CONVEX POLYTOPES IN RIEMANNIAN MANIFOLDS

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1. INTRODUCTION. Let $M^n$ be a $n$-dimensional Riemannian manifold. By a $m$-dimensional convex polytope $p^m$ embedded in $M^n$ ($2 \leq m \leq n$) we mean a convex Riemannian polyhedron (for definition, see [1]) embedded in $M^n$ bounded by a finite number of totally geodesic submanifolds $p^{m-1}_\lambda$ of dimension $m-1$ such that $p^{m-1}_\lambda$ intersect at lower dimensional totally geodesic submanifolds $p^r_u$ ($0 \leq r \leq m-2$).

Let various dimensional outer angles of $p^m$ be given. One question is to find the volume $V(p^m)$ of $p^m$ in terms of the given outer angles of $p^m$. When $m$ is even ($m = 2p$) and $M^n$ is of constant sectional curvature $K \neq 0$, the Gauss-Bonnet formula of Allendoerfer, Chern, Fenchel and Weil ([1] and [2]) implies such a volume formula which might be interesting and seems not to have appeared in given classical literatures on convex polytopes.

2. GAUSS-BONNET FORMULA OF RIEMANNIAN POLYHEDRA IN RIEMANNIAN MANIFOLDS.

A Riemannian polyhedron $p^m$ is a Riemannian manifold with a boundary consisting of polyhedra $p^r_\lambda$ of lower dimensions for $0 \leq r \leq m-1$. We denote by $X'(p^m)$ the inner characteristic of $p^m$, that is, the Euler-Poincaré characteristic of the open complex consisting of all inner cells in an arbitrary simplicial or cellular subdivision of $p^m$.

From now on we shall assume $m = 2p$, that is, $m$ is even.

Let $S(p^m)$ be the tangent sphere bundle over $p^m$ that is the bundle of unit tangent vectors of $p^m$. Let $\sigma: S(p^m) \longrightarrow p^m$ be the projection. Let $\in_{i_1 \ldots i_k}$ be the Kronecker index which is equal to $+1$ or $-1$ according as $i_1 \ldots i_k$ constitute an even or odd permu-
In [2], Chern constructed a \((m-1)\)-form

\[
\Phi = \frac{1}{\pi^p} \sum_{\lambda=0}^{p-1} (-1)^\lambda \frac{1}{1.3...(2p-2\lambda-1)2^{p+\lambda}} \Phi_\lambda
\]
on \(S(P^m)\), where for \(\lambda = 0,1,...,p-1,

\[
\Phi_\lambda = \sum_{i_1,...,i_{2p-1}} \Omega_{i_1}^{i_1} \wedge ... \wedge \Omega_{i_{2\lambda-1}}^{i_{2\lambda-1}} \wedge \Omega_{i_{2\lambda}}^{i_{2\lambda}} \wedge ... \wedge \Omega_{i_{2p-1}}^{i_{2p-1}}.
\]

There exists a unique closed \(m\)-form \(\Psi\) on \(P^m\) such that

\[
\sigma^*(\Psi) = \frac{(-1)^p}{2^{2p}p!} \sum_{i_1,...,i_{2p-1}} \Omega_{i_1}^{i_1} \wedge ... \wedge \Omega_{i_{2p-1}}^{i_{2p-1}}
\]

Let \(\Gamma(P^m)\) be a outer normal vector field on \(P^m\) in \(S(P^m)\). Then the Gauss-Bonnet formula for Riemannian polyhedra \(P^m\) \((m = 2p)\) in Riemannian manifolds is given by (\([1]\) and \([2]\))

\[
\int_{\partial P^m} \Psi = \int_{\gamma(P^m)} \sigma^* \Psi = \int_{\partial P^m} \int_{\Gamma(P^m)} \Phi - \chi'(P^m)
\]

where \(\Gamma(\partial P^m)\) denotes the outer angle at an arbitrary point \(x\) of \(\partial P^m\) which is a spherical cell on the unit sphere \(S^{m-r-1}\) in the normal linear manifold to \(P^m\) at \(x\).

3. CONVEX POLYTOPES IN RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE \(K(\neq 0)\).

Let \(P^m\) be a convex polytope in a Riemannian manifold \(M^n\) of constant sectional curvature \(K(\neq 0)\). We shall consider \(P^m\) as a convex polytope in a totally geodesic submanifold \(N^m\) of \(M^n\). The curvature form \(\Omega = (\Omega^i_j)\) in the principal bundle \(O(N^m)\) satisfies

\[
\Omega^i_j = K \theta^i \wedge \theta^j
\]

where \(\theta = (\theta^i)\) is the canonical form in \(O(N^m)\).

Consequently,

\[
\int_{P^m} \Psi = \frac{(-1)^p}{2^{2p}p!} \int_{\gamma(P^m)} \sum_{i_1,...,i_{2p}} \Omega_{i_1}^{i_1} \wedge ... \wedge \Omega_{i_{2p-1}}^{i_{2p-1}} =
\]
Since $p^m$ is convex, $X'(p^m) = 1$. Hence (1) becomes

$$\int_{p^m} \Psi + 1 = \int_{3p^m} \int_{r(3p^m)} \Phi.$$ 

Let $3p^m = \bigcup_{r=0}^{m-1} p^r_u$. Then we have

$$\int_{3p^m} \int_{r(3p^m)} \Phi = \sum_{u=0}^{m-1} \int_{p^r_u} \int_{r(p^r_u)} \Phi.$$ 

Since $p^r_u$ are totally geodesic in $N^m$, we may choose a suitable frame $\{e_1, \ldots, e_r\}$ on a coordinate neighborhood $U$ in $p^r_u$ such that $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_m\}$ is a frame for $N^m$ and the Christoffel symbol

$$\Gamma_{\alpha \beta} = 0 \quad 1 \leq \alpha, \beta \leq r, \ r+1 \leq \delta \leq m. \quad \text{(see [3]).}$$

We remark that under the spherical map $\eta$ from $r(p^r_u)$ to $S^{m-r-1}_u$, $\eta^*(d\sigma) = \omega^r u^{2p-1} \ldots u^{2p-1}$, where $d\sigma$ is the surface area element of $S^{m-r-1}$.

It is not difficult to see that

$$\int_{r(p^r_u)} \int_{r(p^r_u)} \Phi = (-1)^{ \lambda \frac{1}{\pi^p} } \frac{1}{1.3 \ldots (2p-2\lambda-1)2^p+\lambda!} \Gamma_{\mu} \int_{r(p^r_u)} \Phi = \frac{(-1)^{ \lambda \frac{1}{\pi^p} } }{1.3 \ldots (2p-2\lambda-1)2^p+\lambda!} \frac{1}{(2\lambda)!} \frac{(2\lambda)!}{(2p-2\lambda-1)!} K^p V(p^2) r(p^2)$$

when $r = 2\lambda$, otherwise

$$\int_{r(p^r_u)} \int_{r(p^r_u)} \Phi = 0.$$

Consequently, we get from (1) and (2) the following

$$\frac{(-1)^{ \lambda \frac{1}{\pi^p} } }{2^p \pi^p} \frac{(2p)!}{p!} K^p V(p^{2p}) + 1 =$$
Thus, we can express the volume $V(p^{2p})$ in terms of outer angles of $p^{2p}$ and $p^{2\lambda}$, for $\lambda = 0, \ldots, p-1$. This will be achieved inductively. When $p = 1$, we get the usual Gauss formula for geodesic polygons.

REFERENCES

