Revista de la
Unión Matemática Argentina
Volumen 26, 1973.

A GEOMETRIC APPROACH TO INNER FUNCTION-OPERATORS AND THEIR DIFFERENTIAL EQUATIONS

Domingo A. Herrero*

0. ABSTRACT. An *inner function-operator* in a (complex separable) Hilbert space K is a function U(x) defined on the real line **R**, taking values in the set $\mathcal{U}(K)$ of unitary operators in K, weakly measurable and such that U(x) = (strong) lim $(y\downarrow 0)$ U(x+iy) (a.e., dx), for some uniformly bounded analytic operator-valued function U(z) defined in the upper half-plane. If U(z) can be continued analytically to **R** and at $z=\infty$, then (for real x) it satisfies the differential equation

$$(1)\quad U'(x) = iM(x)U(x)\ ,$$

where M(x) is a (norm) continuous function in **R**, whose values are non-negative hermitian operators in K; moreover,

$\|M\|_1 = \int_{-\infty}^{+\infty} \|M(x)\|\, dx < \infty$. Let $A^1 = \{M(x)\}$, where M(x) satisfies the above requirements, with the metric induced by $\|.\|_1$. By considering U(x) as a continuous (smooth) curve in $\mathcal{U}(K)$, it is shown that, either $M(x) \equiv 0$, or the curve defined by U(x) has diameter 2 and $\|M\|_1 \geq 2\pi$; furthermore, the infimum (2π) can be attained if and only if $U(x) = [(I-P) + (x-\lambda)/(x-\bar{\lambda})P]X$, where I is the identity operator, P is a non-zero (orthogonal) projection in K,

KEY WORDS AND SENTENCES: Inner function-operator, differential equation, analyticity on the closed unit disc, smooth closed curve, geodesic, unique geodesic, argument, gradient, diameter, length, multiplicative integral, upper and lower estimates for the derivative, completeness, analytic continuation, continuity on the boundary.

AMS-MOS Classification Numbers: Primary: 47A45. Secondary: 47A10, 47B05, 47B10, 34D99.

* This research was supported by NSF Grant GU-3171.

$X \in \mathcal{U}(K)$ and Im $\lambda > 0$. A^1 is a complete metric space and, for each $M \in A^1$, $M(x) \not\equiv 0$, for any λ, Im $\lambda > 0$, such that $U(\lambda)$ is not invertible in K, and for any $\varepsilon > 0$, there exists a vector $\varphi \in K$, $\|\varphi\| = 1$, such that $(M(x)\varphi,\varphi) > (2\text{Im}\,\lambda - \varepsilon)/|x-\lambda|^2$, for all $x \in \mathbf{R}$. Finally, it is shown that, if U(z) cannot be continued analytically to z=0, then there is no continuous $\mathcal{U}(K)$-valued function V(x) such that U(x) = V(x) a.e. in $(-\varepsilon,\varepsilon)$, for any $\varepsilon > 0$.

1. INTRODUCTION AND NOTATION.

The basic properties of the inner function-operators can be found in [7].

For a given subset Σ of the complex plane **C**, Σ^- and $\partial\Sigma$ denote the closure and the boundary of Σ, respectively.

We find it very convenient to use the double notation of [1]: u will always be the complex variable in the unit disc $D=\{u:|u|< 1\}$ (more exactly, by f(u) we shall denote the value of the analytic function f, originally defined on D, at the point $u \in R(f)$ = the Riemann surface -or, the domain of analyticity- of f); z will play the same role for analytic functions originally defined on the upper half-plane UHP = {z: Im z > 0}. Let f(u) be defined on D; then f(w) denotes the limit value of f(u) as u approaches non-tangentially to $w \in \partial D$ (in what follows these limits will be always well-defined a.e., and in the case of operator-valued functions, f(w) will denote the limit in the *strong operator topology*). Similarly, if f(z) is defined on UHP, then its non-tangential limit values are denoted by f(x), $x \in \mathbf{R}$ (x and y are the real and imaginary components of z). $u \in D^-$ and $z \in \text{UHP}^- \cup \{\infty\}$ are always assumed to be related by the equations

$$(2) \quad u = (i-z)/(i+z) \quad , \quad z = i(1-u)/(1+u)$$

The set of all inner function-operators will denoted by $\mathcal{F}$; in the above notation, U(z) (U(u), resp.) denotes an element of $\mathcal{F}$, thought as an inner function-operator defined on UHP (on D, resp.). As in [1], the set of all "analytic" inner function-operators is

$$(AI) = \{U \in \mathcal{F}: U(u) \text{ can be continued analytically to } D^-\}$$

If, during the proof of some result we have to use both expressions of the same $U \in F$, then the value of $U(w)$ at $w=1$ will be denoted by $U(w=1)$, etc., to avoid confusions.

$\|.\|$ and $(.,.)$ denote the norm of a vector of (or, an operator acting on) K and the inner product of K, resp..

Finally, $L(K)$ will denote the algebra of all (bounded linear) operators in K and $K^1 = \{\varphi \in K: \|\varphi\|=1\}$ is the unit sphere of K.

It was shown in [8] that, if $U \in F$, then U satisfies the differential equation (1), where $M(x)$ is a continuous function (unless otherwise stated, *continuity* of an operator-valued function means *continuity in the norm*) defined on the open intervals of $R \cap R(U)$, whose values are non-negative hermitian operators in K. Assume that K is one-dimensional; then $U(z)$ is a scalar inner function on UHP and $M(x)$ is, precisely, the derivative of arg $U(x)$. Thus, if $U \in (AI)$, then U is a finite Blaschke product (see [3;7] for definition) and

$$\|M\|_1 = \int_{-\infty}^{+\infty} \|M(x)\| \, dx = 2\pi N ,$$

where N is the number of zeroes (counted with multiplicity) of $U(z)$ in UHP. In particular, the set of values $\{\|M\|_1 : M \in A^1\}$ is discrete in $\mathbf{R}$. If dim $K \geqslant 2$ and P is a non-trivial projection, then

$$U_{N,r}(z) = [(z-i)/(z+i)]^N P + [(z-ri)/(z+ri)](I-P) \in (AI) ,$$

for all integers $N \geqslant 0$ and for all $r > 0$. We have

$$U'_{N,r}(x) = iM_{N,r}(x) U_{N,r}(x)$$

where $$M_{N,r}(x) = 2N/(1+x^2)\, P + 2r/(r^2+x^2)(I-P)$$

and $$\|M_{N,r}(x)\| = \max\{2N/(1+x^2), 2r/(r^2+x^2)\} .$$

It is clear that $\{\|M_{N,r}\|_1 : N \geqslant 0, r > 0\} = [2\pi, +\infty)$; hence, $\{\|M\|_1 : M \in A^1\} \supset \{0\} \cup [2\pi, +\infty)$. We shall prove (*sect.*3) that the inclusion can be actually replaced by equality and that, for non constant U, the lower bound 2π can be attained if and only if U has a trivial form. For this (and for further purposes) we shall need some auxiliary results, which are contained in the next section.

The differential equation (1) was first studied by H.Helson ([8]). Many of the results of this paper can be considered as extensions of the results of S.L.Campbell ([1]). In particular, the idea of analyzing the $\|.\|_1$-norm in A^1 is due to him, but our point of view is more geometric: the meaning of the differential equation (1) for one-dimensional K suggests that M(x) can be considered as "the derivative of the argument", or as "the gradient" of the *analytic curve* $\eta:\mathbb{R}\cup\{\infty\} \longrightarrow \mathcal{U}(K)$ defined by $\eta(x)=U(x)$, $U \in (AI)$, $x \in \mathbb{R}\cup\{\infty\}$. This geometric approach is systematically exploited here. Part of the results have been announced in [9].

We want to thank Professors H.Helson, S.L.Campbell, M.J.Sherman and M.Imina for several helpful discussions.

2. GEODESICS IN K^1 AND $\mathcal{U}(K)$.

The results of this section do not depend on the structure of the inner function-operators.
Thus, they have an independent interest. If R_1 and R_2 are two rotations on $\mathbb{R}^3$, with rotation angles ω_1 and ω_2, resp., then their composition $R=R_1R_2$ is a rotation with angle $\omega \leqslant \omega_1 + \omega_2$.
This simple geometric fact has the following operator theoretical analog:

LEMMA 2.1. *Let* $A,B \in \mathcal{U}(K)$ *and assume that* $\sigma(A) \subset \Gamma(\alpha_1,\alpha_2)$ *and* $\sigma(B) \subset \Gamma(\beta_1,\beta_2)$ *(where* $\sigma(T)$ *denotes the spectrum of* $T \in L(K)$ *and* $\Gamma(\omega_1,\omega_2) = \{e^{i\theta}:\omega_1 \leqslant \theta \leqslant \omega_2\}$*). Then* $\sigma(AB)=\sigma(BA)$ *is contained in* $\Gamma(\alpha_1+\beta_1,\alpha_2+\beta_2)$.

Proof. Since both AB and BA are unitary operators, it is clear from [6, *prob*.61] that $\sigma(AB)=\sigma(BA)$ and that this set is contained in ∂D.

If $(\alpha_2-\alpha_1)+(\beta_2-\beta_1) \geqslant 2\pi$, then there is nothing to prove.

Therefore, we can assume that $\alpha_2 - \alpha_1 < \pi$ (or $\beta_2 - \beta_1 < \pi$).

FIRST CASE: $0 \leqslant \beta_2 - \beta_1 < \pi$. Since $\sigma(e^{i\theta}T) = \{e^{i\theta}\lambda : \lambda \in \sigma(T)\}$, for all $T \in L(K)$, we can replace (if necessary) A and B by $e^{i\theta_1}A$ and $e^{i\theta_2}B$ and assume, without loss of generality, that $\sigma(A) \subset \Gamma(-\alpha,\alpha)$ *and* $\sigma(B) \subset \Gamma(-\beta,\beta)$, where $0 \leqslant \alpha,\beta < \pi/2$.

Since $AB \in U(K)$, $\partial\sigma(AB) = \sigma(AB)$ and therefore every point of $\sigma(AB)$ is an *approximate eigenvalue* of this operator; i.e., given $\lambda \in \sigma(AB)$ and $\varepsilon > 0$, there exists $\varphi_0 \in K^1$ such that $\|(AB-\lambda)\varphi_0\| < \varepsilon$. We have $B\varphi_0 = b\varphi_0 + d\varphi_1$; $b = (B\varphi_0,\varphi_0) \in W(B)$, $|b|^2 + |d|^2 = \|\varphi_0\|^2 = 1$, and

$A^*\varphi_0 = a\varphi_0 + c\varphi_1 + f\varphi_2$, $a = (A^*\varphi_0,\varphi_0) \in W(A^*)$, $|a|^2 + |c|^2 + |f|^2 = 1$,

where $W(T) = \{(T\varphi,\varphi) : \varphi \in K^1\}$ is the *numerical range* of $T \in L(K)$ (see [6]), A^* is the adjoint of the operator A, $\varphi_1 \in K^1$ (or $\varphi_1 = 0$, if $|b| = 1$), $\varphi_2 \in K^1$ (or $\varphi_2 = 0$, if $|a|^2 + |c|^2 = 1$) and $\{\varphi_0,\varphi_1,\varphi_2\}$ is an orthogonal system. Recall that $A^*, B \in U(K)$; hence, they are normal operators and therefore the closure of $W(A^*)$ ($W(B)$) coincides with the convex hull of $\sigma(A^*)$ ($\sigma(B)$, resp.)([6,*prob*.171]). Therefore, a (b) belongs to the convex hull of $\Gamma(-\alpha,\alpha)$ ($\Gamma(-\beta,\beta)$, resp.); in particular $|ab| \geqslant \cos\alpha . \cos\beta > 0$.

We have
$(AB\varphi_0,\varphi_0) = (B\varphi_0, A^*\varphi_0) = \bar{a}b + \bar{c}d$, $((AB-\lambda)\varphi_0,\varphi_0) = \bar{a}b + \bar{c}d - \lambda$.

By Schwartz'inequality, $|\bar{a}b + \bar{c}d| \leqslant 1$. On the other hand, since φ_0 is an ε-approximate eigenvector with eigenvalue λ,

$$\varepsilon > \|(AB-\lambda)\varphi_0\| \geqslant |((AB-\lambda)\varphi_0,\varphi_0)| = |\bar{a}b + \bar{c}d - \lambda|.$$

Thus we have proved that: 1) $\bar{a}b \neq 0$ (in fact, $|\bar{a}b|$ is uniformly bounded below away from zero for all a in $W(A^*)$ and all b in $W(B)$); 2) $|\bar{a}b + \bar{c}d| \leqslant 1$; 3) $|\bar{a}b + \bar{c}d - \lambda| < \varepsilon$. Since $|\lambda| = 1$, it is not difficult to conclude from 1), 2) and 3) that $|\lambda - \exp\{i(\arg b - \arg a)\}| = O(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lambda \in \Gamma(-\alpha-\beta, \alpha+\beta)$. This proves the result for the case when $\alpha_2 - \alpha_1 < \pi$ and $\beta_2 - \beta_1 < \pi$.

SECOND CASE: $\pi \leqslant \beta_2 - \beta_1 < 2\pi - (\alpha_2 - \alpha_1)$. Let $\gamma = (1/2)[(\alpha_2 - \alpha_1) + (\beta_2 - \beta_1)]$. An elementary application of the spectral theorem for unitary operators shows that B can be factores as $B=CB_1$, where $C,B_1 \in \mathcal{U}(K)$, $\sigma(B_1) \subset \Gamma(\beta_2-\gamma,\beta_2)$ and $\sigma(C) \subset \Gamma(\beta_1,\beta_2-\gamma)$ (see, e.g., [5]). Now the result follows applying the first case to $A_1 = AC$ and then to $A_1B_1 = AB$.

qed.

REMARKS. a) An alternate proof of this lemma can be given using *thm.1* of [17]. b) This result is clearly sharp; in fact, it cannot be improved even in the case when $\sigma(A)$ and $\sigma(B)$ are "very small" subsets of ∂D. To see this, observe that the bilateral shift S in ℓ^2 can be written as the product of two symmetries P and Q (see [6, p.269]); thus $\sigma(P)=\sigma(Q)=\{-1,1\}$, while $\sigma(S)=\sigma(PQ)=\partial D$!

Lemma 2.1. can be extended to finite or infinite convergent discrete products of unitary operators. Moreover, it can be also extended to *continuous products*:

COROLLARY 2.2. *Let* $M(x)$ *be a continuous function defined on the (finite or infinite) real interval* $(-a,b)$ $(0 < a,b \leqslant +\infty)$, *whose values are non-negative hermitian operators in* K, *and let* $U(x)$ *be the continuous product (or multiplicative integral) defined by*:

$$(3)\quad \begin{aligned} U(x) &= \overleftarrow{\int_0^x} \exp\{iM(t)dt\} = \lim(N\to\infty) \overleftarrow{\prod}_{j=1}^{N} e^{iM_j} = \\ &= \lim(N\to\infty)\, e^{iM_N}.e^{iM_{N-1}}.\,\ldots\,.e^{iM_2}.e^{iM_1}\ , \end{aligned}$$

where

$$(4)\quad M_j = \int_{(j-1)x/N}^{jx/N} M(t)\,dt \quad , \quad j = 1,2,\ldots,N\ ,$$

and the limits in (3) *are taken in the sense of the norm topology. These limits are well-defined and* $U(x) \in \mathcal{U}(K)$ *for all* $x \in (-a,b)$. *Furthermore,*

$$(5)\quad \begin{aligned} \sigma(U(x)) &\subset \Gamma\Big(0,\int_0^x \|M(t)\|dt\Big)\ , \text{ if } 0 \leqslant x < b \\ \sigma(U(x)) &\subset \Gamma\Big(-\int_x^0 \|M(t)\|dt,0\Big)\ , \text{ if } -a < x \leqslant 0. \end{aligned}$$

For the existence of the limit in (3), see [4;15]. Since every *approximating product* $\overleftarrow{\prod}_{j=1}^{N} e^{iM_j}$ is clearly unitary, the uniform limit $U(x)$ must be necessarily unitary. Finally, since

$$\sigma(e^{iM_j}) \subset \Gamma(0,\|M_j\|) \subset \Gamma(0,\int_{(j-1)x/N}^{jx/N} \|M(t)\|dt), \text{ if } x > 0,$$

and

$$\sigma(e^{iM_j}) \subset \Gamma(-\|M_j\|,0) \subset \Gamma(-\int_{jx/N}^{(j-1)x/N} \|M(t)\|dt,0), \text{ if } x < 0,$$

the proof of (5) follows by induction on *lemma 2.1* and an obvious continuity argument. This proves *cor. 2.2.*

It is worth noting that, in (4), M_j can be also taken equal to $(x/N)M(x_j)$, for some x_j in the interval determined by $(j-1)x/N$ and jx/N; however, the expression (4) is more convenient for our purposes.

If (in *cor.2.2*) $a = +\infty$ and $\lim (x\to-\infty)\ U(x) = U(-\infty)$ does exist, then we can define

$$(6) \quad V(x)=U(x)U(-\infty)^*=\overleftarrow{\int_{-\infty}^{x}} \exp\{iM(t)dt\} \ , \quad x < b \ , \quad V(-\infty)=I \ ;$$

similarly, we can write

$$(6') \quad W(x)=U(x)U(+\infty)^*=[\overleftarrow{\int_{x}^{+\infty}} \exp\{iM(t)dt\}]^* \quad , \ x > -a \quad , \quad W(+\infty)=I$$

in the case when $b=+\infty$ and $\lim(x\to+\infty)\ U(x)=U(+\infty)$ does exist.

In particular, $\int_{-\infty}^{0} \|M(t)\|dt < \infty$ ($\int_{0}^{+\infty} \|M(t)\|dt < \infty$, resp.) is a sufficient condition for the existence of $U(-\infty)$ ($U(+\infty)$,resp.), as it immediately follows from *cor.2.2* (see also [2,p.431).

THEOREM 2.3. *Let* $M(x)$ *be a continuous function defined on the real interval* $(-a,b)$ $(0 < a,b \leq +\infty)$, *whose values are hermitian operators in* K *and let* $X \in U(K)$. *Then the differential equation* (1) *has a unique solution such that* $U(0)=X$, *which is given by*

$$U(x) = [\overleftarrow{\int_{0}^{x}} \exp\{iM(t)dt\}]X \ .$$

Furthermore, if $\{M_k(x)\}$ *is a sequence of functions satisfying the above conditions,*

$$\lim(m,p\to\infty) \int_{-a}^{b} \|M_m(x)-M_p(x)\|dx = 0 \ ,$$

and $U_k(x)$ *is the solution of the equation* $U_k'(x)=iM_k(x)U_k(x)$ *satisfying* $U_k(0) = I$ *, for* $k=1,2,3,\dots$, *then* $\{U_k(x)\}$ *converges uniformly on* $(-a,b)$ *to a* $U(K)$*-valued function* $U(x)$.

NOTE. We are assuming neither the boundedness of the $\|M_k(x)\|'$ nor the integrability of $\|M_k(x)\|$.

Proof. The existence and uniqueness of the solution was proved in [8]; it is straightforward that the above multiplicative integral satisfies (1) (see [4]).

Let $M_k(x)$ and $U_k(x)$ be as indicated; by the definition of the multiplicative integral (3)-(4), for fixed m,p and x in (-a,b), we have

$$\|U_m(x)-U_p(x)\| = \|I - U_p(x)U_m(x)^*\| =$$

$$= \lim_{N\to\infty} \|I-[\overleftarrow{\prod}_{j=1}^{N} e^{iM_{p,j}}][\overleftarrow{\prod}_{j=1}^{N} e^{iM_{m,j}}]^*\| =$$

$$= \lim_{N\to\infty} \|\textstyle\sum_{j=1}^{N} e^{iM_{p,N}}.\ \dots\ .e^{iM_{p,j+1}}(I-e^{iM_{p,j}}.e^{-iM_{m,j}})\times$$

$$\times\ e^{-iM_{m,j+1}}.\ \dots\ .e^{-iM_{m,N}}\| \leqslant \lim_{N\to\infty}\textstyle\sum_{j=1}^{N}\|I-e^{iM_{p,j}}.e^{-iM_{m,j}}\|.$$

Since $M_p(t)$ and $M_m(t)$ are continuous and $|x| < \infty$, there exists a constant $C(x;m,p) < \infty$ such that $\|M_p(t)\| \leqslant C(x;p,m)$ and $\|M_m(t)\| \leqslant C(x;p,m)$, for all t in the interval determined by 0 and x. Now it is easy to see that

$$I - e^{iM_{p,j}}.e^{-iM_{m,j}} = i(M_{m,j} - M_{p,j}) + 0(N^{-2}) ,$$

and therefore

$$\|U_m(x)-U_p(x)\| \leqslant \lim_{N\to\infty} \{\textstyle\sum_{j=1}^{N}\|M_{m,j}-M_{p,j}\|+NO(N^{-2})\} =$$

$$= \lim_{N\to\infty} \sum_{j=1}^{N}\left\|\int_{(j-1)x/N}^{jx/N}[M_m(t)-M_p(t)]dt\right\| \leqslant$$

$$\leqslant \int_0^x\|M_m(t)- M_p(t)\|dt \leqslant \int_0^b\|M_m(t)-M_p(t)\|dt, \text{ for } 0 < x < b,$$

and similarly,

$$\|U_m(x)-U_p(x)\| \leqslant \int_x^0 \|M_m(t)-M_p(t)\|dt \leqslant \int_{-a}^0 \|M_m(t)-M_p(t)\|dt \quad ,$$

for $-a < x < 0$.

Therefore, $\{U_k(x)\}$ is a Cauchy sequence in the space of all $\mathcal{U}(K)$-valued functions, continuous in $(-a,b)$. Since this is a complete metric space under the uniform topology, the result follows.

qed.

Consider the real Hilbert space structure of K given by the inner product $(\varphi,\psi)_R = \text{Re}\ (\varphi,\psi)$. It is clear that $K^1_R = K^1$ and $\|\varphi\|_R = \|\varphi\|$, for every $\varphi \in K_R$ (= K under the real structure). Let $\gamma:[0,1] \to K^1$ be a continuous mapping; then the "length" of the curve γ is defined by

$$(7) \quad \kappa(\gamma)=\sup\{\textstyle\sum_{j=1}^N \|\gamma_j-\gamma_{j-1}\| : t_0=0 < t_1 < \ \dots \ < t_{N-1} < t_N=1\} \ ,$$

where $\gamma_j=\gamma(t_j)$, $j=0,1,\dots,N$, and the supremum is taken over all partitions.

LEMMA 2.4. *Let* γ *be a continuous mapping from* $[0,1]$ *into* K^1 *and assume that* $-1 < \text{Re}\ (\gamma(1),\gamma(0)) < 1$. *Then* $\kappa(\gamma) \geqslant \omega$, *where* $0 < \omega < \pi$ *and* $\cos\omega = \text{Re}\ (\gamma(1),\gamma(0))$.

Furthermore, the lower bound ω *is attained if and only if there is a continuous non-decreasing function* $f(t)$ *from* $[0,1]$ *onto* $[0,\omega]$ *such that*

$$(8) \quad \gamma(t)=(\cos f(t))\ \varphi_0 + (\sin f(t))\ \varphi_1 \ , \quad t \in [0,1] \ ,$$

where $\varphi_0=\gamma(0)$, $\psi=\gamma(1)$ *and* $\varphi_1 \in K^1_R$ *is defined by the conditions:* $(\varphi_1,\varphi_0)_R=0$ *and* $\psi=(\cos\omega)\varphi_0 + (\sin\omega)\varphi_1$.

Proof. If $\kappa(\gamma)=+\infty$, then there is nothing to prove; so we can directly assume that $\kappa(\gamma) < +\infty$. It is clear from the above comments that we can consider K under its real Hilbert space structure K_R; it is also immediate that the sum corresponding to a given partition is always smaller than $\kappa(\gamma)$ and that this sum increases by a refinement of the partition. We recall that the *norm* of a partition is the maximum of the numbers t_j-t_{j-1}, $j=1,2,\dots,N$. Given

$\varepsilon > 0$, let R_j be (for a given partition of $[0,1]$ and for each j, $j=0,1,\dots,N$) the rotation of K_R defined by $R_j\varphi_0 = \varphi_0$; $R_j\varphi = \varphi$, if $(\varphi,\varphi_0)_R = (\varphi,\gamma_j)_R = 0$, and $R_j\gamma_j = \gamma_j'$, where $\gamma_j' = (\gamma_j,\varphi_0)_R\varphi_0 + [1-(\gamma_j,\varphi_0)_R^2]^{1/2}\varphi_1$.

Then,

$$\kappa(\gamma) > \sum_{j=1}^N \{\|R_j\gamma_j - R_{j-1}\gamma_{j-1}\|^2 + \|(R_j - R_{j-1})\gamma_{j-1}\|^2\}^{1/2} - \varepsilon =$$

$$= \sum_{j=1}^N \{\|\gamma_j' - \gamma_{j-1}'\|^2 + \|(R_j - R_{j-1})\gamma_{j-1}\|^2\}^{1/2} - \varepsilon \geqslant$$

$$\geqslant \sum_{j=1}^N \|\gamma_j' - \gamma_{j-1}'\| - \varepsilon \geqslant \sum_{j=1}^N \|\gamma_j'' - \gamma_{j-1}''\| - \varepsilon ,$$

where $\gamma_0'' = \gamma_0' = \varphi_0$, $\gamma_1'' = \gamma_1'$ and γ_j'', $j=2,3,\dots,N$ is defined by induction, as follows: assume that $\gamma_0'',\gamma_1'',\dots,\gamma_{j-1}''$ has been chosen, then set $\gamma_j'' = \gamma_j'$, if $(\gamma_j',\varphi_0)_R \leqslant (\gamma_{j-1}'',\varphi_0)_R$, or $\gamma_j'' = \gamma_{j-1}''$, if $(\gamma_j',\varphi_0)_R > (\gamma_{j-1}'',\varphi_0)_R$.

Thus, if the norm of the partition is small enough,

$$\kappa(\gamma) > \sum_{j=1}^N \|\gamma_j'' - \gamma_{j-1}''\| - \varepsilon \geqslant \omega - 2\varepsilon .$$

Since ε is arbitrary, we conclude that $\kappa(\gamma) \geqslant \omega$.

Moreover, since $R_N = I$ is the only rotation of the above described type that fixes ψ (to see this, recall that $0 < \omega < \pi$!), it is not difficult to infer from the above inequalities that the infimum can be attained if and only if $R_j = I$ and $\gamma_j'' = \gamma_j' = \gamma_j$, for all $j=1,2,\dots,N$, and for all partitions of $[0,1]$; i.e., if and only if γ has the form (8).

qed.

REMARK. The geometric meaning of *lemma 2.4* is the following: if $\varphi,\psi \in K^1$ and $\psi \neq -\varphi$ (hence, $\|\varphi-\psi\| < 2$), then there exists a unique geodesic curve $\gamma:[0,1] \to K^1$ joining them and the length of this geodesic is equal to: arc cos Re (φ,ψ) (it is trivial that if $\psi=-\varphi$ there exist infinitely many geodesics in K^1 joining these two points). The fact that K^1 is "geometrically homogeneous" is *most important* here, and we guess that analogous results can be proved in any uniformly convex (or even strictly convex)

Banach space (i.e., that given two-sufficiently close-different points of the unit sphere of the space, there exists a unique geodesic in that sphere joining the two points and, moreover, that this geodesic is a smooth curve). But none of these properties is true in the general case; for example, if $X = L^1(R,dx)$ and $f_\delta(x)$ is the characteristic function of the interval $(\delta,\delta+1)$, then $f_\delta \in X^1$ (the unit sphere of X) and $\|f_0-f_\delta\|_X = 2\delta$, if $0 \leqslant \delta \leqslant 1$; this shows, in particular, that f_0 and f_δ can be taken "arbitrarily close". Fix δ, $0 < \delta < 1$; then $\gamma_0:[0,1] \to X^1$, $\gamma_0(t)=(1-t)f_0+ tf_\delta$, and $\gamma_1:[0,1] \to X^1$, $\gamma_1(t) = f_{t\delta}$, satisfy

$$\kappa(\gamma_0) = \kappa(\gamma_1) = \|f_0-f_\delta\|_X = 2\delta$$

and therefore, they are geodesics joining f_0 with f_δ; in fact, there are infinitely many geodesics joining these two points. Furthermore, the strong derivative of $\gamma_0(t)$ is well-defined and $\gamma_0'(t) = f_\delta - f_0$ (for $0 < t < 1$), but $\gamma_1(t)$ is differentiable nowhere (not even in the weak sense!) in $(0,1)$.

COROLLARY 2.5. *Let* $\eta:[0,1] \to \mathcal{U}(K)$ *be a continuous mapping such that* $\|\eta(0) - \eta(1)\| = R$, $0 < R < 2$; *then the "length" of the curve* η *(defined by* (7)), *satisfies the inequality* $\kappa(\eta) \geqslant \omega$, *where* $0 < \omega < \pi$ *and* $|1-e^{i\omega}| = R$.

Proof. Observe that $\|\eta(0)-\eta(1)\|=\|I-U\|=R$, where $U=\eta(1)\eta(0)^* \in \mathcal{U}(K)$; this implies that $\sigma(U) \subset \Gamma(-\omega,\omega)$ and, moreover, either $e^{i\omega} \in \sigma(U)$ or $e^{-i\omega} \in \sigma(U)$. We shall assume that $e^{i\omega}$ is in the spectrum of U; the other case can be similarly analyzed; then, as in the *proof* of *lemma 2.1, first case*, we can see that $e^{i\omega}$ is an approximate eigenvalue of U. Hence, given any $\varepsilon > 0$, there exists $\varphi \in K^1$ such that $\|(U-e^{i\omega}I)\varphi\| < \varepsilon$.

Define $\gamma:[0,1] \to K^1$ by $\gamma(t) = \eta(t)\eta(0)^*\varphi$; γ is obviously continuous and satisfies

$$|\cos\omega - \mathrm{Re}\,(\gamma(1),\gamma(0))| = |\cos\omega - (\gamma(1),\gamma(0))_R| < \varepsilon$$

Hence, by *lemma 2.4*, $\kappa(\gamma) > \omega - \varepsilon$ and, since ε is arbitrary we conclude that $\kappa(\eta\eta(0)^*) = \kappa(\eta) \geqslant \omega$.

qed.

In general, two different points of $U(K)$ can be joined by infinitely many geodesics. However, there is exactly one particular case in which we have exactly one geodesic; this is the case of the following:

COROLLARY 2.6. *Let* η *be as in cor.2.5 and assume that* $\kappa(\eta) = \omega$ *and* $\sigma(\eta(1)\eta(0)^*) \subset \{1, e^{i\omega}\}$. *Then there exists a non-zero projection* P *in* K *and a continuous non-decreasing function* f(t) *from* $[0,1]$ *onto* $[0,\omega]$ *such that*

$$\eta(t) = [e^{if(t)}P + (I-P)]\eta(0) \quad , \quad t \in [0,1] \ .$$

Proof. Without loss of generality we can assume that $\eta(0)=I$; then our hypothesis on η says that $\eta(1) = e^{i\omega} P + (I-P)$, for some non-zero projection P.

Let $\varphi \in K^1 \cap P(K)$; then $\eta(t)\varphi$ describes a geodesic in K^1 joining φ with $e^{i\omega}\varphi$. Thus, by *lemma 2.4*, $\eta(t)\varphi = e^{if(t)}\varphi$, for some continuous non-decreasing function f(t) from $[0,1]$ onto $[0,\omega]$.

Assume that $e^{i\alpha} \in \sigma(\eta(t_0))$, for some $t_0 \in (0,1)$ and some α such that $e^{i\alpha} \neq 1, \neq e^{if(t_0)}$. Then, if $0 < \sigma < f(t_0) \pmod{2\pi}$, *cor.2.5* implies that

$$\omega = \kappa(\eta) = \kappa(\eta : t \in [0,t_0]) + \kappa(\eta : t \in [t_0,1]) \geqslant f(t_0) + \omega - \alpha > \omega \quad ,$$

a contradiction. By considering separately all possible cases, we conclude that $\sigma(\eta(t)) \subset \{1, e^{if(t)}\}$, for all $t \in [0,1]$; i.e.,

$$\eta(t) = e^{if(t)} P(t) + [I - P(t)] \ ,$$

where P(t) is a projection in K, depending continuously on t. Moreover, the first part of the proof shows that, if $0 < t_1 < t_2 \leqslant 1$, then $P(t_1) \geqslant P(t_2)$ (observe that every $\psi \in K^1 \cap P(K)$ is an eigenvector of $\eta(t)$, with eigenvalue $e^{if(t)}$).

Since $0 < \omega < \pi$, we can find $t_0 = 0 < t_1 < t_2 < t_3 < t_4 = \omega$, such that $f(t_j) - f(t_{j-1}) < 1$, for j=1,2,3,4. If $P(t_{j_0}) \neq P(t_{j_0 - 1})$, for some j_0, $1 \leqslant j_0 \leqslant 4$, then

$$\kappa(\eta) = \omega = \sum_{j=1}^{4} [f(t_j) - f(t_{j-1})] < 1 + \sum_{j \neq j_0} \kappa(\eta : t \in [t_{j-1}, t_j]) \leqslant$$

$$\leqslant \|\eta(t_{j_0}) - \eta(t_{j_0 - 1})\| + \sum_{j \neq j_0} \kappa(\eta : t \in [t_{j-1}, t_j]) \leqslant \kappa(\eta) \ ,$$

a contradiction.

Therefore, $P(t) \equiv P(1) = P$, for $0 \leqslant t \leqslant 1$.

qed.

REMARK. It is not difficult to see, using the spectral theorem for unitary operators, that the result of *cor.2.6* is sharp in the sense that, if η satisfies the requirements of *cor.2.5* and $\kappa(\eta) = \omega$ $(0 < \omega < \pi)$, but $\sigma(\eta(1)\eta(0)^*)$ contains a point $e^{i\alpha} \neq 1, \neq e^{i\omega}, \neq e^{-i\omega}$, then there exists infinitely many geodesics in $U(K)$ joining $\eta(0)$ with $\eta(1)$.

3. THE BEST LOWER BOUND FOR $\|M\|_1$.

In this section we improve *thm.10* of [1]:

THEOREM 3.1. *If* $M \in A^1$ *and* $M(x) \not\equiv 0$, *then* $\|M\|_1 \geqslant 2\pi$. *Furthermore,* $\|M\|_1 = 2\pi$ *if and only if* $U'(x) = iM(x)U(x)$ *for some* $U \in (AI)$ *of the form*

$$(9) \qquad U(z) = [\,(z-\lambda)/(z-\bar{\lambda})P + (I-P)]X \quad ,$$

where $\lambda \in$ UHP , P *is a non-zero projection in* K *and* $X \in U(K)$.

Proof. Let U be a non-constant inner function-operator in (AI) satisfying the differential equation (1). It follows from *thm. 6.1,i)* of [11] that

$$\|X - U\| = \sup\{\|X-U(x)\| : x \in \mathbf{R}\} = 2 \;,$$

for every X in $U(K)$. This means that the continuous (furthermore, analytic) *closed curve* $\eta : \mathbf{R}^* \longrightarrow U(K)$ (where $\mathbf{R}^*$ is the one-point compactification of the reals) defined by $\eta(x) = U(x)$ has diameter 2 ($U(-\infty) = U(+\infty) = U(\infty)$). Therefore, there exists a point $x_0 \in \mathbf{R}$ such that $\|U(\infty) - U(x_0)\| = 2$; i.e., $-1 \in \sigma(U(x_0)U(\infty)^*)$.

Thus, by *cor.2.5*, the total length of η satisfies

$$\kappa(\eta) = \kappa(\eta : -\infty \leqslant x \leqslant x_0) + \kappa(\eta : x_0 \leqslant x \leqslant +\infty) \geqslant 2\pi \;.$$

But, since η is smooth, $\kappa(\eta)$ is indeed equal to

$$\kappa(\eta) = \int_{-\infty}^{+\infty} \|U'(x)\|\,dx = \int_{-\infty}^{+\infty} \|M(x)\|\,dx = \|M\|_1 \;.$$

Therefore, $\|M\|_1 \geqslant 2\pi$ (this is also a consequence of *cor. 2.2*.
If U(z) has the form (9), it is completely apparent that $\|M\|_1 = 2\pi$.
Thus, in order to complete the proof we only have to show that, if $\|M\|_1 = 2\pi$, then U(z) has the form (9).

First of all, observe that M(x) cannot be identically zero on a non-empty open subinterval of R (otherwise, the analyticity of U(z) would imply that $M(x) \equiv 0$ on R). Therefore,

$$x \longrightarrow \omega(x) = \int_{-\infty}^{x} \|M(t)\| \, dt$$

is a continuous and strictly increasing function from $[-\infty, +\infty]$ onto $[0, 2\pi]$.

Since $U \in (AI)$ it is easy to see, using *cor. 2.2* (and comments following it!), *thm. 2.3* and the above observations about $\omega(x)$, that $U(x) = V(x)U(\infty) = W(x)U(\infty)$, where V(x) and W(x) are defined by (6) and (6'), resp., and

$$\sigma(U(x)U(\infty)^*) = \sigma(V(x)) = \sigma(W(x)) \subset \Gamma(0, \omega(x)) \cap \Gamma(2\pi - \omega(x), 0) =$$

$$= \{1, e^{i\omega(x)}\} \ , \ x \in \mathbf{R}^*$$

Using *cor. 2.6* we can easily follow that

$$U(x) = [e^{i\omega(x)} P + (I-P)] U(\infty) \ ,$$

for some non-zero projection P in K.
Since $U \in (AI)$, $e^{i\omega(x)}$ must coincide with the limits on the real axis of a inner function (in the UHP) b(z); moreover, b(u) must be analytic on D^-. Hence, b(z) is a finite Blaschke product such that $b(\infty) = e^{i\omega(\infty)} = 1$. On the other hand, by our observations of *section 1*, the number of zeroes of b(z) is equal to

$$N = (1/2\pi) \int_{-\infty}^{+\infty} d \arg b(x) = (1/2\pi) \int_{-\infty}^{+\infty} \|M(x)\| \, dx = 1 \ .$$

We conclude that $b(z) = (z-\lambda)/(z-\bar{\lambda})$, for some $\lambda \in$ UHP; i.e., U(z) has the form (9).

qed.

4. THE BEST LOWER BOUND FOR $\|M(x)\|$.

Thm. 3.1 provides the best lower "global" bound for M(x) and its main interest lies in the fact that the lower bound can only be attained by a class of particularly simple inner function-operators. This lower bound for $\|M\|_1$ corresponds to "the total variation of the argument on ∂D is $\geqslant 2\pi$", for the scalar case.

We want to show here that, if M(x) is interpreted as "the gradient" of the inner function-operator U(x), this makes it easier to obtain *pointwise* lower bounds for $\|M(x)\|$. *Theorem 4.1* below was suggested by an observation of Prof. H. Helson.

If b(z) is a finite Blaschke product on the UHP with a zero at $z=\lambda$, then

$$|\text{grad } b(x)|=|(d/dx)b(x)| \geqslant |(d/dx)(x-\lambda)/(x-\bar{\lambda})| \geqslant 2\text{Im}\lambda/|x-\lambda|^2.$$

The vectorial analog of $|\text{grad } b(x)|$ is $\|M(x)\|$; thus, the above scalar result suggests that $\|M(x)\| \geqslant 2\text{Im}\lambda/|x-\lambda|^2$, where $\lambda \in$ UHP is any point such that $U(\lambda)$ is not invertible in K. We shall prove that this is actually true; furthermore, we have

THEOREM 4.1. *Let* $U \in (AI)$ *be a non-constant inner function-operator and assume that* U(x) *satisfies the differential equation* (1), *for* $x \in R$. *Then, for any* λ *in the* UHP *such that* $U(\lambda)$ *is not invertible in* K *and any* $\varepsilon > 0$, *there exists* $\varphi \in K^1$ *such that*

$$(M(x)\varphi,\varphi) > (2\text{Im}\lambda-\varepsilon)/|x-\lambda|^2 \quad , \; x \in \mathbf{R}.$$

In particular,

$$\|M(x)\| \geqslant \sup\{2\text{Im}\lambda/|x-\lambda|^2 : U(z=\lambda) \textit{ is not invertible}\}$$

Proof. We shall prove the result for the case when $\lambda=i$; equivalently: when U(u=0) is not invertible in K. The general case will follow by a conformal transformation of the UHP onto itself.

By (2),

$$(10) \quad U'(x)=iM(x)U(x)=-2i/(i+x)^2(d/dw)U(w=(i-x)/(i+x)).$$

Let $U^{\sim}(u)=U(\bar{u})^*$; it is easy to check that $U^{\sim}(u) \in (AI)$, $U^{\sim}(z) = U(-1/\bar{z})^*$ And $U^{\sim\prime}(x)=iM^{\sim}(x)U^{\sim}(x)$, where (using (10)) $M^{\sim}(x) = (1/x^2)M(-1/x)$, for all real $x\neq 0$, and $M^{\sim}(0)=\lim(x\to 0)(1/x^2)M(-1/x) = 2[(d/dw)U(w=-1)]U(w=-1)^*$.

Now it is clear that, for $\varphi \in K^1$ and $\varepsilon > 0$,
$(M^\sim(x)\varphi,\varphi) > (2-\varepsilon)/(1+x^2)$, for all real x , if and only if
$(M(x)\varphi,\varphi) > (2-\varepsilon)/(1+x^2)$, for all real x.

Therefore, it is equivalent to prove the result for M(x) or for $M^\sim(x)$.

FIRST CASE. Ker $U(u=0)^*$=Ker $U^\sim(u=0)=K_0\neq\{0\}$.

Then (see [12;13;16]), U(u) can be factored as U(u)=B(u)C(u) where $B(u)=uP+(I-P)$ (P=the projection of K onto K_0) and $C \in$ (AI).

Thus, in UHP we shall have $U(z)=B(z)C(z)=[(z-i)/(z+i)P+(I-P)]C(z)$.

If M(x), M(x;B) and M(x;C) are the hermitian valued functions associated to U, B and C, resp., by means of the differential equation (1), then (see [8,*thm.3*])

$$M(x)=M(x;B)+B(x)M(x;C)B(x)^* \geqslant M(x;B).$$

Hence, if $\varphi \in K^1 \cap K_0$, then $(M(x)\varphi,\varphi) \geqslant (M(x;B)\varphi,\varphi)=2/(1+x^2)$.

SECOND CASE. Ker U(u=0)=Ker $U^\sim(u=0)^*=K_1\neq\{0\}$.

Applying the first case to $U^\sim(z)$, we obtain
$(M^\sim(x)\varphi,\varphi) \geqslant 2/(1+x^2)$ (and therefore, $(M(x)\varphi,\varphi) \geqslant 2/(1+x^2)$), for all $\varphi \in K^1 \cap K_1$ and all $x \in \mathbf{R}$.

THIRD CASE. U(u=0) is not invertible, but Ker U(u=0)=Ker $U(u=0)^*$= $=\{0\}$.

In this case we shall prove that U(u) can be approximated by elements of (AI) satisfying the conditions of the first case, uniformly on $|u| \leqslant R$, for some $R > 1$. Without loss of generality we can assume that U(w=1)=I; then K admits the orthogonal direct sum decomposition $K=K_T\oplus K_T^\perp$ reducing U(u), where $U(u)|_{K_T^\perp}$ (= the restriction of U(u) to $K_T^\perp$) is the identity I_T of $K_T^\perp$ and $U(u)|_{K_T}$ = $= A_T\theta_T(u)B_T$, where

$$\theta_T(u)=[-T+uD_{T*}(I-uT^*)^{-1}D_T]|_{\mathcal{D}_T}:\mathcal{D}_T \to \mathcal{D}_{T*}$$

is the *characteristic function* of a C_{00}-contraction T on a Hilbert space H, $D_T=(I-T^*T)^{1/2}$, $D_{T*}=(I-TT^*)^{1/2}$, $\mathcal{D}_T$= closed Range(D_T) , $\mathcal{D}_{T*}$= closed Range(D_{T*}) and A_T and B_T are unitary maps (independent of u) from $\mathcal{D}_{T*}$ onto K_T and from K_T onto $\mathcal{D}_T$, resp.. For definitions and properties of the characteristic function of a contraction, C_{00}-contractions, etc., see [16]; for details about the decomposition $K=K_T \oplus K_T^{\perp}$, see [13]. Here, we want to use those results without specifying details: since $U(u) \in (AI)$, $\sigma(T)=\{\sigma \in D : U(\sigma)$ is not invertible$\}$ in particular, $0 \in \sigma(T)$, i.e., T is not invertible in H. Moreover, our hypothesis on $U(u=0)$ and *thm. VI.4.1* of [16] imply that the polar decomposition of T has the form $T=XH$, where $X \in U(H)$ and $H \in L(H)$ is a non-invertible hermitian non-negative operator such that $\operatorname{Ker} H = \{0\}$ and $\|H\| \leqslant 1$. Therefore, by the spectral theorem for hermitian operators (see [5]), given any $\delta > 0$ there exists an hermitian operator $H_\delta \in L(H)$ such that $\|H_\delta\| < \delta$, $\|T-T_\delta\| < \delta$ (where $T_\delta = X(H-H_\delta)$), $\operatorname{Ker}(H-H_\delta) \neq \{0\}$ and $\sigma(H-H_\delta) \subset \{0\} \cup [\delta/2, 1]$. We can assume that $\delta < 1$; then D_T and $D_{T_\delta} = (I-T_\delta^* T_\delta)^{1/2} = I-(H-H_\delta)$ have the same range and, similarly D_{T^*} and $D_{T_\delta^*}$ have the same range. Thus, if we set

$$U_\delta(u) = A_T \theta_{T_\delta}(u) B_T \oplus I_T ,$$

then $U_\delta(u) \in (AI)$, $\operatorname{Ker} U_\delta(0)^* = A_T X[\operatorname{Ker}(H-H_\delta)] = K_\delta \neq \{0\}$, and

$$U(u)-U_\delta(u) = A_T[\theta_T(u)-\theta_{T_\delta}(u)] B_T \oplus 0 =$$

$$= A_T\{XH_\delta + uD_{T^*}[(I-uT^*)^{-1}-(I-uT_\delta^*)^{-1}]D_T\}B_T \oplus 0 ;$$

clearly, since $U, U_\delta \in (AI)$, $\|U(u)-U_\delta(u)\| < \delta'$ on $|u| \leqslant R$ (for some $R > 1$), where $\delta' \to 0$, as $\delta \to 0$.

From the Cauchy formula

$$(11) \quad (d^n/du^n)U(u) = (n!/2\pi i)\int_{|t|=R} U(t)(t-u)^{-n-1}\, dt , \quad |u| < R,$$

for the derivatives of a function analytic on $|u| \leqslant R$, it follows that

$$(12) \qquad \|(d/dw)[U(w)-U_\delta(w)]\| < \delta'/(R-1)^2 < \varepsilon ,$$

and therefore

$$(1+x^2)\|M(x)-M_\delta(x)\| < \varepsilon$$

(where M_δ has the obvious meaning), for all $x \in \mathbf{R}$, provided δ (and hence δ' too) is small enough.

Applying the first case to this U_δ, for all $\varphi \in K^1 \cap K_\delta$ and for all $x \in \mathbf{R}$, we obtain

$$(M(x)\varphi,\varphi) > (M_\delta\varphi,\varphi)-\varepsilon/(1+x^2) \geqslant (2-\varepsilon)/(1+x^2).$$

qed.

REMARKS. a) It was proved in [1,*p.9*] that, if $U \in (AI)$, then $\|M(x)\| \leqslant (2\sum_{n=1}^{+\infty} n\|U_n\|)/(1+x^2)$, where $U(u)=\sum_{n=0}^{+\infty} u^n U_n$ is the Taylor series of $U(u)$ about the origin (here $U_n \in L(K)$; since the Taylor series of $U(u)$ converges in a closed disc of radius larger than one, the sum of the norms in the upper bound of $\|M(x)\|$ also converges). Here is an alternate proof for the existence of an upper bound: it follows from (10)-(11)-(12) that

$$\|M(x)\|=\|U'(x)\| \leqslant [2 \inf\{(R-1)^{-2} \max_{w\varepsilon\partial D} \|U(Rw)\|\}]/(1+x^2) ,$$

where the infimum is taken over all $R > 1$ such that $U(u)$ can be continued analytically to $|u| \leqslant R$.

b) The result of *thm.4.1* is obviously sharp (and it provides a new proof of: "$\|M\|_1 \geqslant 2\pi$, for all $M \in A^1$, $M(x) \not\equiv 0$"). We can say even more; it was proved in [1,*cor.1*] (using a result due to Potapov, [15,*p.154*]; see also *lemma 4.2* and *cor.4.3*, below) that, if $M \in A^1$ then $\operatorname{Ker} M(x) \equiv \operatorname{Ker} M(0)$, for all real x. This result might suggest that, in *thm.4.1*, one can replace "there exists a $\varphi \in K^1$" by "for every $\varphi \in K^1$, $\varphi \perp \operatorname{Ker} M(x)$ ..."; however, the analogy with the scalar case cannot be carried to this point, as it is shown by the following example: let $K=\mathbf{C}^2$ (i.e., dim K=2) with orthonormal basis $\{\psi_1,\psi_2\}$ and let $0 < \omega < \pi/2$; define $U(w) \in (AI)$ by

$$U(w) = \begin{Vmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{Vmatrix} \begin{Vmatrix} w & 0 \\ 0 & 1 \end{Vmatrix} \begin{Vmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{Vmatrix} \begin{Vmatrix} w & 0 \\ 0 & 1 \end{Vmatrix} .$$

Then, a straightforward computation shows that $\operatorname{Ker} M(x) \equiv \{0\}$; however, $\kappa(U(w)\psi_2) = O(\omega)$, and therefore (for small values of ω) ψ_2 cannot satisfy the thesis of *thm.4.1*. Furthermore, according to the *proof* of the *theorem* (*first case*), there exists a vector ψ in K^1, such that $(M(x)\varphi,\varphi) \geqslant 2/(1+x^2)$, for all $x \in \mathbf{R}$ and therefore, $\kappa(U(w)\varphi) \geqslant 2\pi$; however, there is no $\psi \in K^1$ such that the curve $\{U(w)\psi\}$ has diameter 2.

c) For the scalar case, $|\operatorname{grad} b(x)|$ is always determined by its values on a small segment of the reals. The situation is different in the vectorial case: given any closed subset Σ of **R** and an orthonormal basis $\{\varphi_n\}_{n=1}^{+\infty}$ of K, it is always possible to select a sequence $\{\lambda_n \in \text{UHP}\}$ in such a way that, if $U(z)\varphi_n=(z-\lambda_n)/(z-\bar{\lambda}_n)\varphi_n$, $n=1,2,3,\ldots$, then $\|M(x)\|=2/(1+x^2)$ if and only $x \in \Sigma$.

We are going to close this section with an improvement (in fact, a "localization" of the result) of the above mentioned result of Potapov (*remark b*) to *thm.4.1*).

LEMMA 4.2. *Let* $\varphi(z)$ *be a* K*-valued analytic function in the* UHP *and assume that*: 1) $\|\varphi(z)\| \leqslant 1$; 2) *for some* $\psi \in K^1$, $\lim\,(y\downarrow 0)\;(\varphi(iy),\psi)=1$ *and, either* $\lim\,(y\downarrow 0)\;(d/dz)(\varphi(z=iy),\psi)=0$, *or* $\lim\,(y\downarrow 0)\;[(\varphi(iy),\psi)-1]/y=0$. *Then* $\varphi(z) \equiv \psi$.

Proof. It follows from the theorem of Julia-Carathéodory (see,e.g, [14,p.57]) and 1) and 2), that the scalar analytic function $f(z)= (\varphi(z),\psi) \equiv 1$. Hence, $\varphi(z)=\psi+\psi(z)$, where $(\psi(z),\psi) \equiv 0$, in UHP. This last expression of $\varphi(z)$ and 1) imply that, for all $z \in \text{UHP}$, $1 \geqslant \|\varphi(z)\|^2 = 1 + \|\psi(z)\|^2$, which is clearly impossible, unless $\psi(z) \equiv 0$; i.e., $\varphi(z) \equiv \psi$ on UHP.

qed.

From this *lemma* and *thm.2.3*, we obtain the following

COROLLARY 4.3. *Let* $U \in F$ *and* $\varphi \in K^1$, *and assume that, for some* $\psi \in K^1$, $(U(z)\varphi,\psi)$ *satisfies the condition* 2) *of lemma 4.2; then* $U(z)\varphi \equiv \psi$. *Moreover, if* $U(z)$ *is analytic at* $z=0$, *then* $\varphi \in \text{Ker}U'(x)$ *and* $\text{Ker}\,M(x) = U(0)[\text{Ker}\,U'(0)] \equiv \text{Ker}\,M(0)$, *for all* $x \in \mathbf{R} \cap R(U)$.

5. COMPLETENESS OF A^1.

THEOREM 5.1. A^1 *is a complete metric space.*
Furthermore, if $\{M_k\}$ *is a Cauchy sequence in* A^1 *and* $U_k \in (AI)$ *is the solution of* $U_k'(x)=iM_k(x)U_k(x)$, $U_k(0)=I$, *then there exists* $U \in (AI)$ *satisfying* (1), *such that* $U(0)=I$ *and*

i) *For each* $n \geqslant 0$, $\|(d^n/du^n)[U_k(u)-U(u)]\| \to 0\;(k \to \infty)$, *uniformly for* $|u| \leq R$, *for some* $R > 1$;

ii) $(1+x^2)\|M_k(x)-M(x)\| \to 0$ *and* $\|M_k-M\|_1 \to 0$ $(k \to \infty)$.

Proof. i) By *thm.2.3*, the sequence $\{U_k(x)\}$ converges uniformly on **R** to a $\mathcal{U}(K)$-valued function $U(x)$. Hence, $\|U_k(w)-U(w)\| \to 0$, uniformly on ∂D; since F is complete, it follows that $U \in F$ and that $\|U_k(u)-U(u)\| \to 0$, uniformly on D^- (Clearly, $U(w=1) = U(x=0) = I$). Therefore, there is an m such that $\|U_k(u)-U(u)\| < 1/4$, for all $k \geq m$ and all $u \in D^-$.

Since $U_m \in (AI)$, there exists $\varepsilon > 0$ such that $U_m(u)$ is invertible in K and, moreover, $\|U_m(u)^{-1}\| < 4/3$, in the annulus $1-\varepsilon \leq |u| \leq 1$. Hence, $\|U_k(u)^{-1}\| < 2$ and $\|U(u)^{-1}\| \leq 2$, on that annulus. Thus, it follows from [10] that $U_k(u)$ $(k \geq m)$ and $U(u)$ can be continued analytically to the closed disc of radius $R_0=1/(1-\varepsilon)$; moreover, $\|U_k(u)-U(u)\| \to 0$, uniformly for $|u| \leq R_0$. Now i) follows from the Cauchy formula (11) for the derivatives of an analytic function, by choosing as R any real number such that $1 < R < R_0$.

ii) In particular, for the first derivative, (10)-(11)-(12) show that

$$(1+x^2)\|M_k(x)-M(x)\| \leq (1+x^2)\{\|U_k'(x)-U'(x)\|+\|U'(x)[U_k(x)^*-U(x)^*]\|\} \leq$$

$$\leq \max_{w\in\partial D}\{\|(d/dw)[U_k(w)-U(w)]\|\}+C\max_{w\in\partial D}\{\|U_k(w)-U(w)\|\},$$

where $\|M(x)\| \leq C/(1+x^2)$, as in *thm.4.1* (and M is related with U by (1)). It follows that (using i))

$$(1+x^2)\|M_k(x)-M(x)\| \to 0 \quad (k \to \infty),$$

uniformly on **R**, and therefore

$$\|M_k-M\|_1 \leq \pi \sup\{(1+x^2)\|M_k(x)-M(x)\| : x \in \mathbf{R}\} \to 0$$

qed.

NOTE. The completeness of A^1 has been independently proved by S.L.Campbell (personal communication).

6. CONTINUITY ON THE BOUNDARY.

The result of this section has an independent character. Let $U \in F$.

It was shown in [8] that, if $U(z)$ is continuous on $\{z:\text{Im } z = 0, |z| < \varepsilon\}$, then $U(z)$ can be continued analytically to $z=0$. In fact, a stronger result is true: $U(z)$ can be continued analytically to $z=0$ if and only if $U(z)$ is invertible in K and $\|U(z)^{-1}\|$ is uniformly bounded for all z in the set $\{z:\text{Im } z > 0, |z| < \varepsilon\}$ for some $\varepsilon > 0$ (see [10]). The result of [8] can be improved in a different direction; namely:

THEOREM 6.1. *Let* $U \in F$ *and assume that* $U(x)$ *coincides a.e. with a continuous* $\mathcal{U}(K)$*-valued function* $V(x)$ *on* $(-\varepsilon,\varepsilon)$ *(for some* $\varepsilon > 0$*); then* $U(z)$ *can be continued analytically to that real interval and* $U(x)=V(x)$, *for all* $x \in (-\varepsilon,\varepsilon)$.

Furthermore, if $0 \notin R(U)$*,* Σ *is any null set (with respect to the Lebesgue measure) and* $w \in \partial D$*, then given any* $\delta > 0$*, there exist* $\xi_0,\xi_1 \in (-\varepsilon,\varepsilon)\setminus\Sigma$ *such that* $U(\xi_j)=\lim(y\downarrow 0)\ U(\xi_j+iy) \in \mathcal{U}(K)$, $j=0,1$, *and*

$$\text{dist}[w,\sigma(U(\xi_1)U(\xi_0)^*)] < \delta .$$

Proof. Clearly, it is enough to prove the last statement.

Without loss of generality, we can assume that $U(x) = \lim(y\downarrow 0)\ U(x+iy) \in \mathcal{U}(K)$, for all $x \in (-\varepsilon,\varepsilon)\setminus\Sigma$ and that $U(\xi_0)=I$, for some ξ_0 in that set. Assume that, for some $e^{i\theta} \in \partial D$ and some $\delta > 0, \sigma(U(x)) \cap \Gamma(\theta-\delta,\theta+\delta) = \emptyset$, for all $x \in (-\varepsilon,\varepsilon)\setminus\Sigma$.

Let $a \in D$ and consider the inner function-operator

$$(13)\qquad U_a(z) = [U(z)-aI][I-\bar{a}U(z)]^{-1}\ e^{-i\theta}.$$

It follows from the previous observations and the *spectral mapping theorem* (see, e.g., [6,*prob.59*]), that if a is close enough to $e^{i\theta}$, then $\sigma(U_a(x)) \subset \Gamma(-\pi/4,\pi/4)$, for all $x \in (-\varepsilon,\varepsilon)\setminus\Sigma$. Hence,

$$(14)\qquad \text{Re } (U_a(x)\varphi,\varphi) \geqslant 1/\sqrt{2}\ , \quad \text{for all } \varphi \in K^1$$

(and a.e. $x \in (-\varepsilon,\varepsilon)$).

Recall that, for $z=x+iy \in$ UHP, $(U_a(z)\varphi,\varphi)$ is given by the convolution of $(U_a(x)\varphi,\varphi)$ with the Poisson kernel $y/\pi(x^2+y^2)$.

Since $|(U_a(x)\varphi,\varphi)| \leqslant 1$, it follows from the properties of the Poisson kernel (see, e.g., [18]) and (14) that there exists a positive η such that Re $(U_a(z)\varphi,\varphi) \geqslant 1/2$, uniformly with respect to

$\varphi \in K^1$ and $z \in F = \{x+iy\colon |x| < \varepsilon/2,\ 0 < y < \eta\}$; this shows, in particular, that $U_a(z)$ is invertible in K and $\|U_a(z)^{-1}\| \leqslant 2$, for all $z \in F$.
Thus, it follows from the above mentioned result of [10] that $0 \in R(U_a)$. From (13), we conclude that $0 \in R(U)$.

qed.

REMARK. *Continuity* cannot be replaced by *strong continuity* in *thm. 6.1.* In fact, if $\{\psi_n\}_{n=1}^{\infty}$ is an orthonormal basis of K and $U \in F$ is defined by $U(u)\psi_n = b_n(u)\psi_n$, where $b_n(u)$ is a finite Blaschke product (n=1,2,...), then U(u) is strongly continuous on the closed unit disc. If, for example, $b_n(u) = u^n$, then U(u) is compact for all $u \in D$ and $R(U) = D$ (i.e., every point of ∂D is a singularity of U). If $b_n(z) = (z-i-n)/(z+i-n)$, n=1,2,... , then U(z) is analytic on R and it satisfies the differential equation (1), where

$$M(x) = \sum_{n=1}^{+\infty} 2/[1+(x-n)^2]\, P_n$$

(P_n is the orthogonal projection of K onto the one-dimensional subspace spanned by ψ_n), and

$$0 \leqslant N(x) = \int_{-\infty}^{x} M(t)\, dt \leqslant 2\pi I \ ;$$

moreover, N(x) converges strongly to $2\pi I$ as x converges to $+\infty$. However, $U \notin$ (AI).

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State University of New York at Albany
Department of Mathematics

Recibido en enero de 1973.