1. INTRODUCTION.

A) We shall be concerned with the linear problem:

"to maximize $L(t) = \sum_{i=1}^{n} c_i t_i$

subject to the restraints:

a) $t_j - t_i \leq q_{ij}$, $i, j = 0, 1, \ldots, n$.

b) $t_0 = 0$.

This problem is seen to be equivalent to the "potential's restricted problem" (see [1], p.205).

The polyhedron $Z_q$ of solutions of a) is assumed non void; and we will denote by $Z_q^0$ the set of solutions of a) and b). Note that $Z_q$ is a cylinder, parallel to the vector $e = (1,1,\ldots,1)$.

Henceforth, we set $X = \{0,1,\ldots, n\}$; and a "potential of $X$" will be any real function $t$, eventually extended either by $+\infty$ or by $-\infty$ values.

B) In the inequalities a), their bounds $q_{ij}$ can be replaced by their "strictest bounds" $\epsilon_{ij}$, obtained either by

$$
\epsilon_{ij} = \inf_{j_1 \ldots, j_n} (q_{ij_1} + q_{j_1j_2} + \ldots + q_{j_{n-1}j}) , \quad (1)
$$

([2]), or by some alternative way (see [3]). Hence, $Z_q = Z_{\epsilon}, Z_q^0 = Z_{\epsilon}^0$.

As in [3], we shall use the term semitensions for these strictest bounds $\epsilon_{ij}$. They have the basic properties:

* Comunicado previamente en la reunión anual de la UMA de 1971.
i) $\varepsilon_{ii} = 0$

ii) $\varepsilon_{ij} + \varepsilon_{jk} \geq \varepsilon_{ik}$

and we summarize below the few other properties that will henceforth be needed.

For a fixed potential $t$ of $X$, the highest potential in $Z_\varepsilon$ that precedes $t$ is $E(t)$, given by

$$E(t)_j = \inf_{i} (t_i + \varepsilon_{ij}).$$

Thus, $E(t) \leq t$, and $E(t) = t$ is equivalent to $t \in Z_\varepsilon$.

Alternatively, the lowest potential in $Z_\varepsilon$ that follows $t$ is

$$F(t)_j = \sup_{i} (t_i - \varepsilon_{ji}).$$

Thus, $F(t) \geq t$, and $F(t) = t$ iff $t \in Z_\varepsilon$.

Now, if $\delta_j$ is the potential that has the value 0 at $j$ and $+\infty$ at each $i \neq j$:

$$E(\delta_j)_i = \varepsilon_{ji}, \quad F(-\delta_j)_i = -\varepsilon_{ij}$$

2. THE $(A,B)$-MINIMAX POINTS OF $Z_\varepsilon^0$.

A) In this section, we will show that: if $t$ is a potential defined on a subset $A$ of $X$, where it satisfies the inequalities a), then $t$ has both a maximum and a minimum extension on $X$, where they also satisfy a).

Let $E_A$ denote the operator defined as in (2), but "on A"; that is, relative to the restriction $\varepsilon/A$. For any potential $t$ we set

$$t^A_j = \begin{cases} t_j, & \text{when } j \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly:

$$E(t^A)/A = E_A(t),$$

since for $j \in A$, $E(t^A)_j = \inf_{i \in X} (t^A_i + \varepsilon_{ij}) = \inf_{i \in A} (t_i + \varepsilon_{ij}) = E_A(t)_j$.

From (3) it follows that if $t \in Z_\varepsilon/A$, then $E(t^A)$ is the maximum
extension of \( t \) on \( X \). In particular, for any \( t \in Z_e \), \( E(t^A) \geq t \), 
\[ E(t^A)/A = t_{A}^A. \]

In analogous manner, we define \( t_A \) as:
\[ (t_A)_j = \begin{cases} 
  t, & \text{if } j \in A, \\
  \infty, & \text{otherwise}. 
\end{cases} \]

Thus, \( F(t^A)/A = F_A(t) \), and therefore, if \( t \in Z_e/A \), then \( F(t^A) \in Z_e \)
is the minimum extension of \( t \) on \( X \). Moreover, for any \( t \in Z_e \),
\[ F(t^A) \leq t, \quad F(t^A)/A = t_A. \]

We add a remark on the monotony of extensions: \( t \geq t' \) implies both
\[ E(t^A) \geq E(t'^A) \text{ and } F(t^A) \geq F(t'^A). \]

B) With every \( t \in Z_e \), we now associate a relation \( R_t \subset X \times X \), by
means of:
\[ (i,j) \in R_t \text{ if } t_j = t_i + e_{ij}. \]

Now, it is clear that, for \( t, t' \in Z_e \), we shall have \( R_t = R_{t'} \) if
and only if both \( t \) and \( t' \) belong to the same open face of \( Z_e \); by
reason of that, and since \( c = \{x/x=x_i+e_{ij}, (i,j) \in R_t \cap Z_e \} \) is a
closed face, the \( R = R_t \) is seen to be associated to the closed
face \( c \) of \( Z_e \).

Moreover, notice that if \( A_1, A_2, \ldots, A_s \) are the connected compo-
nents of \( R \) - i.e., of its representative graph - the dimension of \( c \)
is precisely \( s \). This conclusion relies on the fact that the solutions of the linear system
\[ x_j = x_i + e_{ij}, (i,j) \in R, \]
can be given by:
\[ x = \lambda_1(t^A_1) + \lambda_2(t^A_2) + \ldots + \lambda_s(t^A_s), \]
where \( \lambda_A_j \) is the
characteristic function of the set \( A_j \).

Also, it is clear that the dimension of a face \( c^0 \) of \( Z_0 \) is one
unity less than the dimension of the corresponding face \( c \) of \( Z_e \),
since \( Z_0 \) is the intersection of \( Z_e \) with the hyperplane \( x_0 = 0 \).

In particular, a vertex \( t^0 \in Z_0 \) has a connected associated relation \( R^0 = R_{t^0} \); an edge of \( Z_0 \), a relation with two connected compo-
nents. We note that \( R^0 \) determines uniquely \( t^0 \).
In this way, the relations $R$ associated to edges meeting a given vertex $t^0 \in Z_e^0$, can be characterized as follows:

"$R \subseteq R^0$ has two connected components $C \ni 0$ and $D = X - C$, with $R/C = R^0/C$ and $R/D = R^0/D$; and moreover satisfies either I) $(i,j) \in R^0 \iff i \in C, j \in D,$ (α)
or II) $(i,j) \in R^0 \iff i \in D, j \in C$.

For brevity, we shall speak of the "blocks" $C$ and $D$, "fixed" and "loose", respectively.

The edge associated to this $R$ has the supporting line $t = t^0 + \lambda^1 d$. Thus, $t$ is a point in this edge iff either

a) case I: $\lambda^1 \leq \lambda \leq 0$, where $\lambda^1 = \inf_{i \in D, j \in C} (\varepsilon_{ij} - (t^0_j - t^0_i))$, or

b) case II: $0 \leq \lambda \leq \lambda^2$, where $\lambda^2 = \inf_{i \in C, j \in D} (\varepsilon_{ij} - (t^0_j - t^0_i))$. (β)

Joining to $R$ the pairs $(i,j)$ where (β) is realized, we obtain the relation $R_{t^1}$ associated to the opposed vertex $t^1$ in this edge.

We see that case I (resp. II) corresponds to a "descent" ("ascend") of $t^0$ on the loose block $D$.

For a geometric view, it could be useful to consider the line $t = t^0 + \lambda^1 d$ even when $R$ is neither in case I nor in case II. This situation may be named case III, and corresponds to:

"There are some $(i,j) \in R^0, i \in C, j \in D$, and some $(k,l) \in R^0, k \in D, l \in C$.

In this case, the line has in common with $Z_e^0$ the only point $t^0$, and thus can be most conveniently considered as supporting a "virtual edge" - i.e., reduced to $t^0$.

C. The relation $R = R_t$ associated to a point $t \in Z_e$, has the following properties of interest:

1. $R$ is a preorder in $X$.

Clearly, reflexivity is immediate. For transitivity, we have:

a) for $(i,j), (j,k) \in R$, it must be $t_j = t_i + \varepsilon_{ij}$, $t_k = t_j + \varepsilon_{jk}$.
and thus \( t_k = t_i + \varepsilon_{ij} + \varepsilon_{jk} \geq t_i + \varepsilon_{ik} \) from property (ii) of \( \varepsilon \).

b) \( t_k \leq t_i + \varepsilon_{ik} \). From a) and b), it follows that \((i,k) \in R\).

2. If \( \varepsilon_{ij} + \varepsilon_{jk} = \varepsilon_{ik} \), then: \((i,k) \in R \iff (i,j),(j,k) \in R\).

D. Let \( A,B \) be sets of \( X \) such that \( A \cup B = X \), \( A \cap B = \{0\} \). A point \( t \in Z^0_\varepsilon \) will be called an "\((A,B)\)-minimax" in notation, \( t \in M_{AB} = M \) if for any \( t' \in Z^0_\varepsilon \) such that \( t'/A \geq t_A \) and \( t'/B \leq t_B \), necessarily \( t = t' \).

Observe that this can be easily expressed in terms of the extensions above considered. In fact, \( t \in Z^0_\varepsilon \) is an \((A,B)\)-minimax iff \( t = E(t_B^B) = F(t_A^A) \). From that, \( t \in M \) implies: 1) For any \( j \in A \), \( t_j = \inf_{i \in B} (t_i + \varepsilon_{ij}) \) and 2) for any \( j \in B \), \( t_j = \sup_{i \in A} (t_i - \varepsilon_{ij}) \).

Thus, when \( t \in M \), its associated relation \( R = R_t \) verifies: 1) for any \( j \in A \), there is some \( i \in B \) such that \((i,j) \in R \), and 2) for any \( j \in B \), there is some \( i \in A \) such that \((j,i) \in R \). A relation \( R \) with these properties will be conveniently named "\((A,B)\)-total", and we can summarize our conclusions as follows:

"\( t \in Z^0_\varepsilon \) is an \((A,B)\)-minimax point iff its associated relation \( R \) is \((A,B)\)-total".

Incidentally, \( M \) is an union of open faces of the polyhedron \( Z^0_\varepsilon \); and even more, of closed faces, since \( R_t \supset R_t = R \) implies that \( R_t \) is also \((A,B)\)-total.

Moreover, let \( R^0 \) denote the connected and \((A,B)\)-total relation associated to a vertex \( t^0 \in M \). Then, to any edge in \( M \) and meeting \( t^0 \), corresponds a relation \( R \subset R^0 \) that is \((A,B)\)-total and verifies condition \((\alpha)\).

E. For our purposes, it will be useful to consider not only preorder relations \( R \) that are \((A,B)\)-total, but also their "traces", \( S = R \cap (B \times A) \) on \( B \times A \). It is seen than \( R \) and \( S \) have the same connected components.

Furthermore, if \( R \) and \( R^0 \) are two preorder and \((A,B)\)-total relations that both satisfy condition \((\alpha)\), then their respective traces \( S \) and \( S^0 \) are also \((A,B)\)-total, satisfy \((\alpha)\), and moreover to
cases I, II for R corresponds respectively cases I, II for S.

By reason of that, the $\lambda_1, \lambda_2$, values can be obtained by

$$
\lambda_1 = \inf_{i \in C \cap B, j \in C \cap A} (\epsilon_{ij} - (t_j^0 - t_i^0)),
$$

$$
\lambda_2 = \inf_{i \in C \cap B, j \in C \cap A} (\epsilon_{ij} - (t_j^0 - t_i^0)). \quad (\beta')
$$

For brevity, we confine ourselves to the verification in the case $\lambda_2 = \epsilon_{ij} - (t_j^0 - t_i^0), i \in C \cap B, j \in C \cap B$. Since R is $(A,B)$-total and C, D, are their connected components, there is some $k \in D \cap A$ such that $(j,k) \in R$, i.e. $\epsilon_{jk} = t_k^0 - t_j^0$. Then, $\lambda_2 = \epsilon_{ij} - (t_j^0 - t_i^0) = \epsilon_{ij} - (t_j^0 - t_k^0 + t_k^0 - t_i^0) = \epsilon_{ij} + \epsilon_{jk} - (t_k^0 - t_i^0) \geq \epsilon_{ik} - (t_k^0 - t_i^0)$, hence $\lambda_2 = \epsilon_{ik} - (t_k^0 - t_i^0), i \in C \cap B, k \in D \cap A.$

F. We can summarize these results as follows:

To each $t_0 \in M = M_{AB} \subset Z^0$ we assign its "characteristic relation" $S_{t_0} = S^0 \subset B \times A$, defined as $S^0 = \{(i,j)/t_j^0 - t_i^0 = \epsilon_{ij}, (i,j) \in B \times A\}$ and which is $(A,B)$-total: Proj_A(S^0) = A, Proj_B(S^0) = B. The dimension of the open face of $Z^0$ that contains $t_0$ is one unity less than the number of connected components of $S^0$.

In particular, if $S^0$ is connected - and therefore $t_0$ is a vertex-

any $(A,B)$-total subrelation $S$ of $S^0$ which has two connected components determines a partition $C \ni 0, D$, of X.

According to the above considerations, S corresponds to an edge in M whenever we have: either I): $(i,j) \in S^0$ - S iff i \in C, j \in D, or II): $(i,j) \in S^0$ - S iff j \in C, i \in D. The opposed vertex in this edge will be obtained by replacing $\lambda$ in $t_0 + \lambda I_D$ by the corresponding $\lambda_1$ or $\lambda_2$, evaluated from $(\beta')$; and its characteristic relation $S'$ will result by joining to $S$ all the pairs $(i,j)$ that verify either $\lambda_1 = -(\epsilon_{ij} - (t_j^0 - t_i^0))$ or $\lambda_2 = \epsilon_{ij} - (t_j^0 - t_i^0),$
according to the case in consideration.

G. We confine ourselves to a way of associating to a point \( t \in Z_0^\varepsilon \) another point \( t^* \in M \), namely by means of:

\[ t^* = E((F(t_A))^B) \]

By doing that, we will have:

**Proposition 1.**

1) \( t \in Z_0^\varepsilon \) implies \( t^* \in M \),

2) \( \frac{t^*}{A} > \frac{t}{A} \), \( \frac{t^*}{B} < \frac{t}{B} \),

3) \( t = t^* \) iff \( t \in M \),

4) The mapping \( t \mapsto t^* \) is continuous.

**Proof.** First, it is a clear consequence of the properties of the extensions that for any \( t \in Z_0^\varepsilon \) we have:

\[ F(t_A)/A = \frac{t_A}{A} \leq E(t_B^B)/A, \quad F(t_A^A)/B \leq \frac{t}{B} = E(t_B^B)/B \quad (\gamma) \]

Now, is self-evident that \( t^* \in Z_0^\varepsilon \). Therefore, by use of (\( \gamma \)):

a) \( \frac{t^*}{B} = E((F(t_A))^B)/B = F(t_A)/B \), and thus \( t_B^B = (F(t_A))^B \) concludes that \( t^* = E(t_B^B) \).

b) From (\( \gamma \)) and a) we have:

\[ F(t_A^A)/A = \frac{t_A^A}{A}, \quad F(t_A^B)/B \leq \frac{t^*_A}{B} = F(t_A^A)/B. \]

But now \( \frac{t^*_A}{A} = E((F(t_A))^B)/A \geq F(t_A)/A = t_A \) ensures \( t^*_A \geq t_A \) and by monotony \( F(t_A)^B \geq F(t_A) \), whence \( F(t_A)/B \geq F(t_A)/B = t_A^B \). Then \( t^* = F(t_A^A) \).

This proves that \( t^* \in M \), the rest is a straightforward consequence from the definitions for \( E, F \) and minimax.

H. We remark that \( (E(\delta_0^0))^* \) and \( (F(-\delta_0^0))^* \) are both vertices of \( Z_0^\varepsilon \) and, of course, \((A,B)\)-minimax.

In fact, \( \rho = E\delta_0^0 \) has coordinates \( \rho_j = \varepsilon_{0j} \), so that from \( \rho_j - \rho = \varepsilon_{0j} \) it follows that \( (0,j) \in R_\rho \) for any \( j \in X \). Because \( R_\rho \) is
connected, \( \rho \) is a vertex of \( Z_0^* \). Moreover, as \( \rho^*_A \geq \rho^*_B \) and \( \rho \) is the highest potential in \( Z_0^* \), we have \( \rho^*_A = \rho^*_B \), and thus \((0,j) \in R^*_\rho \) for any \( j \in A \).

Furthermore, the fact that \( \rho^* \) is \((A, B)\)-minimax ensures that for any \( i \in B \) there is some \( j \in A \) such that \((i, j) \in R^*_\rho \), and thus \( R^*_\rho \) is connected and \( \rho^* \) is proved to be a vertex of \( Z_0^* \).

The proof for \( \sigma^* = (F(-\delta^0))^* \) proceeds by arguments completely analogous.

3. THE LINEAR FORM \( L \) AND THEIR RELATED MINIMAX.

We start by considering the linear form \( L(t) = \sum_{i=1}^{n} c_i t_i \), and by setting \( A = \{i/c_i > 0\} \cup \{0\} \) and \( B = \{i/c_i < 0\} \cup \{0\} \).

From the definition of \( t^* \), it follows that \( t \in Z_0^* \) implies \( L(t) \leq L(t^*) \). In particular, the maximum of \( L \) is attained at a point \( t \in M \). The converse is ensured whenever \( c_i \neq 0 \) for any \( i \).

PROPOSITION 2. If \( t^0 \in M \) is a local maximum of \( L \) in \( M \), then it is a maximum of \( L \) in \( Z_0^* \).

Proof. Let us consider any \( t^1 \in Z_0^* \), and form \( t^\lambda = t^0 + \lambda(t^1 - t^0) \), \( 0 < \lambda < 1 \). Then, \( \lambda \to 0 \) determines \( t^\lambda \to t^0 \), and from Prop. 1, \( (t^\lambda)^* \to (t^0)^* = t^0 \). From the assumed conditions, \( L((t^\lambda)^*) \leq L(t^0) \) for suitably small values of \( \lambda \), and the above considerations imply that \( L(t^1) \leq L(t^0) \).

B. For a given vertex \( t^0 \in M \) whose characteristic relation is \( S^0 \), let \( S \subset S^0 \) be the characteristic relation of an edge in \( M \) meeting \( t^0 \) and having \( C, D \), as, respectively, fixed and loose blocks.

From \( t = t^0 + \lambda \sum_{i \in D} c_i (\lambda_1 < \lambda < \lambda_2 \text{ when in case I}, 0 < \lambda < \lambda_1 \text{ when in case II}) \), we see that \( L \) increases on the edge iff: either \( L(1_D) < 0 \) in case I, or \( L(1_D) > 0 \) in case II. \( L(1_D) = \sum_{i \in D} c_i \)

In the simplex method, the consideration of all the edges meeting
a vertex is generally out of reach. Very much the same happens with the determination of all relations $S \subseteq S^0$ corresponding to edges in $M$ and meeting $t^0$.

In the following section, we present an algorithm that obviates this difficulty.

4. ALGORITHM.

A) The algorithm proposed for solving the problem posed in § 1, can be summarized in the following steps:

0) The starting point is a vertex $t^0 \in M$, of characteristic relation $S^0$. We construct on $S^0$ a tree $T^0$ which is maximal and without any path of length two. Such a tree will be called an "$(A,B)$-tree" - for short, a TAB of $t^0$.

1) For a given $(h,k) \in T^0$, we consider $T^0-(h,k)$ and the corresponding connected components $C \ni h$ and $D$ that it determines. After doing that, we evaluate $L(1_D) = \sum_{i \in D} c_i$. We have then the alternative:

a) either (I) $L(1_D) < 0$ with $h \in C$, $k \in D$, or (II) $L(1_D) > 0$ with $h \in D$, $k \in C$.

b) The situation a) does not happen.

Clearly, a) expresses the possibility of an increase of $L$ by displacing the loose block $D$.

2) When $(h,k)$ is such that a) holds, we evaluate:

either (case I): $\lambda_1 = \inf_{i \in B \cap D} (e_{ij} - (t_j^0 - t_i^0))$

or (case II) $\lambda_2 = \inf_{i \in B \cap C} (e_{ij} - (t_j^0 - t_i^0))$.

For each case, there will be always at least a pair such that the inf. is realized. From among them, we choose if possible a pair $(i,j)$ that excludes the point $0$.

After this has been carried out, we construct the TAB $T^1 = T^0$ -
- \((h,k) + (i,j)\) and obtain its corresponding vertex \(t^1 = t^0 + \lambda D \in M\), where of course \(\lambda\) is either \(\lambda_1\) or \(\lambda_2\).

We return to 1) and reinitiate the procedure, considering now \(T^1\) instead of \(T^0\).

3) If \((h,k)\) corresponds to b) we start again with another pair \((l,m)\) of \(T^0\).

4) The algorithm is concluded when every \((h,k)\) of \(T^0\) appears in the situation b). When that happens, \(T^0\) maximizes \(L\).

REMARKS.

1) For the starting point in 0), we may take either \(p^*\) or \(a^*\) (see 2.H).

2) The construction of the TAB \(T^0\) is always possible not only if a) for any \(i,j\), \((i,0)\) and \((0,j)\) are not both in \(S^0\), but also when b) there are \((1,0)\) and \((0,1)\) in \(S^0\), in which situation it suffices to suppress every \((i,0)\in S^0\) and to take a maximal tree in the residual connected relation.

3) When a) holds, it is clear that if \(\lambda_1\) (resp. \(\lambda_2\)) \(\neq 0\), then \(t^1 \neq t^0\) and \(L(t^1) > L(t^0)\). But when \(\lambda_1(\lambda_2) = 0\), although \(t^1 = t^0\) we nevertheless have \(T^1 \neq T^0\).

4) The instructions in step 3 preclude the possibility of \(T^1\) having a path of length 2, so that \(T^1\) is truly a TAB. In fact, assume case I and let \((1,0)\) be a pair that realizes the infimum: if moreover there is some \((0,j)\in T^0\), we single out \((i,j)\) that also realizes the inf. (see 2.C).

Case II is similar.

To justify the algorithm, it remains to prove: first, the second assertion of step 4); and next, that the algorithm can conclude in a finite number of steps.

B. We begin by considering a relation \(U\) in \(X\), which is connected and without paths of length 2 - i.e., the occurrence of both \((i,j)\) and \((j,k)\) in \(U\) is excluded. Then \(A' = \{i/(i,j) \not\in U, for any j\}\), \(B' = \{j/(i,j) \not\in U, for any i\}\) determine a partition of \(X\).

We now define the mapping \(\eta: XxX \rightarrow R\), by setting: \(\eta_{ij} = 0\) when-
ever either \((i,j) \in U\) or \(i = j\); and for the rest: \(\eta_{ij} = 1\) when \(i \in B'\) and \(j \in A'\), \(\eta_{ij} = 2\) when \(i,j\) are either both in \(A'\) or both in \(B'\), and finally \(\eta_{ij} = 3\) when \(i \in A'\) and \(j \in B'\).

By a rather tedious discussion of cases, we can verify the following:

**Lemma 1.** \(\eta\) is a semitension on \(X\), such that: \(\eta_{ij} = 0\), \(i \neq j\), iff \((i,j) \in U\); \(\eta_{ij} > 1\) when \((i,j) \notin U\), and \(\eta_{ij} + \eta_{jk} - \eta_{ik} > 1\) when \(i \neq j\) and \(j \neq k\).

As the foregoing conditions hold in particular for any arbitrary TAB, this Lemma permits us to assert:

**Proposition 3.** Let \(t^0\) be a vertex in \(M\). If \(T^0\) is a TAB of \(t^0\), and \(b)\) holds for it, then \(t^0\) is optimal for \(L\).

**Proof.** The semitension \(\varepsilon' = \varepsilon + \eta\) coincides with \(\varepsilon\) on \(T^0\) and is strictly greater than \(\varepsilon\) in the remnant. First, we note that \(Z^0_\varepsilon \subset Z^0_{\varepsilon'}\). Next, \(t^0 \in Z^0_{\varepsilon'}\), and its associated relation is precisely \(T^0\), since \(t^0_j - t^0_i = \varepsilon'_{ij}\) whenever \((i,j) \in T^0\) and furthermore \(t^0_j - t^0_i < \varepsilon'_{ij}\) for any \((i,j) \notin T^0\).

From the above it follows that, as \(t^0\) is connected, \(t^0\) is a vertex of \(Z^0_{\varepsilon'}\) (and even more, is a regular vertex). By reason of that, \(b)\) implies (3.B) that \(t^0\) is optimal of \(L\) on \(Z^0_{\varepsilon'}\), and thus also on \(Z^0_\varepsilon\).

C. Let \(t^0\) be a vertex in \(M\), and \(T^0\) a TAB of \(t^0\). As seen before, if \(T^0\) is in \(b)\) of the alternative, then \(t^0\) is optimal for \(L\). But whenever \(T^0\) is in \(a)\) of the alternative, one of two possibilities happens: Whether

- \(a_1\) It is possible to determine an "algorithmic sequence" \(T^0, T^1, T^2, \ldots\), of TABs - i.e., each one is obtained from the precedent by means of the algorithm - for which there is some \(T^m\) corresponding to a vertex \(t^m \neq t^0\); and then, \(L(t^m) < L(t^0)\). Or
- \(a_2\) The possibility \(a_1\) is excluded.
When $a_1$ holds, then $T^m$ is discussed as $T^0$ - that is, determining for it the validity of $b), a_1$) or $a_2$).

When $a_2$) appears, we will show below that is possible to determine a (finite) algorithmic sequence $T^0, T^1, T^2, \ldots, T^N$, such that $T^N$ is in $b)$ and thus $t^0 = t^N$ is optimal for $L$.

For proving that, we proceed by "perturbating" $\epsilon$ so as to split $t^0$ into a finite number of regular vertices. For doing that, let $U$ be the relation obtained from the characteristic relation $S^0$ of $t^0$ by taking out, whenever there is some $(i, 0) \in S^0$, the pairs $(i, 0) \in S^0$.

Now, let us assume that we can determine - as it will be carried out later - a semitension $\epsilon'$ such that:

A1) $t^0$ is a vertex of $Z^0_{\epsilon'}$, whose associated relation is $T^0$.

A2) If $t^r$ is any vertex of $Z^0_{\epsilon'}$, whose associated relation is a TAB $T^r \subset U$, and $t^{r+1}$ is obtained from $t^r$ by means of the algorithm as applied in $Z^0_{\epsilon'}$, then the relation associated to $t^{r+1}$ in $Z^0_{\epsilon'}$ is also a TAB $T^{r+1} \subset U$, and moreover $T^{r+1}$ proceeds from $T^r$ by means of the algorithm in $Z^0_{\epsilon'}$.

By assuming this for the moment, the desired conclusion follows. In fact, note first that A2) implies $L(t^{r+1}) > L(t^r)$, hence it precludes repeated elements in the sequence $t^0, t^1, t^2, \ldots$ and therefore any two of the corresponding TABs $T^0, T^1, T^2, \ldots$ are different.

From that, and since $U$ contains only a finite number of TABs, the sequence must end in some TAB $T^N$, so that the passing of $t^N$ to an other vertex $t^{N+1}$ of $Z^0_{\epsilon'}$ is not possible by means of the algorithm in $Z^0_{\epsilon'}$. The conclusion follows that $T^N$ must be, and is, in $b)$ of the alternative.

Now for the proof that we can determine $\epsilon'$ complying with the aforesaid requirements. First, and as the starting assumption is that we cannot by means of the algorithm in $Z^0_{\epsilon'}$ pass from a vertex
t^0 to another one t, we will modify ε with a view to maintain this impossibility on the future "perturbated" t^ρ. This may, and shall, be done by adding to ε the semitension η defined as in Lemma 1. Finally, we "perturbate" ε + η by means of a suitably small function, so as to split t^0 into regular vertices of 2^0.ε.

For this we state:

**Lemma 2. (Variation of η).** In the conditions of Lemma 1, and if x satisfies: x_{ii} = 0, 0 ≤ x_{ij} < 1/3 for i ≠ j, then η + x is a positive semitension.

With that, we have the starting point for the proof; however, we introduce previously the following notation.

Let γ = [β_0 α_0 β_1 α_1 ... β_n α_n β_{n+1}], with β_0 = β_{n+1}, be a cycle in U such that β_i ∈ B, α_i ∈ A. We denote by x_γ the sum of the x_{β_i a_i} and the -x_{a_i β_{i+1}}. In particular, we note that ε_γ = 0, and also (ε + η)_γ = 0, since ε_β_i a_i = t^0_{a_i} - t^0_{β_i}, -ε_α_i β_{i+1} = t^0_{a_i} - t^0_{β_{i+1}} and η/γ = 0.

Let us prove that we can determine x complying with the requirements:

- x_{ij} = 0 for (i,j) ∈ T^0,
- 0 < x_{ij} < 1/3 for (i,j) ∈ U - T^0,
- x_γ ≠ 0 for any cycle γ of U.

In fact, x_γ = 0 determines an hyperplane in R^{U-T^0}, since x_γ = 0 would imply γ ⊂ T^0 and that is impossible. As there is a finite number of cycles γ in U, we cannot cover with the corresponding hyperplanes the open set \{x/0 < x_{ij} < 1/3, (i,j) ∈ U-T^0\}.

Once selected one such x, and for 0 < δ < 1, we set ε' = ε + η + δx. Since δx remains in the conditions of Lemma 2, ε' is a semitension.

Now, it only remains to prove that ε' satisfies the above requirements A1 and A2.
A1) \( \eta + \delta x = 0 \) on \( T^0 \), \( \eta + \delta x > 0 \) elsewhere. Therefore, \( T^0 \) is the associated relation of \( t^0 \) in \( Z^0_e \), and \( t^0 \) is a vertex.

A2) For this property - more lengthy to prove - we suppose inductively that there is \( \delta_r > 0 \) such that, for any \( \delta \in (0, \delta_r) \), \( T^0 = t^0(\delta) \) is a vertex in \( Z^0_e \), whose associated relation \( T^r \) is a TAB in \( U \) which does not depend on \( \delta \); and, moreover, such that \( t^r(\delta) \) is continuous, with \( t^r(0) = t^0 \).

Now, let \( t^{r+1} \) be a vertex in \( Z^0_e \), obtained from \( t^r \) by means of the algorithm.

We assert that \( t^{r+1} \) satisfies the same conditions that \( t^r \), and furthermore than its TAB \( T^{r+1} \) is obtained from \( T^r \) through the algorithm in \( Z^0_e \).

For that, note first that from \( t^{r+1} = t^r + \lambda_1 \delta \), if follows that \( t^{r+1} \) is a continuous function of \( \delta \) in \( [0, \delta_r] \), with \( t^{r+1}(0) = t^0 \).

In fact, \( \lambda \) is an infimum of continuous functions of \( \delta \), each having the expression \( \epsilon'_{ij} - (t^r_j - t^r_i) \); and for \( \delta = 0 \) and \( (i,j) \in U \), we have \( \epsilon'_{ij} - (t^r_j - t^r_i) = \epsilon_{ij} \) \( (t^0_j - t^0_i) = 0 \), thus ensuring \( \lambda(0) = 0 \).

Next, suppose that the algorithmic passing from \( t^r \) to \( t^{r+1} \) corresponds to case II - the case I is clearly analogous. Then, we have:

\[
\lambda = \lambda_2 = \inf_{(i,j) \in B \cap C} \left( \epsilon'_{ij} - (t^r_j - t^r_i) \right)
\]

We once more consider the pairs \( (i,j) \), \( i \in B \cap C \), \( j \in A \cap D \).

1) For \( (i,j) \notin U \), \( \epsilon'_{ij} - (t^r_j - t^r_i) = \epsilon_{ij} + \delta x_{ij} + \eta_{ij} \) \( (t^0_j - t^0_i) + (t^0_j - t^r_j - (t^0_j - t^r_i)) \geq 1 \) \( (t^0_j - t^r_j - (t^0_j - t^r_i)) - (t^0_j - t^r_j) \) since \( \eta_{ij} \geq 1 \), \( x_{ij} > 0 \), \( \epsilon_{ij} - (t^0_j - t^0_i) > 0 \), and thus, for suitably small \( \delta \) we have: \( \epsilon'_{ij} - (t^r_j - t^r_i) \geq 1/2 \).

2) For \( (i,j) \in U \), \( \epsilon'_{ij} - (t^r_j - t^r_i) \) converges to 0 = \( \epsilon_{ij} - (t^0_j - t^0_i) \).

Then, for suitably small \( \delta_{r+1} \) and \( \delta \in [0, \delta_{r+1}] \), \( \lambda(\delta) = \epsilon'_{ij} - \)
- \((t_j^r - t_i^r)\), with \((i,j) \in U, i \in B \cap C, j \in A \cap D\). Assuredly that, for any \(\delta \in [0, \delta_{r+1})\), there is an unique pair \((i,j)\) with these requirements. For, if there were two of them, a cycle would appear in the associated relation \(T^{r+1}\) of \(T^{r+1}\), and from that, \(e'_\gamma = 0 = e_\gamma + \eta_\gamma + \delta x_\gamma = \delta x_\gamma\) gives a contradiction.

Finally, let \(\Delta_{ij}(\delta) = e'_{ij} - (t_j^r - t_i^r), (i,j) \in U, i \in B \cap C, j \in A \cap D\).

From the results above, it is not possible for any two of such functions to have the same value at any \(\delta \in [0, \delta_{r+1})\), and therefore one of those functions is strictly lesser than the others.

This shows that we have \(\lambda(\delta) = e'_{ij} - (t_j^r - t_i^r)\) for the same unique pair \((i,j)\) and any \(\delta \in [0, \delta_{r+1})\); in particular, it enables us to conclude not only that \(T^{r+1}\) is a TAB in \(U\) which does not depend on \(\delta\), but furthermore that we can pass from \(T^r\) to \(T^{r+1}\) by means of the algorithm in \(Z_0\).

The existence of \(\delta_0\) for the starting \(T^0\) is easily verifiable.

CONCLUDING REMARK. In the proposed algorithm, the eventuality of a "circling" is not precluded; but we rely on the fact of its highly improbable occurrence in practice. That is why no special corrective devices were considered nor deemed necessary.

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Universidad Nacional del Sur.
Bahía Blanca, Argentina.

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