A NOTE ON THE MAXIMALITY OF THE IDEAL OF COMPACT OPERATORS

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Let $A,B$ be rings, and $\mathcal{L}$ and $A$-$B$-bimodule, i.e., $\mathcal{L}$ is a left $A$-module and a right $B$-module and moreover $s(tu) = (st)u$ for $s \in A$, $t \in \mathcal{L}$ and $u \in B$. A subset $C \subseteq \mathcal{L}$ is a sub-bimodule if it is an additive subgroup and satisfies $sku \in C$ whenever $k \in C$ and $s \in A$, $u \in B$.

If $E,F$ are Banach spaces, we shall denote the space of bounded linear operators $T:E \rightarrow F$ by $\mathcal{L}(E,F)$ (and by $\mathcal{L}(E)$ when $E = F$).

Consider the following situation: $A = \mathcal{L}(\ell^q)$, $B = \mathcal{L}(\ell^p)$, $\mathcal{L} = \mathcal{L}(\ell^p, \ell^q)$, where $\ell^r$, $1 \leq r < +\infty$ denotes the (real or complex) Banach space of numerical $r$-summable sequences.

The bimodule structure is defined by composition

$\ell^p \rightarrow \ell^p \rightarrow \ell^q \rightarrow \ell^q$ (we will use capital letters for operators).

It is clear that the set of compact operators $C = C(\ell^p, \ell^q)$ is a sub-bimodule of $\mathcal{L}$. We aim to make a few remarks on the following results:

a) if $1 < q < p < +\infty$, then $C = \mathcal{L}$;
b) if $1 < p = q < +\infty$, then $C$ is a maximal sub-bimodule ( = two sided ideal) of $\mathcal{L}$;
c) if $1 < p < q < +\infty$, then all sub-bimodules $S \subseteq \mathcal{L}$ satisfying $C \subseteq S$ contain necessarily the identity operator $J: \ell^p \rightarrow \ell^q$.

The statements a) and b) are known; a) goes back to Pitt [3] and is in fact a particular case of Th. A2 in [4], b) coincides with Th. 5.1 in [1] and c) seems to be new.

Our goal here is to observe that a modification of known proofs of b) actually yield c) of which b) is a particular case, and that a) is a corollary of b). This last remark would shorten the proof of Th. A2 in [4] and mildly confirms our suspicion that proving c) first has some methodological advantages. We believe (but have been unable to prove) the following:

CONJECTURE: if $1 < p < q < +\infty$, then $C$ is a maximal sub-bimodule,
from which c) follows trivially.

The proof of c) above is obtained by restating meanderingly the ingredients of the proofs of Lemma 5.1 in [1] and Lemmas 1 and 2 in [2]. We denote by \( \|x\|_s \) the s-norm of \( x = (x_1, x_2, \ldots) \), i.e.,

\[
\|x\|_s = \left( \sum_{j=1}^{n} |x_j|^s \right)^{1/s}
\]

**LEMMA.** Let \( 1 < s < \infty \), \( \epsilon_n > 0 \), \( n = 1, 2, \ldots \), \( x^k \in \ell^s \), \( k = 1, 2, \ldots \) and suppose that \( x^k \rightarrow 0 \) weakly and \( \inf \{ \|x^k\|_s \ ; \ k = 1, 2, \ldots \} = \delta > 0 \). Then there exists an increasing sequence of positive integers \( n_1 < n_2 < \ldots \) and elements \( z^k \in \ell^s \), \( k = 1, 2, \ldots \) such that:

i) \( \|x^k - z^k\|_s \leq \epsilon_k \) for \( k = 1, 2, \ldots \);

ii) the operator \( T_1 \in \ell(\ell^s) \) determined by \( T_1 e^k = \frac{z^k}{\|z^k\|_s} \) (where \( e^k \) is the kth unit vector \((0,0,\ldots,1,0,\ldots)\)) in \( \ell^s \) is an isometry and the image \( E = T_1(\ell^s) \) of \( T_1 \) is a complemented subspace of \( \ell^s \).

**Proof.** Define \( \epsilon'_n = \min(\epsilon_n, \frac{\delta}{2}) \). For \( x = (x_j) \in \ell^s \) and \( n \) a positive integer denote by \( P_n x \) the sequence \( (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \).

Let now \( n_1 \) be large enough for \( \|x^1 - P_{n_1} x^1\|_s \leq \epsilon'_1 \) to be true and define \( z^1 = P_{n_1} x^1 \).

Since \( x^n \rightarrow 0 \) weakly (i.e., coordinate wise) there is an integer \( n_2 \) such that \( \|P_n x^n\|_s \leq \frac{\epsilon'_1}{2} \). Choose \( N \) such that \( \|x^{n_2} - P_N x^{n_2}\|_s \leq \frac{\epsilon'_1}{2} \) and define \( z^2 = P_N x^{n_2} - P_{n_1} x^{n_2} \). Clearly \( \|x^{n_2} - z^2\|_s \leq \|x^{n_2} - P_N x^{n_2}\|_s + \|P_{n_1} x^{n_2}\|_s \leq \epsilon'_1 \). The procedure can be iterated in such a way that \( \|x^n - z^n\|_s \leq \epsilon'_n \) and the vectors \( z^k \) have disjoint support, i.e., for each \( n \) there is at most one \( k \) with \( z^n_k \neq 0 \).

Since we also have \( \|z^n_k\|_s > \|x^n_k\|_s - \|x^n_k - z^n_k\|_s \geq \delta - \frac{\epsilon'_n}{2} \) \( = \frac{1}{2} \delta \) \( > 0 \), (i) and (ii) follow from Lemma 1 in [2].
Proof of a). Let $p^*$ be the conjugate of $p$ defined by $p^* = p/(p - 1)$. First observe that if $T \in \mathcal{L}(\ell^p, \ell^q)$, $1 < p, q < +\infty$, and

$$\|T\|_{p^*} < +\infty,$$

then $T$ is compact.

This is obvious because if $P_n \in \mathcal{L}(\ell^p)$ is the projector on the first $n$ coordinates defined above, then for $x \in \ell^p$ we have $\|T - TP_n\|_q = 0$.

Assume now that $S$ is a sub-bimodule of $\mathcal{L} = \mathcal{L}(\ell^p, \ell^q)$, $1 < p < q < +\infty$ such that $C \subset S$ and $C \not= S$, or equivalently, such that all compact operators belong to $S$ and there is a non-compact $T' \in S$.

For $\epsilon > 0$, choose a sequence $\epsilon_n > 0$ such that $\sum_n \epsilon_n^{p^*} = \epsilon p^*$ and let $n_1 < n_2 < \ldots$ and $z_1, z_2, \ldots$ be as in the lemma above, corresponding to these $\epsilon_n$. It is clear that $\frac{1}{2} \delta < \|z_k\|_p < \Delta$ for some $\Delta$ and all $k$ and therefore the operator $T_1$ in the lemma can be modified by an invertible diagonal operator $D \in \mathcal{L}(\ell^p)$ in such a way that $S_1 = T_1D : \ell^p \rightarrow \ell^p$ satisfies $S_1 e^{k} = z_k$ for all $k = 1, 2, \ldots$. Consider now, for $\lambda_1, \lambda_2, \ldots, \lambda_n$ arbitrary scalars, the estimate
This clearly shows that there is a well defined bounded operator \( S: \ell^p \to \ell^p \) satisfying \( S e^k = x^n_k \) for \( k = 1, 2, \ldots \), and in fact \( \| (S - S_1) e^k \|_p = \| x^n_k - z^n_k \|_p \leq \epsilon_k \). Let now \( T'' = T' S \in S \). Setting \( y^k = T'' e^k = T x^n_k \in \ell^q \), we have \( \| y^k \|_q > \delta > 0 \) for \( k = 1, 2, \ldots \) and since \( e^k \to 0 \) weakly we also have \( y^k \to 0 \) weakly in \( \ell^q \). Hence the lemma above applies again: let \( (y^m_k) \) be a sub-sequence of \( (y^k) \) and \( (w^k) \) satisfy \( \| y^m_k - w^k \|_q \leq \epsilon_k \) with \( (w^k) \) equivalent to the unit basis of \( \ell^q \). If \( S' \in \mathcal{L}(\ell^p) \) is defined by \( S' e^k = e^m_k \) we obviously have \( T = T'' S' \in S \) and \( T e_k = y^m_k \). Let us denote by \( U \in \mathcal{L}(\ell^q) \) the operator (corresponding to \( T_1 \) in the lemma) determined by \( U e^k = w^k \) and by \( J: \ell^p \to \ell^q \) the identity map. We have \( \| U J e^k - T e^k \|_q = \| w^k - T e^k \|_q \leq \epsilon_k \) so that \( U J - T \in \mathcal{L} \) is compact by the first part of this proof. Therefore \( U J = (U J - T) + T \in S \). But the subspace generated by \( (w^k) \) being complemented in \( \ell^q \) (see lemma) and isomorphic to \( \ell^q \), there is a \( U' \in \mathcal{L}(\ell^q) \) such that \( U' U \in \mathcal{L}(\ell^q) \) is the identity operator. Then \( J = (U' U) J = U'(U J) \in S \), as claimed.

**Proof of a.** First let us observe that b) implies that every operator \( W \in \mathcal{L}(\ell^q) \) of the form \( W = W_1 W_2 \), \( W_1 \in \mathcal{L}(\ell^p, \ell^q) \), \( W_2 \in \mathcal{L}(\ell^q, \ell^p) \) for some \( p \neq q \), must necessarily be compact. In fact, the family \( M \) of such operators is a two sided ideal in \( \mathcal{L}(\ell^q) \) which contains all operators of finite rank. Thus, the closure of \( M \) contains \( \mathcal{C}(\ell^q) \). But the closure of \( M \) is different from \( \mathcal{L}(\ell^q) \) because the identity in \( \mathcal{L}(\ell^q) \) is at distance one from any proper ideal such as \( M \). But \( \mathcal{C} \) being maximal by b), it follows that \( M \subseteq \text{closure} \ M = \mathcal{C} \).

Assume now that \( 1 < q < p < +\infty \) and \( T \in \mathcal{L}(\ell^p, \ell^q) \) is not compact. Then there is a sequence \( (x^n) \) in \( \ell^p \) such that \( x^n \to 0 \) weakly and \( \| T x^n \|_q > \delta > 0 \) for some \( \delta \). It follows that \( \| x^n \|_p > \delta' > 0 \) also for an appropriate \( \delta' \). Now we apply the lemma again to produce a
sequence \((z^k)\) in \(\ell^p\) such that: i) there is an operator \(T_1 \in \mathcal{L}(\ell^p)\) satisfying \(Tz^k = z^k\) and ii) \(z^k\) is near \(x^0\), so that also \(\|Tz^k\|_q \geq \delta/2\) for all \(k = 1, 2, \ldots\). Consider now the operator \(W = W_1 W_2\) where \(W_1 = T\) and \(W_2 = T_1 J\) for \(J: \ell^q \to \ell^p\) the identity. From the first remark, \(W\) must be compact, and in particular \(\|Wz^k\|_q \to 0\). But this contradicts \(Wz^k = TT_1 Je^k = TT_1 e^k = Tz^k \to 0\). Then \(T\) is compact, and the proof of a) is complete.

REFERENCES


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