m \in \mathbb{Z}) \]

No example of such operator is known. Here we shall establish a necessary condition for the unicellularity of a B.W.S.

**Corollary 8.** If $B$ is a unicellular B.W.S., then   

$I^+ = Q^- = 0$.  

In particular, an invertible B.W.S. cannot be unicellular.

**Proof.** If $\lambda_0 \in \sigma_p(T^*)$ and $h_{\lambda_0}$ is a non-zero eigenvector of $T^*$
with eigenvalue \( \lambda_0 \), then

\[
M_{\lambda_0} = \{ g \in X^+; (g, h_{\lambda_0}) = 0 \} = \{ g \in X^+ : g(\lambda_0) = 0 \}
\]

is an invariant subspace for \( T^+ \); hence \( M_{\lambda_0} \) is an invariant subspace for \( B \) and it is not hard to check that if \( \lambda_0 \neq 0 \), then \( M \) can not be of the form (12).

Similarly, if \( g_0 \neq 0 \) is an eigenvector of \( T^- \) with eigenvalue \( \lambda_0 \neq 0 \), then

\[
C_{g_0} = \bigvee_{k=0}^{\infty} B^k g_0
\]

is an invariant subspace of \( B \), not of the form (12). In fact, \( P^r C_{g_0} = \bigvee g_0 \) is a one-dimensional subspace of \( X^- \); since \( \lambda_0 \neq 0 \), \( g_0 \) cannot be a multiple of \( e_1 \); therefore \( C_{g_0} \) is a non-trivial invariant subspace and \( C_{g_0} \neq X_m \), for all \( m \in \mathbb{Z} \).

Therefore, if \( B \) is unicellular, \( \sigma_p(T^+) = \sigma_p(T^-) = \{ 0 \} \); now, by the observations following thm.7, this is equivalent to: \( r^+ = Q^- = 0 \).

\[ \text{q.e.d.} \]

Cor. 8 also shows that if \( r^+ > 0 \) or \( Q^- > 0 \), then \( B \) has "many" invariant subspaces (the lattice of invariant subspaces of \( B \) has the power of the continuum!). Now we want to show that some of these subspaces have an invariant topological complement (however, as we shall see in the next section, an invariant subspace of \( B \) has no invariant algebraic complement, in general). The next lemma has some interest in itself: it says (in the terminology of [8]) that the multiplicity of a B.W.S. cannot be greater than 2.

**Lemma 9.** If \( B \) is a B.W.S. on \( X \), then there exists a vector \( g \in X^- \) such that

\[
X = \bigvee_{k=0}^{\infty} \{ B^k e_0, B^k g \}.
\]

**Proof.** A simple modification of the argument given in [5; prob. 126] shows that a "backward" unilateral weighted shift operator always has a cyclic vector; since \( T^- \) (acting on \( X^- \)) belongs to this class of operators, it follows that there exists \( g \in X^- \)
such that
\[ X^- = \bigvee_{k=0}^\infty (T^-)^k g. \]
Assume that \( g \) satisfies the above condition; since \( X^+ = \bigvee_{k=0}^\infty B^k e_0 \), we have that
\[ (T^-)^n g = p^* B^n g = B^n g - p^* B^n g \]
belongs to the subspace spanned by \( \{B^k e_0, B^k g : k = 0, 1, 2, \ldots \} \) for all \( n = 0, 1, 2, \ldots \), and therefore
\[ X = X^- \oplus X^+ \cap \bigvee_{k=0}^\infty \{B^k e_0, B^k g \} \subset X. \]
q.e.d.

**THEOREM 10.** Let \( B \) be a B.W.S. such that \( \sigma_p(B^*) \supset \Pi \) and let
\[ C_f = \bigvee_{k=0}^\infty B^k f \]
be the cyclic invariant subspace generated by an element \( f \) of \( X \).
Assume that \( f, g \in X \) satisfy the conditions:
1) The continuous functions \( f(\lambda), g(\lambda) (\lambda \in \Pi) \) never vanish;
2) \( f(\lambda) g^{-1}(\lambda) \) are not the radial limits (a.e. with respect to the Lebesgue measure) of a function meromorphic on \( |z| < 1 \).
Then
\[ C_f \cap C_g = \{0\}. \]

**Proof.** The modulus maximum theorem, inequality (9) and 1) imply that if \( h \in C_f \) then
\[ h(\lambda) = f(\lambda) h_1(\lambda) \quad (\lambda \in \Pi) \]
where \( h_1(\lambda) = \lim h_1(r\lambda) \quad (0 \leq r < 1, r \to 1) \), \( h_1(z) \) being a function analytic on \( |z| < 1 \).
Let \( 0 \neq h \in C_f \cap C_g \); the above result shows that
\[ h(\lambda) = f(\lambda) h_1(\lambda) = g(\lambda) h_2(\lambda) \neq 0, \]
where \( h_1(z) \) and \( h_2(z) \) are two bounded and non-identically zero functions analytic on \( |z| < 1 \). Thus (see, e.g. [6]), the radial limits
\[ h_2(\lambda) h_1^{-1}(\lambda) = f(\lambda) g^{-1}(\lambda) \]
are well-defined almost everywhere on \( \Pi \); but the function
\( h_2(z) h_1^{-1}(z) \) is meromorphic on \( |z| < 1 \), contradicting 2). This
contradiction proves that \( h(\lambda) \equiv 0 \) and therefore, by (10), iii),
\( h = 0 \), which proves (13).

\[ \text{q.e.d.} \]

COROLLARY 11. Let \( B \) be a B.W.S. on \( X \) such that \( \sigma_p(B^*) \supseteq \Pi \) and let
\( g \in X \) be a vector satisfying the conditions:

1) \( P^- g \) is a cyclic vector for \( T^- \) (acting on \( X^- \));
2) \( g(\lambda) \) never vanishes (\( \lambda \in \Pi \)) and
3) \( g(\lambda) \) are not the radial limits (a.e.) of a function meromorphic on \( |z| < 1 \), or
3') \( g(\lambda) \) is the restriction to \( \Pi \) of a function analytic on some
neighborhood of the unit circle.

Then
\[ C_g \cap X_m = \{0\}, \quad C_g \lor X_m = X, \]
for every \( m \in \mathbb{Z} \). However, the algebraic direct sum \( C_g \oplus X_m \) is
never closed.

Proof. To see that \( C_g \lor X_m = X \) we only have to repeat the proof
of lemma 9 with minor changes (use 1).

Condition 2) implies (as in the proof of thm.10), that if
\( C_g \cap X_m \neq \{0\} \), then
\[ g(\lambda) = \lambda^m h_1(\lambda) h_2^{-1}(\lambda), \]
where \( h_1(z) \) and \( h_2(z) \) are two bounded and non-identically zero
functions, analytic on \( |z| < 1 \). If \( g \) satisfies 3'), the above ex-
pression shows that there exist a polynomial \( p(z) \) such that
\( p(z) g(z) \) is analytic on \( |z| < 1 \); but this implies that the di-
mension of the subspace
\[ P^- C_g = \lor_{k=0}^= (T^-)^k g^- \]
of \( X^- \) cannot be larger than degree \( (p) \leq \infty \), contradicting 1).

We concluded that 1), 2), and 3') imply 3); now, if \( g \) satisfies 1),
2) and 3), then \( C_g \cap X_m \) must be equal to \( \{0\} \) by thm. 10.

Finally, observe that if \( C_g \oplus X_m \) is closed, then
\[ X = C_g \oplus X_m = C_g \oplus X_{m-1}; \]

hence \( e_{m-1} = f + h \), where \( f \in C_g \), \( h \in X_{m-1} \), and therefore \( e_{m-1} \) has
two different expressions as an element of the direct sum.
C \ominus X_{m-1}; this contradiction shows that C \ominus X_m cannot be closed in X.

q.e.d.

The hypothesis \( \sigma_p(B^*) \neq \emptyset \) is sufficient but not necessary for the existence of two topologically, but not algebraically, complementary invariant subspaces.

EXAMPLE. If \( U \) is the (unweighted) bilateral shift acting on \( \ell_2 \), then \( U \) is unitarily equivalent to multiplication by \( \lambda \) on \( L^2(\mathbb{R}, dm) \) (where \( dm \) denotes the normalized Lebesgue measure on the unit circle); if \( f(\lambda) \) is a function of modulus one (a.e., \( dm \)) and \( g(\lambda) \) is the characteristic function of a measurable set \( E \subset \mathbb{R} \) such that \( 0 < m(E) < 1 \), then

\[
C_f \cap C_g = \{0\} \quad \text{and} \quad C_f \vee C_g = L^2,
\]

but \( C_f \ominus C_g \) is not closed in \( L^2 \) (the proof follows from the results contained in [6]).

CONJECTURE. If \( B \) is a B.W.S. such that \( I^+ > 0 \), then \( B \) has two topologically, but not algebraically, invariant subspaces.

We are going to close this section with a more precise relation between (8) and (8").

**Lemma 12.** Assume that for every \( g \in X \) the series (8") is Cesàro summable to a finite limit for \( \lambda = \lambda_0 \); then \( \lambda_0 \in \sigma_p(B^*) \).

**Proof.** The operators \( C_N \) defined by

\[
C_N g = g_N = \sum_{n=-N}^{+N} C_n (1 - \frac{|n|}{N+1}) e_n, \quad (N=1,2,\ldots)
\]

have finite dimensional range and therefore they are bounded on \( X \) (moreover, \( \|C_N\| = 1 \), for all \( N \)). It follows that the linear functionals

\[
j_N(g) = g_N(\lambda_0)
\]

are also bounded.

By hypothesis, \( j_N(g) = g_N(\lambda_0) \) converges to a finite limit; hence for every \( g \in X \) there is a constant \( K_g \) such that

\[
|j_N(g)| \leq K_g, \quad \text{for all } N.
\]

From this and the uniform boundedness principle, we conclude that
for some positive constant $K$, independent of $N$. Hence

$$|i_N^*(g)| = |g_N^*(\lambda_0)| \leq K \|g\|$$

and therefore

$$|g(\lambda_0)| \leq K \|g\|, \text{ for all } g \in X.$$

Let $h_{\lambda_0}^* \in X^*$ be the bounded linear functional defined by $g \mapsto g(\lambda_0)$; then $g(\lambda_0) = (g, h_{\lambda_0}^*)$, for all $g \in X$ (it is clear that $h_{\lambda_0}^* \neq 0$), and therefore

$$(g, \lambda_0 h_{\lambda_0}^*) = \lambda_0 (g, h_{\lambda_0}^*) = \lambda_0 g(\lambda_0) = (Bg)(\lambda_0) =$$

$$= (B \lambda_0, h_{\lambda_0}^*) = (g, B^* h_{\lambda_0}^*);$$

i.e., $B^* h_{\lambda_0}^* = \lambda_0 h_{\lambda_0}^*$ and therefore $\lambda_0 \in \sigma_p(B^*)$.

q.e.d.

It is not difficult to conclude (using the results of sections 1 and 2) that, if $g(\lambda_0)$ (given by (8'')) is Cesàro summable for all $g \in X$ and for some $\lambda_0 \in \pi$, then $\pi \subset \sigma_p(B^*)$ and $g(\lambda) \in C(\pi)$.

5. OPERATORS COMMUTING WITH $B$.

Assume that $\sigma_p(B) \supset \pi$ and let $R \in A^*_\pi$, the commutant of $B$; then

$$0 = (BR - RB)g_\lambda = (B - \lambda) R g_\lambda.$$  

Since every $\lambda \in \sigma_p(B)$ is a simple eigenvalue, the above equality implies

(14)  

$$R g_\lambda = c(R, \lambda) g_\lambda,$$

for some complex number $c(R, \lambda)$. An elementary analysis of the results of sections 2-3 shows that

THEOREM 13. If $\sigma_p(B) \supset \pi$, to every operator $R$ commuting with $B$ corresponds a function $c(R, \lambda) \in C(\pi)$; the mapping $\gamma: A^*_\pi \rightarrow C(\pi)$ defined by $\gamma(R) = c(R, \lambda)$ is an algebra isomorphism (of $A^*_\pi$ into $C(\pi)$) and

$$\|c(R, \lambda)\|_\infty \leq K \|R\|,$$
Proof. The existence of \( C(R,A) \) is clear from the previous observations. The continuity of \( C(R,A) \) is a consequence of (2), (6) and the fact that \( R \) itself is continuous.

That \( \gamma \) is an algebra homomorphism is also clear. If \( C(R,\lambda) = 0 \), then \( Rg_\lambda = 0 \), for all \( \lambda \in \Pi \), and this implies (since the set \( \{g_\lambda, \lambda \in \Pi\} \) is complete in \( X \), as we saw in the proof of thm.5) that \( R = 0 \); hence, \( \gamma \) is one-to-one.

Finally, the inequality \( \|C(R,\lambda)\|_\infty \leq K\|R\| \) follows immediately from (14), and the results of sect.1.

q.e.d.

COROLLARY 14. If \( \sigma_p(B) \supset \Pi \), then:

i) \( B \) has no non-trivial complementary invariant subspaces;

ii) If \( |z| < 1 \), \( (B - z) \) is rootless;

iii) If \( |z| \leq 1 \), \( (B - z) \) is logarithmless.

Proof.

i) Assume that \( P \in \mathbb{A}_B \) satisfies the equation \( P^2 = P \); then, by thm.13, \( c^2(P,\lambda) = c(P,\lambda) \in C(\Pi) \) and therefore \( c(P,\lambda) \equiv 0 \) (and \( P=0 \)) or \( c(P,\lambda) \equiv 1 \) (and \( P=I \)). Therefore \( \mathbb{A}_B \) contains no non-trivial idempotents and this condition is equivalent (see [2; exerc.3, p. 70]) to the non-existence of two non-trivial complementary invariant subspaces for \( B \).

ii) If \( R^k = (B - z) \), then it is clear that \( R \in \mathbb{A}_B \) and, by thm.13, \( c^k(R,\lambda) = c(B - z,\lambda) = (\lambda - z) \); since \( |z| < 1 \), the equation \( c^k(R,\lambda) = (\lambda - z) \) has no solutions in \( C(\Pi) \), for any \( k > 1 \). Therefore \( (B-z) \) is rootless.

iii) Similarly, if \( e^R = \sum_{n=0}^{\infty} (1/n!) R^n = (B - z) \), then \( R \in \mathbb{A}_B \) and \( c(R,\lambda) = \log (\lambda - z) \); but this equation has no solutions in \( C(\Pi) \), unless \( |z| \) is strictly larger than 1.

q.e.d.

COROLLARY 15. The conclusions of thm.13 and cor.14 remain true when the hypothesis "\( \sigma_p(B) \supset \Pi \)" is replaced by "\( \sigma_p(B^*) \supset \Pi \)."

Proof. Let \( R \in \mathbb{A}_B^* \); then \( R^k \in \mathbb{A}_B^* \) and we conclude as in (14) that

\[
R^*h_\lambda = c(R^*,\lambda)h_\lambda, \quad (\lambda \in \Pi).
\]
However (as it is shown by the example at the end of seat.3), the mapping $\lambda \mapsto h_\lambda$ is highly discontinuous in general, and we cannot apply the arguments of thm.13. The solution arrives by using the duality between $X$ and $X^*$; in fact, (8)-(10) and the equalities

$$c(R^*, \lambda) = c(R^*, \lambda) (e_0, h_\lambda) = (e_0, R^*h_\lambda) = (Re_0, h_\lambda),$$

show that $c(R^*, \lambda) \in C(\mathbb{N})$.

The remaining statements are clear now. q.e.d.

6. COMPLEMENTARY REMARKS.

The results contained in [3],[4] and those of this paper show that, if $B$ is a B.W.S. in a Banach space $X$ satisfying our conditions and, moreover,

1) $B$ has two non-trivial complementary invariant subspaces, or
2) $B$ has a $k^{th}$ root for some integer $k > 1$, or
3) $B$ has a logarithm, then

$\sigma(B)$ is a circle and $\sigma_p(B) = \sigma_p(B^*) = \phi$

CONJECTURE. If both $X$ and $X^*$ satisfy our conditions, $X$ is reflexive and $B$ is a B.W.S. on $X$, then each of the three above conditions is equivalent to

4) $B$ is similar to a positive multiple of $U$, i.e., there exists an invertible operator $V$ on $X$ and a positive $r$ such that $B = rVU^{-1}$.

All the results of this paper can be easily extended to a larger class of B.W.S., and even to a class of operators related with them.

For example, if $Y$ is a Banach space satisfying (1) and (2), and the $Y$-norm satisfies all the conditions that we asked for the $X$-norm of the intermediate subspaces and if the B.W.S. $B$ is well defined and continuous on $Y$, then all the results can be extended to $B$ acting on $Y$. This is true, in particular, for all B.W.S. $B$ such that

$$B(C_0) \subset \ell_1.$$ 

EXAMPLE 1. Let $\{X_k\}$ be a finite (or denumerable) family of dif-
ferent intermediate spaces between $C_0$ and $\ell_1$ in the conditions of sect.1, and let $Z = U_k Z_k$ be a partition of $Z$ (where $k$ ranges over the same set of indices as for the $X_k$'s). Let $P_k$ be the projection defined by

$$\sum_{n \in Z_k} c_n e_n = \sum_{n \in Z_k} e_n$$

and let $Y$ be the completion of $\sum P_k X_k$ under some "suitable" norm (e.g., $\|g\|_Y = \sum_k \|P_k g\|_{X_k}$). Observe that, in general, the $W$'s of sect.1 are not unitary operators in $Y$, even if $Y$ is re-normed with an equivalent norm. However, $Y$ satisfies our requirements and, if $B$ is a B.W.S. whose restriction to $Y$ maps $Y$ continuously into itself, then the results of the paper apply to $B$ restricted to $Y$, except perhaps cor.3.

If, e.g., $Y = P^+ C_0 \oplus P^- \ell_1$, then every B.W.S. defines an operator in $Y$. If $Y = P^e C_0 \oplus P^0 \ell_1$, where $P^e \sum_n c_n e_n = \sum c_{2n} e_{2n}$, $P^0 = I - P^e$, then the only B.W.S. that we can "interpolate" in this $Y$ are those satisfying the condition (15) (for each of these operators, $\sigma(B) = \{0\}$).

EXAMPLE 2. The results also apply to $Y = C(\mathbb{N})$, though as the space of all sequences of Fourier coefficients of continuous functions on $\mathbb{N}$, even when the projections $P^+$ and $P^-$ are not bounded here and, moreover, (2) is not satisfied in this space (see [9; Chap.II and VIII]).
REFERENCES


