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SOME ISOTHERMAL PROPERTIES OF CARTOGRAMS T AND DENSITY TRANSFORMATIONS T*

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1. ABSOLUTE DERIVATIVES AND NATURAL FAMILIES UNDER A CONFORMAL CARTOGRAM T.

Consider a conformal cartogram T between two Riemannian spaces V_n and $\bar{V}_n$, each of dimension $n \geqslant 2$, for which the scale $\rho = e^{\mu} = d\bar{s}/ds > 0$, where $\mu = \mu(x)$ is a point function. Under the conformal cartogram T two corresponding *unit* contravariant vectors λ^i and $\bar{\lambda}^i$ of V_n and $\bar{V}_n$ respectively transform according to the law

$$(1.1) \qquad \bar{\lambda}^i = e^{-\mu} \lambda^i .$$

Under T the arc length absolute derivative $\frac{\bar{D}\bar{\lambda}^i}{d\bar{s}}$ of a contravariant vector $\bar{\lambda}^i$ of $\bar{V}_n$ is expressed in terms of $\frac{D\bar{\lambda}^i}{ds}$ when it is considered as a vector of V_n, by the set of relations

$$(1.2) \qquad \frac{\bar{D}\bar{\lambda}^i}{d\bar{s}} = e^{-\mu} \frac{D\bar{\lambda}^i}{ds} + e^{-\mu} \left[\frac{d\mu}{ds} + \frac{\partial\mu}{\partial x^{\alpha}} \bar{\lambda}^{\alpha} \frac{dx^i}{ds} - (g_{jk}\bar{\lambda}^j \frac{dx^k}{ds})(g^{i\alpha} \frac{\partial\mu}{\partial x^{\alpha}})\right].$$

In particular if λ^i and $\bar{\lambda}^i$ are two corresponding *unit* contravariant vectors then

$$(1.3) \qquad \frac{D\bar{\lambda}^i}{d\bar{s}} = e^{-2\mu} \frac{D\lambda^i}{ds} + e^{-2\mu} \left[\frac{\partial\mu}{\partial x^{\alpha}} \lambda^{\alpha} \frac{dx^i}{ds} - (g_{jk} \lambda^j \frac{dx^k}{ds})(g^{i\alpha} \frac{\partial\mu}{\partial x^{\alpha}})\right].$$

Consider two curves $C: x^i = x^i(s)$ and $\bar{C}: \bar{x}^i = \bar{x}^i(\bar{s}) = x^i(\bar{s})$ which correspond under the conformal cartogram T between V_n and $\bar{V}_n$. Their two unit contravariant tangent vectors satisfy the relations $dx^i/d\bar{s} = e^{-\mu}(dx^i/ds)$ and the two corresponding contravariant geodesic curvature vectors K^i and $\bar{K}^i$ obey the law of transformation [1]

$$(1.4) \qquad \overline{K}^i = e^{-2\mu} K^i + e^{-2\mu} \left[\frac{d\mu}{ds} \frac{dx^i}{ds} - g^{i\alpha} \frac{\partial \mu}{\partial x^\alpha} \right].$$

A *natural family* Ω [2] of ∞^{2n-2} curves C of Riemannian space V_n is such that every curve C of Ω corresponds under a conformal cartogram T on V_n onto a Riemannian space $\overline{V}_n$ to a geodesic $\overline{C}$ of $\overline{V}_n$.

In particular, the set of ∞^{2n-2} geodesics C of a Riemannian space V_n is a natural family.

THEOREM 1.1. *A natural family* Ω *of a Riemannian space* V_n *is composed of all the* ∞^{2n-2} *integral solutions of the set of* n *second order ordinary differential equations*

$$(1.5) \qquad K^i = \frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = g^{i\alpha} \frac{\partial \mu}{\partial x^\alpha} - \frac{d\mu}{ds} \frac{dx^i}{ds} .$$

This is obtained from (1.4) by setting $\overline{K}^i = 0$.

It is evident that under a conformal cartogram T that a natural family Ω of V_n corresponds to a natural family $\overline{\Omega}$ of $\overline{V}_n$.

2. THE SYMBOLS A^i_{jk} AND B_{jk} OF A CONFORMAL SPACE Γ_n.

Consider a Riemannian space V_n of dimension $n \geqslant 2$. The totality of all Riemannian spaces $\overline{V}_n$ such that there exists a conformal cartogram T between V_n and $\overline{V}_n$ is termed a *conformal space* Γ_n.

If V_n and $\overline{V}_n$ belong to the same conformal space Γ_n then their affine connections are related by the law

$$(2.1) \qquad \overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \frac{\partial \mu}{\partial x^k} + \delta^i_k \frac{\partial \mu}{\partial x^j} - g_{jk}\, g^{i\alpha} \frac{\partial \mu}{\partial x^\alpha} ;$$

where μ is the scale of the conformal cartogram T relating V_n and $\overline{V}_n$.

It is an immediate consequence of these relations that

$$(2.2) \qquad \frac{\partial \mu}{\partial x^j} = \frac{1}{n} \left[\overline{\Gamma}^\alpha_{\alpha j} - \Gamma^\alpha_{\alpha j} \right] \qquad j = 1,2,\ldots,n.$$

Consider a fixed Riemannian space V_n, for $n \geqslant 2$. The symbols A^i_{jk}

and B_{jk} are defined by the expressions

$$(2.3)\qquad A^i_{jk} = \Gamma^i_{jk} - \frac{1}{n}(\delta^i_j \Gamma^\alpha_{\alpha k} - \delta^i_k \Gamma^\alpha_{\alpha j}) + \frac{1}{n} g_{jk} g^{i\beta} \Gamma^\alpha_{\alpha\beta},$$

$$(2.4)\qquad B_{jk} = \frac{\partial \Gamma^\alpha_{\alpha j}}{\partial x^k} - \frac{\partial \Gamma^\alpha_{\alpha k}}{\partial x^j}.$$

THEOREM 2.1. *Two Riemannian spaces* V_n *and* $\overline{V}_n$, *for* $n \geqslant 2$, *are conformally equivalent if and only if the two sets of symbols* A^i_{jk} *and* B_{jk} *are the same in every admissible coordinate system* (x), *provided that the set of initial conditions* $\overline{g}_{ij}(x_0) = e^{2\mu(x_0)} g_{ij}(x_0)$ *is satisfied for* $i,j = 1,2,\ldots,n$ *at some fixed point* P_0, *where* $\mu(x_0)$ *is a fixed real constant.*

For, by means of (2.1) and (2.2) it is found that $\overline{A}^i_{jk} = A^i_{jk}$ and $\overline{B}_{jk} = B_{jk}$ for two conformally equivalent Riemannian spaces V_n and $\overline{V}_n$.

Conversely, suppose that $\overline{A}^i_{jk} = A^i_{jk}$ and $\overline{B}_{jk} = B_{jk}$ for two Riemannian spaces V_n and $\overline{V}_n$. The second set represent integrability conditions for the equations (2.2). Thus, let $\mu = \mu(x)$ represent a solution of (2.2). There exists one and only one solution satisfying the prescribed set of initial conditions at the fixed point P_0. By use of the conditions $\overline{A}^i_{jk} = A^i_{jk}$ the law of transformation (2.1) is found. It then is easily deduced that $\overline{g}_{ij} = e^{2\mu(x)} g_{ij}$ where $\mu = \mu(x)$ is the unique solution discussed above. Consequently V_n and $\overline{V}_n$ are conformally equivalent.

It is noted that the symbols A^i_{jk} are symmetric in the lower indices and that $A^i_{ik} = A^i_{ki} = 0$. However the A^i_{jk} do *not* form a tensor. The symbols B_{jk} are skew symmetric, that is $B_{jk} = -B_{kj}$, and form a skew symmetric covariant tensor of second order.

3. COVARIANT DIFFERENTIATION IN A CONFORMAL SPACE Γ_n.

In a Riemannian space V_n the covariant derivatives $\lambda^i_{,k}$ and $\lambda_{i,k}$ of a contravariant vector λ^i and a covariant vector λ_i may be given by

$$\begin{aligned}
\lambda^i_{,k} &= \frac{\partial\lambda^i}{\partial x^k} + A^i_{ak}\,\lambda^a + \frac{1}{n}\,\lambda^i\,\Gamma^\alpha_{\alpha k} + \frac{1}{n}\,\delta^i_k\,\Gamma^\alpha_{\alpha a}\,\lambda^a - \\
&\quad - \frac{1}{n}\,g^{ia}\,g_{kb}\,\Gamma^c_{ac}\,\lambda^b\,, \\
\lambda_{i,k} &= \frac{\partial\lambda_i}{\partial x^k} - A^a_{ik}\,\lambda_a - \frac{1}{n}\,\lambda_i\,\Gamma^\alpha_{\alpha k} - \frac{1}{n}\,\lambda_k\,\Gamma^\alpha_{\alpha i} + \\
&\quad + \frac{1}{n}\,g_{ik}\,g^{ab}\,\Gamma^c_{ac}\,\lambda_b\,.
\end{aligned} \tag{3.1}$$

The two corresponding absolute differentials are $D\lambda^i = \lambda^i_{,k}\,dx^k$ and $D\lambda_i = \lambda_{i,k}\,dx^k$.

In a conformal space Γ_n, $n \geqslant 2$, with invariant symbols A^i_{jk} and B_{jk} the conformal covariant derivatives $\Delta_k\lambda^i$ and $\Delta_k\lambda_i$ of any contravariant vector λ^i and any covariant vector λ_i are defined by

$$\Delta_k\lambda^i = \frac{\partial\lambda^i}{\partial x^k} + A^i_{ka}\lambda^a\,, \qquad \Delta_k\lambda_i = \frac{\partial\lambda_i}{\partial x^k} - A^a_{ik}\lambda_a\,. \tag{3.2}$$

The corresponding conformal absolute differentials are $\Delta\lambda^i = \Delta_k\lambda^i dx^k$ and $\Delta\lambda_i = \Delta_k\lambda_i dx^k$.

THEOREM 3.1. *The four sets of quantities,* $\Delta_k\lambda^i$, $\Delta_k\lambda_i$, $\Delta\lambda^i$, $\Delta\lambda_i$ *are all invariant under the conformal group* G *of the conformal group* G *of* Γ_n. *If* V_n *is an element of* Γ_n *then the relationships between these conformal covariant derivatives and the covariant derivatives relative to* V_n *are found by substituting the relations* (3.2) *into the equations* (3.1).

It is observed that the differential $d(\phi,\psi)$ of the inner product $(\phi,\psi) = \phi^i\psi_i$ of any contravariant vector ϕ^i and any covariant vector ψ_i is

$$(3.3)\quad d(\phi,\psi) = D(\phi,\psi) = \phi^i D\psi_i + \psi_i D\phi^i = \phi^i \Delta\psi_i + \psi_i \Delta\phi^i = \Delta(\phi,\psi).$$

THEOREM 3.2. *If two Riemannian spaces* V_n *and* $\overline{V}_n$, *for* $n \geqslant 2$, *belong to the same conformal space* Γ_n *and correspond by a conformal cartogram* T *for which the scale is* $\rho = e^{\mu} = d\overline{s}/ds > 0$, *then for every geometric vector* $\lambda = \lambda^i = \lambda_i$ *with* $|\lambda| > 0$ *of* V_n *there exists a point function* $r = r(x)$ *depending on* λ *such that by* T *the images* $\overline{\lambda}^i$ *and* $\overline{\lambda}_i$ *of* λ^i *and* λ_i *in* V_n *are* $\overline{\lambda}^i = e^{r-\mu}\lambda^i$ *and* $\overline{\lambda}_i = e^{r+\mu}\lambda_i$. *The two covariant derivatives* $\lambda^i_{,k}$ *and* $\lambda_{i,k}$ *obey the laws of transformation*

$$(3.4)\quad \begin{aligned} \overline{\lambda}^i_{,k} &= e^{r-\mu}\left[\lambda^i_{,k} + \lambda^i \frac{\partial r}{\partial x^k} + \delta^i_k \lambda^a \frac{\partial\mu}{\partial x^a} - g^{ia} g_{kb} \frac{\partial\mu}{\partial x^a}\lambda^b\right], \\ \overline{\lambda}_{i,k} &= e^{r+\mu}\left[\lambda_{i,k} + \lambda_i \frac{\partial r}{\partial x^k} - \lambda_k \frac{\partial\mu}{\partial x^i} + g_{ik} g^{ab} \lambda_a \frac{\partial\mu}{\partial x^b}\right]. \end{aligned}$$

This result is a consequence of the previous discussion.

It is noted that when $r = 0$, we obtain the laws of transformation for unit vectors.

4. SOME CONFORMAL PROPERTIES OF THE LAME DIFFERENTIAL PARAMETERS $\Delta_1(U,V)$ AND $\Delta_2(V)$ [3].

In a Riemannian space V_n, for $n \geqslant 2$, the *Lamé differential parameter* $\Delta_1(U,V)$ of order one, of two scalars $U = U(x)$ and $V = V(x)$ is the scalar

$$(4.1)\quad \Delta_1(U,V) = g^{jk} \frac{\partial U}{\partial x^j}\frac{\partial U}{\partial x^k} = (\text{grad } U, \text{ grad } V),$$

In particular, if $U = V$, then $\Delta_1(V) = \Delta_1(V,V) = |\text{grad } V|^2$.

The *Lamé differential parameter of second order* $\Delta_2(V) = \nabla^2(U)$ is the Laplacean and is defined by the scalar

(4.2) $\Delta_2(V) = \nabla^2(V) = g^{jk}\, V_{,jk} = \frac{1}{\sqrt{g}}\ \frac{\partial}{\partial x^j}[\sqrt{g}\ g^{jk}\ \frac{\partial V}{\partial x^k}]$;

where $g = |g_{ij}| > 0$.

If two Riemannian spaces correspond by a conformal cartogram T and if V = V(x) is a scalar then

(4.3) $\overline{V}_{,j} = V_{,j}$, $\overline{V}_{,jk} = V_{,jk} - V_{,j}\ \mu_{,k} + g_{jk}\ \Delta_1(\mu,V)$.

The three laws of transformation for the Lamé differential parameters are

$$\overline{\Delta_1(U,V)} = e^{-2\mu}\ \Delta_1(U,V) \quad , \quad \overline{\Delta_1(V)} = e^{-2\mu}\ \Delta_1(V) \tag{4.4}$$
$$\overline{\Delta_2(V)} = e^{-2\mu}\ [\Delta_2(V) + (n-2)\ \Delta_1(\mu,V)] .$$

These are obtained by means of Theorem 3.2, where the scalar $r = r(x)$ is replaced by $-\mu(x)$.

If a scalar $V = V(x)$ with $\Delta_1(V) > 0$, is a harmonic function in a Riemannian space V_n, then the equation $V = V(x) =$ constant defines an *isothermal family* of ∞^1 surfaces $\sum_{n-1}$, each of deficiency one, in V_n.

It may be proved that a simple family of ∞^1 surfaces $\sum_{n-1}$, each of deficiency one in a Riemannian space V_n, for $n \geqslant 2$, is an isothermal family $V = V(x) = C =$ constant if and only if $V = V(x)$ obeys a partial differential equation of second order of the form

$$\frac{\Delta_2(V)}{\Delta_1(V)} = \frac{g^{jk}\ V_{,jk}}{g^{jk}\ V_{,j}V_{,k}} = F(V) , \tag{4.5}$$

where $F = F(V)$ is a scalar depending *only* on $V = V(x)$.

THEOREM 4.1.[4] *If two Riemannian spaces* V_n *and* $\overline{V}_n$, *for* $n \geqslant 2$, *correspond by a conformal cartogram* T *then every isothermal family in* V_n *is converted into an isothermal family in* $\overline{V}_n$ *if and only if either* $n = 2$ *or else if* $n \geqslant 3$, T *is a homothetic cartogram for which* μ *is a real constant.*

This proposition is established by means of the conditions (4.4) and (4.5).

5. DENSITY TRANSFORMATIONS BETWEEN TWO RIEMANNIAN SPACES V_n AND $\overline{V}_n$.

Let two Riemannian spaces V_n and $\overline{V}_n$, for $n \geqslant 2$, correspond by a cartogram T, conformal or not, for which the differential ds and $d\overline{s}$ of arc length along corresponding curves C and $\overline{C}$ of V_n and $\overline{V}_n$ are defined by $ds^2 = g_{ij}\, dx^i\, dx^j$ and $d\overline{s}^2 = \overline{g}_{ij}\, dx^i\, dx^j$.

A *density transformation* T* between V_n and $\overline{V}_n$ is such that their respective points correspond by a cartogram T, either conformal or not, and a scalar $V = V(x)$, which is evaluated at a point P of V_n is converted into the scalar

$$\overline{V} = \overline{V}(x) = F(V;x) \ , \tag{5.1}$$

for which $\dfrac{\partial\overline{V}}{\partial V} = \dfrac{\partial F}{\partial V} \neq 0$, whose value is associated with the corresponding point $\overline{P}$ of $\overline{V}_n$. A scalar $V = V(x)$ calculated at P in V_n is called a *density* V of P.

THEOREM 5.1. *Under a density transformation* T* *between two Riemannian spaces* V_n *and* $\overline{V}_n$, *for* $n \geqslant 2$, *the Lamé differential parameters* $\Delta_1(V)$ *and* $\Delta_2(V)$ *of a density transform according to the laws*

$$\begin{aligned}
\overline{\Delta}_1(\overline{V}) &= \left(\frac{\partial F}{\partial V}\right)^2 \overline{\Delta}_1(V) + 2\,\frac{\partial F}{\partial V}\,\overline{\Delta}_1(V,F) + \overline{\Delta}_1(F) \ ,\\
\Delta_1(\overline{V}) &= \left(\frac{\partial F}{\partial V}\right)^2 \Delta_1(V) + 2\,\frac{\partial F}{\partial V}\,\Delta_1(V,F) + \Delta_1(F) \ ,\\
\overline{\Delta}_2(\overline{V}) &= \frac{\partial F}{\partial V}\,\overline{\Delta}_2(V) + \overline{\Delta}_2(F) + 2\,\overline{\Delta}_1\left(\frac{\partial F}{\partial V}\,,\ V\right) + \frac{\partial^2 F}{\partial V^2}\,\overline{\Delta}_1(V) \ ,\\
\Delta_2(\overline{V}) &= \frac{\partial F}{\partial V}\,\Delta_2(V) + \Delta_2(F) + 2\,\Delta_1\left(\frac{\partial F}{\partial V}\,,\ V\right) + \frac{\partial^2 F}{\partial V^2}\,\Delta_1(V).
\end{aligned} \tag{5.2}$$

A *conformal density transformation* T* is one for which the associated cartogram T is conformal. Under a conformal density transformation T* the preceding result yields

THEOREM 5.2. *Under a conformal density transformation* T* *for which the scale of the associated conformal cartogram* T *is* $\rho = e^{\mu} = d\overline{s}/ds > 0$, *the Lamé differential parameters transform according* *to the rules*

$$e^{2\mu}\,\bar{\Delta}_1(\bar{V}) = \left(\frac{\partial F}{\partial V}\right)^2 \Delta_1(V) + 2\,\frac{\partial F}{\partial V}\,\Delta_1(V,F) + \Delta_1(F)$$

(5.3) $$e^{2\mu}\,\bar{\Delta}_2(\bar{V}) = \frac{\partial F}{\partial V}\,\Delta_2(V) + \Delta_2(F) + 2\,\Delta_1\left(\frac{\partial F}{\partial V}\,,\,V\right) + \frac{\partial^2 F}{\partial V^2}\,\Delta_1(V) +$$

$$+ (n-2)\left[\frac{\partial F}{\partial V}\,\Delta_1(V,\mu) + \Delta_1(F,\mu)\right].$$

This is established by means of equations (4.4) and Theorem 5.1. If a conformal density transformation T* between two Riemannian spaces is such that the scale of the associated cartogram T is $\rho = e^{\mu} = d\bar{s}/ds > 0$ and the law of change for the density V is

(5.4) $$G\,\bar{V} = G(x)\,\bar{V}(x) = V(x) = V$$

where $G = G(x)$ is a fixed positive scalar, then from Theorem 5.2 the Lamé differential parameters transform as follows

$$G^4 e^{2\mu}\,\bar{\Delta}_1(\bar{V}) = G^2\,\Delta_1(V) - 2\,G\,V\,\Delta_1(V,G) + V^2\,\Delta_1(G)\ ,$$

(5.5) $$G^2 e^{2\mu}\,\bar{\Delta}_2(\bar{V}) = [G\,\Delta_2(V) - V\,\Delta_2(G)] + (n-2)[G\,\Delta_1(V,\mu) -$$

$$- V\,\Delta_1(G,\mu)] + \frac{2}{G}\,[V\,\Delta_1(G) - G\,\Delta_1(V,G)]\,.$$

The following proposition is an extension to Riemannian space V_n, for $n \geqslant 2$, of the Kelvin transformation T*[5] of a Euclidean space E_n.

THEOREM 5.3. *The conformal density transformation* T* *whose density transforms according to the law* (5.4) *is such that the Lamé differential parameter of second order obeys*

(5.6) $$G^2 e^{2\mu}\,\bar{\Delta}_2(\bar{V}) = G\,\Delta_2(V) - V\,\Delta_2(G)$$

if and only if, except for a real positive multiplicative constant, the scale of the associated cartogram T *and the law for the change of density are*

(5.7) $$\rho = e^{\mu} = 1/R^2 = d\bar{s}/ds > 0\quad ,\quad \bar{V} = R^{n-2}\,V\ ,$$

where $R = R(x)$ *is a real positive scalar. The rules for the change* *of the Lamé differential parameters are*

$$\overline{\Delta}_1(\overline{V}) = R^{2n} [\Delta_1(V) - 2 V R^{n-2} \Delta_1(V , 1/R^{n-2}) +$$

$$(5.8) \qquad + V^2 R^{2n-4} \Delta_1(1/R^{n-2})] ,$$

$$\overline{\Delta}_2(\overline{V}) = R^{n+2} \Delta_2(V) - V R^{2n} \Delta_2(1/R^{n-2}) .$$

For, the given conformal density transformation T* possesses the stated property if and only if

$$(5.9) \quad (n-2)[G \Delta_1(V,\mu) - V \Delta_1(G,\mu)] + \frac{2}{G} [V \Delta_1(G) - G \Delta_1(V,G)] = 0$$

is an identity. If $\rho = e^{\mu} = 1/R^2$, where $R = R(x)$ is a real positive scalar, then the preceding identity is valid if and only if $G = e^{\frac{1}{2}(n-2)\mu} = 1/R^{n-2}$, except for a real positive multiplicative constant. Upon substituting this value for G in equations (5.3) the result follows.

In terms of cartesian coordinates of a point P in a Euclidean space E_n, for $n \geqslant 2$, an inversion T with respect to a sphere $\sum_{n-1}$ of dimension $n-1 \geqslant 1$, with center at a fixed point P_0 given by $(x_0^i) = (x_0^1,\ldots,x_0^n)$ and radius $a > 0$ is given by the set of n equations

$$(5.10) \qquad X^i - x_0^i = \frac{a^2}{R^2} (x^i - x_0^i) , \quad i = 1,2,\ldots,n ,$$

where $R^2 = \delta_{ij}(x^i - x_0^i)(x^j - x_0^j) > 0$. The scale of this inversion T is $\rho = d\overline{s}/ds = a^2/R^2 > 0$.

Therefore, every such T is a conformal cartogram T of the Euclidean space E_n onto itself, except for the center (x_0^i) of the sphere $\sum_{n-1}$

Since $1/R^{n-2} > 0$, is a harmonic function in E_n it is found that

$$(5.11) \qquad \overline{V} = R^{n-2} V , \qquad \overline{\Delta}_2(\overline{V}) = R^{n+2} \Delta_2(V) .$$

The system of equations (5.10) and (5.11) forms the Kelvin transformation T* for the Euclidean space E_n. The importance of such a transformation is that it converts every isothermal family Ω of ∞^1 surfaces $\sum_{n-1}$ into another family $\overline{\Omega}$ of ∞^1 surfaces in E_n.

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