1. ABSOLUTE DERIVATIVES AND NATURAL FAMILIES UNDER A CONFORMAL CARTOGRAM T.

Consider a conformal cartogram T between two Riemannian spaces $V_n$ and $\tilde{V}_n$, each of dimension $n \geq 2$, for which the scale $\rho = e^\mu = ds/d\tilde{s} > 0$, where $\mu = \mu(x)$ is a point function. Under the conformal cartogram T two corresponding unit contravariant vectors $\lambda^i$ and $\tilde{\lambda}^i$ of $V_n$ and $\tilde{V}_n$ respectively transform according to the law

\begin{equation}
\tilde{\lambda}^i = e^{-\mu} \lambda^i.
\end{equation}

Under T the arc length absolute derivative $\frac{\Delta \lambda^i}{ds}$ of a contravariant vector $\lambda^i$ of $V_n$ is expressed in terms of $\frac{\Delta \tilde{\lambda}^i}{ds}$ when it is considered as a vector of $\tilde{V}_n$, by the set of relations

\begin{equation}
\frac{\Delta \lambda^i}{ds} = e^{-\mu} \frac{\Delta \lambda^i}{ds} + e^{-\mu} \left( \frac{\partial u}{\partial x^a} \lambda^a \frac{dx^i}{ds} - (g_{jk} \lambda^j \frac{dx^k}{ds})(\lambda^a \frac{\partial u}{\partial x^a}) \right).
\end{equation}

In particular if $\lambda^i$ and $\tilde{\lambda}^i$ are two corresponding unit contravariant vectors then

\begin{equation}
\frac{\Delta \lambda^i}{ds} = e^{-2\mu} \frac{\Delta \lambda^i}{ds} + e^{-2\mu} \left( \frac{\partial u}{\partial x^a} \lambda^a \frac{dx^i}{ds} - (g_{jk} \lambda^j \frac{dx^k}{ds})(\lambda^a \frac{\partial u}{\partial x^a}) \right).
\end{equation}

Consider two curves $C: x^i = x^i(s)$ and $\tilde{C}: \tilde{x}^i = \tilde{x}^i(s) = x^i(\tilde{s})$ which correspond under the conformal cartogram T between $V_n$ and $\tilde{V}_n$. Their two unit contravariant tangent vectors satisfy the relations $\frac{dx^i}{ds} = e^\mu (\frac{dx^i}{ds})$ and the two corresponding contravariant geodesic curvature vectors $\kappa^i$ and $\tilde{\kappa}^i$ obey the law of transformation \cite{1}.
(1.4) \[ \overrightarrow{K}^i = e^{-2\mu} K^i + e^{-2\mu} \left( \frac{d\nu}{ds} \frac{dx^i}{ds} - g^{ia} \frac{\partial \nu}{\partial x^a} \right). \]

A natural family \( \mathcal{Q} \) [2] of \( \infty^{2n-2} \) curves \( C \) of Riemannian space \( V_n \) is such that every curve \( C \) of \( \mathcal{Q} \) corresponds under a conformal cartogram \( T \) on \( V \) onto a Riemannian space \( \overline{V}_n \) to a geodesic \( \overline{C} \) of \( \overline{V}_n \).

In particular, the set of \( \infty^{2n-2} \) geodesics \( C \) of a Riemannian space \( V_n \) is a natural family.

**THEOREM 1.1.** A natural family \( \mathcal{Q} \) of a Riemannian space \( V_n \) is composed of all the \( \infty^{2n-2} \) integral solutions of the set of \( n \) second order ordinary differential equations

(1.5) \[ K^i = \frac{d^2x^i}{ds^2} + \gamma^{i} jk \frac{dx^i}{ds} \frac{dx^k}{ds} = g^{ia} \frac{\partial \nu}{\partial x^a} - \frac{d\nu}{ds} \frac{dx^i}{ds}. \]

This is obtained from (1.4) by setting \( \overrightarrow{K}^i = 0 \).

It is evident that under a conformal cartogram \( T \) that a natural family \( \mathcal{Q} \) of \( V_n \) corresponds to a natural family \( \overline{\mathcal{Q}} \) of \( \overline{V}_n \).

### 2. THE SYMBOLS \( A_{jk}^i \) AND \( B_{jk} \) OF A CONFORMAL SPACE \( \Gamma_n \)

Consider a Riemannian space \( V_n \) of dimension \( n \geq 2 \). The totality of all Riemannian spaces \( \overline{V}_n \) such that there exists a conformal cartogram \( T \) between \( V_n \) and \( \overline{V}_n \) is termed a conformal space \( \Gamma_n \).

If \( V_n \) and \( \overline{V}_n \) belong to the same conformal space \( \Gamma_n \) then their affine connections are related by the law

(2.1) \[ \overrightarrow{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \frac{\partial \nu}{\partial x^k} + \delta^i_k \frac{\partial \nu}{\partial x^j} - \delta^i_{jk} \frac{\partial \nu}{\partial x^a} \frac{\partial x^a}{\partial x^i}; \]

where \( \nu \) is the scale of the conformal cartogram \( T \) relating \( V_n \) and \( \overline{V}_n \).

It is an immediate consequence of these relations that

(2.2) \[ \frac{\partial \nu}{\partial x^j} = \frac{1}{n} \left[ \overrightarrow{\gamma}^a_j - \gamma^a_j \right] \quad j = 1, 2, \ldots, n. \]

Consider a fixed Riemannian space \( V_n \), for \( n \geq 2 \). The symbols \( A_{jk}^i \)
and $B_{jk}$ are defined by the expressions

$$
\begin{align*}
A^i_{jk} &= r^i_{jk} - \frac{1}{n} (\delta^i_j r^a_{ak} - \delta^i_k r^a_{aj}) + \frac{1}{n} g^i_{jk} g^a_{a\delta} r^a_{a\delta}, \\
B_{jk} &= \frac{3r^a_{aj}}{3x^k} - \frac{3r^a_{ak}}{3x^j}.
\end{align*}
$$

THEOREM 2.1. Two Riemannian spaces $V_n$ and $\overline{V}_n$, for $n \geq 2$, are conformally equivalent if and only if the two sets of symbols $A^i_{jk}$ and $B_{jk}$ are the same in every admissible coordinate system $(x)$, provided that the set of initial conditions $\overline{g}_{ij}(x_0) = e^{2\nu(x_0)} g_{ij}(x_0)$ is satisfied for $i, j = 1, 2, \ldots, n$ at some fixed point $P_0$, where $\nu(x_0)$ is a fixed real constant.

For, by means of (2.1) and (2.2) it is found that $A^i_{jk} = A^i_{jk}$ and $B_{jk} = B_{jk}$ for two conformally equivalent Riemannian spaces $V_n$ and $\overline{V}_n$.

Conversely, suppose that $\overline{A}^i_{jk} = A^i_{jk}$ and $\overline{B}_{jk} = B_{jk}$ for two Riemannian spaces $V_n$ and $\overline{V}_n$. The second set represent integrability conditions for the equations (2.2). Thus, let $\nu = \nu(x)$ represent a solution of (2.2). There exists one and only one solution satisfying the prescribed set of initial conditions at the fixed point $P_0$. By use of the conditions $A^i_{jk} = A^i_{jk}$ the law of transformation (2.1) is found. It then is easily deduced that $\overline{g}_{ij} = e^{2\nu(x)} g_{ij}$ where $\nu = \nu(x)$ is the unique solution discussed above. Consequently $V_n$ and $\overline{V}_n$ are conformally equivalent.

It is noted that the symbols $A^i_{jk}$ are symmetric in the lower indices and that $A^i_{ik} = A^i_{ki} = 0$. However the $A^i_{jk}$ do not form a tensor. The symbols $B_{jk}$ are skew symmetric, that is $B_{jk} = -B_{kj}$, and form a skew symmetric covariant tensor of second order.
3. COVARIANT DIFFERENTIATION IN A CONFORMAL SPACE $\Gamma_n$.

In a Riemannian space $V_n$ the covariant derivatives $\lambda^i_{,k}$ and $\lambda^i_{,k}$ of a contravariant vector $\lambda^i$ and a covariant vector $\lambda_i$ may be given by

$$\lambda^i_{,k} = \frac{\partial \lambda^i}{\partial x^k} + A^i_{ak} \lambda^a + \frac{1}{n} \lambda^i \rho^a_{ak} + \frac{1}{n} \delta^i_{k} \rho^a_{aa} \lambda^a - \frac{1}{n} g^{ia} g_{kB} r_{ac} \lambda^b,$$

(3.1)

$$\lambda^i_{,k} = \frac{\partial \lambda^i}{\partial x^k} - A^i_{ik} \lambda - \frac{1}{n} \lambda^i \rho^a_{ak} - \frac{1}{n} \lambda^i \rho^a_{ai} + \frac{1}{n} g^{ia} g_{ik} g_{ab} r_{ac} \lambda^b.$$

The two corresponding absolute differentials are $D\lambda^i = \lambda^i_{,k} dx^k$ and $D\lambda_i = \lambda_i_{,k} dx^k$.

In a conformal space $\Gamma_n$, $n \geq 2$, with invariant symbols $A^i_{jk}$ and $B_{ik}$ the conformal covariant derivatives $\Delta^i_k \lambda^i$ and $\Delta_k \lambda_i$ of any contravariant vector $\lambda^i$ and any covariant vector $\lambda_i$ are defined by

$$\Delta^i_k \lambda^i = \frac{\partial \lambda^i}{\partial x^k} + A^i_{ik} \lambda^a, \quad \Delta_k \lambda_i = \frac{\partial \lambda_i}{\partial x^k} - A^a_{ik} \lambda^a.$$

(3.2)

The corresponding conformal absolute differentials are $\Delta \lambda^i = \Delta^i_k \lambda^i dx^k$ and $\Delta \lambda_i = \Delta_k \lambda_i dx^k$.

**THEOREM 3.1.** The four sets of quantities, $\Delta^i_k \lambda^i$, $\Delta_k \lambda_i$, $\Delta \lambda^i$, $\Delta \lambda_i$ are all invariant under the conformal group $G$ of the conformal group $G$ of $\Gamma_n$. If $V_n$ is an element of $\Gamma_n$ then the relationships between these conformal covariant derivatives and the covariant derivatives relative to $V_n$ are found by substituting the relations (3.2) into the equations (3.1).

It is observed that the differential $d(\phi, \psi)$ of the inner product $(\phi, \psi) = \phi^i \psi_i$ of any contravariant vector $\phi^i$ and any covariant vector $\psi_i$ is
THEOREM 3.2. If two Riemannian spaces $V_n$ and $\overline{V}_n$, for $n \geq 2$, belong to the same conformal space $\Gamma_n$ and correspond by a conformal cartogram $T$ for which the scale is $\rho = e^{u} = ds/ds > 0$, then for every geometric vector $\lambda = \lambda^i = \lambda_i$ with $|\lambda| > 0$ of $V_n$ there exists a point function $r = r(x)$ depending on $\lambda$ such that by $T$ the images $\overline{\lambda}^i$ and $\overline{\lambda}_i$ of $\lambda^i$ and $\lambda_i$ in $\overline{V}_n$ are $\overline{\lambda}^i = e^{-r}\lambda^i$ and $\overline{\lambda}_i = e^{-r}\lambda_i$. The two covariant derivatives $\lambda^i$ and $\lambda_i$ obey the laws of transformation

\begin{align*}
\overline{\lambda}^i_{,k} &= e^{-r-u} \left[ \lambda^i_{,k} + \lambda_i \frac{\partial u}{\partial x^k} + g^{ia} \frac{\partial u}{\partial x^a} \right], \\
\overline{\lambda}_i \kappa &= e^{-r+u} \left[ \lambda_i \kappa + \lambda_i \frac{\partial u}{\partial x^i} - \lambda_k \frac{\partial u}{\partial x^i} + g_{ik} \frac{\partial u}{\partial x^b} \right].
\end{align*}

This result is a consequence of the previous discussion.

It is noted that when $r = 0$, we obtain the laws of transformation for unit vectors.

4. SOME CONFORMAL PROPERTIES OF THE LAME DIFFERENTIAL PARAMETERS $\Delta_1(U,V)$ AND $\Delta_2(V)^{[3]}$.

In a Riemannian space $V_n$, for $n \geq 2$, the Lamé differential parameter $\Delta_1(U,V)$ of order one, of two scalars $U = U(x)$ and $V = V(x)$ is the scalar

\begin{equation}
\Delta_1(U,V) = g^{jk} \frac{\partial U}{\partial x^j} \frac{\partial U}{\partial x^k} = (\text{grad } U, \text{grad } V),
\end{equation}

In particular, if $U = V$, then $\Delta_1(V) = \Delta_1(V,V) = |\text{grad } V|^2$.

The Lamé differential parameter of second order $\Delta_2(V) = \nabla^2(U)$ is the Laplacean and is defined by the scalar
(4.2) \[ \Delta_2(V) = \nabla^2(V) = g^{jk} V_{,jk} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left( \sqrt{g} g^{jk} \frac{\partial V}{\partial x^k} \right); \]

where \( g = |g_{ij}| > 0. \)

If two Riemannian spaces correspond by a conformal cartogram \( T \) and if \( V = V(x) \) is a scalar then

(4.3) \[ \overline{V}_{,j} = V_{,j}, \quad \overline{V}_{,jk} = V_{,jk} - V_{,j} \mu_{,k} + g_{jk} \Lambda_1(\mu, V). \]

The three laws of transformation for the Lamé differential parameters are

(4.4) \[ \overline{\Lambda}_1(U,V) = e^{-2\mu} \Lambda_1(U,V), \quad \overline{\Lambda}_1(V) = e^{-2\mu} \Lambda_1(V) \]

\[ \overline{\Lambda}_2(V) = e^{-2\mu} [\Lambda_2(V) + (n-2) \Lambda_1(\mu, V)]. \]

These are obtained by means of Theorem 3.2, where the scalar \( r = r(x) \) is replaced by \(-\mu(x)\).

If a scalar \( V = V(x) \) with \( \Delta_1(V) > 0 \), is a harmonic function in a Riemannian space \( V_n \), then the equation \( V = V(x) = \text{constant} \) defines an isothermal family of \( \omega^1 \) surfaces \( \Gamma_{n-1} \), each of deficiency one, in \( V_n \).

It may be proved that a simple family of \( \omega^1 \) surfaces \( \Gamma_{n-1} \), each of deficiency one in a Riemannian space \( V_n \), for \( n \geq 2 \), is an isothermal family \( V = V(x) = C = \text{constant} \) if and only if \( V = V(x) \) obeys a partial differential equation of second order of the form

(4.5) \[ \frac{\Delta_2(V)}{\Delta_1(V)} = \frac{g^{jk} V_{,jk}}{g^{jk} V_{,j,k}} = F(V), \]

where \( F = F(V) \) is a scalar depending only on \( V = V(x) \).

THEOREM 4.1. [4] If two Riemannian spaces \( V_n \) and \( \overline{V}_n \), for \( n \geq 2 \), correspond by a conformal cartogram \( T \) then every isothermal family in \( V_n \) is converted into an isothermal family in \( \overline{V}_n \) if and only if either \( n = 2 \) or else if \( n \geq 3 \), \( T \) is a homothetic cartogram for which \( \mu \) is a real constant.

This proposition is established by means of the conditions (4.4) and (4.5).
5. DENSITY TRANSFORMATIONS BETWEEN TWO RIEMANNIAN SPACES $V_n$ AND $\overline{V}_n$.

Let two Riemannian spaces $V_n$ and $\overline{V}_n$, for $n \geq 2$, correspond by a cartogram $T$, conformal or not, for which the differential $ds$ and $d\overline{s}$ of arc length along corresponding curves $C$ and $\overline{C}$ of $V_n$ and $\overline{V}_n$ are defined by $ds^2 = g_{ij} \, dx^i \, dx^j$ and $d\overline{s}^2 = \overline{g}_{ij} \, dx^i \, dx^j$.

A density transformation $T^*$ between $V_n$ and $\overline{V}_n$ is such that their respective points correspond by a cartogram $T$, either conformal or not, and a scalar $V = V(x)$, which is evaluated at a point $P$ of $V_n$ is converted into the scalar

\begin{equation}
V = \overline{V}(x) = F(V;x),
\end{equation}

for which $\frac{\partial V}{\partial V} \neq 0$, whose value is associated with the corresponding point $\overline{P}$ of $\overline{V}_n$. A scalar $V = V(x)$ calculated at $P$ in $V_n$ is called a density $V$ of $P$.

THEOREM 5.1. Under a density transformation $T^*$ between two Riemannian spaces $V_n$ and $\overline{V}_n$, for $n \geq 2$, the Lamé differential parameters $\Delta_1(V)$ and $\Delta_2(V)$ of a density transform according to the laws

\begin{align}
\Delta_1(\overline{V}) &= \frac{\partial F}{\partial V} \Delta_1(V) + 2 \frac{\partial F}{\partial V} \Delta_1(V,F) + \Delta_1(F), \\
\Delta_2(\overline{V}) &= \frac{\partial F}{\partial V} \Delta_2(V) + \Delta_2(F) + 2 \frac{\partial F}{\partial V} \Delta_2(V,F) + \Delta_2(F), \\
\Delta_1(\overline{V}) &= \frac{\partial F}{\partial V} \Delta_1(V) + \Delta_1(F) + 2 \frac{\partial F}{\partial V} \Delta_1(V,F) + \frac{\partial^2 F}{\partial V^2} \Delta_1(V), \\
\Delta_2(\overline{V}) &= \frac{\partial F}{\partial V} \Delta_2(V) + \Delta_2(F) + 2 \frac{\partial F}{\partial V} \Delta_2(V,F) + \frac{\partial^2 F}{\partial V^2} \Delta_2(V).
\end{align}

A conformal density transformation $T^*$ is one for which the associated cartogram $T$ is conformal. Under a conformal density transformation $T^*$ the preceding result yields

THEOREM 5.2. Under a conformal density transformation $T^*$ for which the scale of the associated conformal cartogram $T$ is $\rho = e^{\mu} = \frac{d\overline{s}}{ds} > 0$, the Lamé differential parameters transform according
to the rules
\[ e^{2\mu} \Delta_1(\nu) = (\frac{3F}{3V})^2 \Delta_1(V) + 2 \frac{3F}{3V} \Delta_1(V,F) + \Delta_1(F) \]
(5.3) \[ e^{2\mu} \Delta_2(\nu) = \frac{3F}{3V} \Delta_2(V) + \Delta_2(F) + 2 \Delta_1(\frac{3F}{3V}, V) + \frac{3F}{3V^2} \Delta_1(V) + \]
\[ + (n-2)(\frac{3F}{3V} \Delta_1(V,\nu) + \Delta_1(F,\nu)) \].

This is established by means of equations (4.4) and Theorem 5.1.

If a conformal density transformation T* between two Riemannian spaces is such that the scale of the associated cartogram T is \( \rho = e^\mu = d\xi/ds > 0 \) and the law of change for the density V is

\[ G \nu = G(x) \nu(x) = V(x) = V \]
where G = G(x) is a fixed positive scalar, then from Theorem 5.2 the Lamé differential parameters transform as follows

\[ G^4 e^{2\mu} \Delta_1(\nu) = G^2 \Delta_1(V) - 2 G V \Delta_1(V,G) + V^2 \Delta_1(G) \]
(5.4) \[ G^2 e^{2\mu} \Delta_2(\nu) = [G \Delta_2(V) - V \Delta_2(G)] + (n-2)[G \Delta_1(V,\nu) - \]
\[ - V \Delta_1(G,\nu)] + \frac{2}{G} [V \Delta_1(G) - G \Delta_1(V,G)] .

The following proposition is an extension to Riemannian space \( V_n \), for \( n \geq 2 \), of the Kelvin transformation T*\(^5\) of a Euclidean space \( E_n \).

THEOREM 5.3. The conformal density transformation T* whose density transforms according to the law (5.4) is such that the Lamé differential parameter of second order obeys

\[ G^2 e^{2\mu} \Delta_2(\nu) = G \Delta_2(V) - V \Delta_2(G) \]
(5.6) \[ if and only if, except for a real positive multiplicative constant, the scale of the associated cartogram T and the law for the change of density are \]
\[ \rho = e^\mu = 1/R^2 = d\xi/ds > 0 \ , \ \nu = R^{n-2} V \ , \]
where R = R(x) is a real positive scalar. The rules for the change
of the Lamé differential parameters are

$$\Delta_1(V) = R^{2n} \left[ \Delta_1(V) - 2 V R^{n-2} \Delta_1(V, 1/R^{n-2}) - \right.\
\left. + V^2 R^{2n-4} \Delta_1(1/R^{n-2}) \right],$$

(5.8) $$\Delta_2(V) = R^{n+2} \Delta_2(V) - V R^{2n} \Delta_2(1/R^{n-2}).$$

For, the given conformal density transformation $T^*$ possesses the stated property if and only if

$$(5.9) \quad (n-2)(G \Delta_1(V, \mu) - V \Delta_1(G, \mu)) + \frac{2}{G} [V \Delta_1(G) - G \Delta_1(V, G)] = 0$$

is an identity. If $\rho = e^\mu = 1/R^2$, where $R = R(x)$ is a real positive scalar, then the preceding identity is valid if and only if $G = e^{1/2 (n-2)\mu} = 1/R^{n-2}$, except for a real positive multiplicative constant. Upon substituting this value for $G$ in equations (5.5) the result follows.

In terms of cartesian coordinates of a point $P$ in a Euclidean space $E_n$, for $n \geq 2$, an inversion $T$ with respect to a sphere $S_{n-1}$ of dimension $n-1 \geq 1$, with center at a fixed point $P_0$ given by $(x_0^i) = (x_0^1, \ldots, x_0^n)$ and radius $a > 0$ is given by the set of $n$ equations

$$(5.10) \quad x^i - x_0^i = \frac{a^2}{R^2} (x^i - x_0^i), \quad i = 1, 2, \ldots, n,$$

where $R^2 = \delta_{ij} (x^i - x_0^i)(x^j - x_0^j) > 0$. The scale of this inversion $T$ is $\rho = d\bar{s}/ds = a^2/R^2 > 0$.

Therefore, every such $T$ is a conformal cartogram $T$ of the Euclidean space $E_n$ onto itself, except for the center $(x_0^i)$ of the sphere $S_{n-1}$.

Since $1/R^{n-2} > 0$, is a harmonic function in $E_n$ it is found that

$$(5.11) \quad \bar{V} = R^{n-2} V, \quad \Delta_2(\bar{V}) = R^{n+2} \Delta_2(V).$$

The system of equations (5.10) and (5.11) forms the Kelvin transformation $T^*$ for the Euclidean space $E_n$. The importance of such a transformation is that it converts every isothermal family $\hat{\alpha}$ of $\omega^1$ surfaces $S_{n-1}$ into another family $\bar{\alpha}$ of $\omega^1$ surfaces in $E_n$. 
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