ACTIONS ON A GRAPH

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ABSTRACT. Part of the theory of flows and tensions on a graph is extended to any kind of actions, i.e. to any subspace of the space of real functions defined on the arcs; in particular, the theorems on the existence of flows or tensions under bilateral restraints.

1. INTRODUCTION.

Our basic setting is a graph $G = (X, S)$; here it means that $S \subseteq X \times X$ verifies: $(i, i) \notin S$ and $(i, j) \in S$ implies $(j, i) \in S$, $i, j \in X$. We assume $G$ connected.

We consider functions $f, g, \ldots$ defined on $S$ and we denote by $f \cdot g = \sum_{ij} f_{ij} g_{ij}$, the usual scalar product.

By $E$ we denote the linear space of anti-symmetric functions:

$e_{ij} + e_{ij} = 0, (i, j) \in S$.

The subspaces of flows and tensions, $\Phi, \Theta \subseteq E$, are defined by:

$\Phi$, if $\sum_j e_{ij} = 0, \forall i \in X$, and $\Theta$, if $\theta_{ij} = t_j - t_i$, $t$ defined on $X$ ($\theta = \Delta t$). $\Phi$ and $\Theta$ are orthogonal complements in $E$.

If $F \subseteq E$ is defined as the set of solutions $f$ of the linear system: $\lambda^\alpha f = 0$, $\alpha \in M$, its orthogonal space $G = F^\perp$ in $E$, is generated by the $\mu^\alpha$, $\alpha \in M$, where $\mu^\alpha_{ij} = \lambda^\alpha_{ji} - \lambda^\alpha_{ij}$ When $M = X$ and the $\lambda^\alpha_{jk}$ vanish except for $j = i$, writing $\lambda^i_{ik} = \lambda^i_{ik}$, we have $\mu^i_{ij} = -\lambda^i_{ij}$, $\mu^i_{ji} = \lambda^i_{ij}$ and $\mu^i_{hk} = 0$ for the remaining $(h, k) \in S$. Then, the elements $g \in G$ are of the form $g_{ij} = t_j \lambda^i_{ji} - t_i \lambda^i_{ij}$ ($g = \Delta^i t$),

where $t$ is a function on $X$. The case $\lambda^i_{ij} = 1$ corresponds to $F = \Phi$, $G = \Theta$. Given an orientation to $G$ - i.e. a subset $U$ of $S$ containing for each $(i, j) \in S$ one and only one of the pairs $(i, j)$, $(j, i)$ and a positive $m_i$, $i \in X$, defining $\lambda^i_{ik} = m_i$, if $(i, k) \in U$, and
\( \lambda_{ij} = 1 \), if \((i,j) \notin U\), we obtain the spaces of multiplicative flows and tensions ([1] pp.225).

Actions of a certain kind can be thought as the elements \( f \) of a subspace \( F \) of \( E - f_{ij} \) representing the intensity of the action \( f \) transmitted from \( i \) to \( j \) through the link \((i,j) \in S\).

Certain notions and results of the theory of flows and tensions on a graph, can be extended to any subspace \( F \) of \( E \). Doing that, a unified linear treatment of the outstanding cases \( F = \emptyset, F = \Theta \) - that may be useful - is obtained.

In 2 we give the notion of elementary action, corresponding to the notions of elementary cycles and cocycles, and a proposition on the decomposition of any action in elementary ones. It gives the known decomposition of a positive flow (tension) - on an oriented graph - as a positive linear combination of elementary cycles (cocycles) ([1] pp.143).

In 3 we prove the analogue of Hoffman and Roy's theorems ([2], [3]) for actions of any kind, using the appropriate geometric version of the consistence theorem of a system of linear inequalities (Farkas-Minkowsky).

2. ELEMENTARY ACTIONS.

For \( f \in E \) we denote \( s(f) = \{(i,j)/f_{ij} > 0\} \), the (effective) support of \( f \).

It is seen that, for \( f, g \in E \):

\[ (A) \quad \emptyset \neq s(g) \subseteq s(f) \Rightarrow s(f-\lambda g) \subseteq s(f), \text{ properly, for the positive number } \lambda = \max \left\{ \frac{f_{ij}}{g_{ij}} \mid g_{ij} > 0 \right\}. \]

A function \( f \in F, f \neq 0 \), is said to be an elementary function of \( F \) if for any \( g \in F, s(g) \subseteq s(f) \) implies \( g = \lambda f \).

This means that \( s(f) \) is a minimal set of \( \{s(g)/g \in F\} \). In fact, \( s(f) \) minimal implies, for each \( g \in F \) with \( s(g) \subseteq s(f) \), that \( s(f-\lambda g) \subseteq s(f) \), properly, (A); since \( f - \lambda g \in F \) it follows \( s(f - \lambda g) = \emptyset \), \( f = \lambda g \). The converse is clear.

Of course, if \( f \) is an elementary function of \( F \), so is \( \lambda f, \lambda \neq 0 \).

PROPOSITION. Any \( f \in F, f \neq 0 \), is a sum of elementary functions.
$\mathcal{F}_a$ of $F$ such that $s(f_a) \subset s(f)$.

**Proof.** Let $g_1$ be an elementary function of $F$ such that $s(g_1) \subset s(f)$. From (A), for some $\lambda_1 > 0$, it is $s(f - \lambda_1 g_1) \subset s(f)$, properly. If $f - \lambda_1 g_1 \neq 0$, we apply to $f - \lambda_1 g_1 \in F$ the same argument and we get an elementary $g_2 \in F$, $\lambda_2 > 0$, such that $s(g_2) \subset s(f - \lambda_1 g_1)$ and $s(f - \lambda_1 g_1 - \lambda_2 g_2) \subset s(f - \lambda_1 g_1)$, properly. After a finite number of steps we have elementary $g_1, \ldots, g_k \in F$, $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$, such that $f - \lambda_1 g_1 - \ldots - \lambda_k g_k = 0$. The proposition follows with $f_a = \lambda_1 g_a$, $1 \leq \alpha \leq k$.

For a set $Z \subset S$, such that $(i,j) \in Z$ implies $(j,i) \notin Z$, we define $\xi = \xi(Z)$ by $\xi_{ij} = 1, -1, 0$ according to $(i,j) \in Z$, $(j,i) \in Z$ or $(i,j), (j,i) \notin Z$, respectively. $\xi \in E$ and $s(\xi) = Z$.

If $Z = \{(i,j), (j,k), \ldots, (h,l), (l,i)\}$ is a cycle, $\xi$ is a flow, if $Z = \{(i,j)/i \in A, j \notin A\}$ ($A, X - A \neq \emptyset$) is a cocycle, $\xi$ is the tension $\Delta(-1)\lambda$. If the sequence $i,j,k,\ldots,h,l$ of the cycle $Z$ has not repeated elements $Z$ is an elementary cycle. If $A$ and $X - A$ are connected, in the graph $G_z$ obtained eliminating the $(i,j), (j,i)$, with $(i,j) \in Z$, the cocycle $Z$ is said to be an elementary one.

It is clear that $\xi = \xi(Z)$ is an elementary flow, when $Z$ is an elementary cycle. For an elementary cocycle $Z$, if $\theta = \Delta t$ is such that $s(\theta) \subset Z = s(\xi)$, the connectedness of $A$ and $X - A$ in $G_z$ implies $t_{\mid A} = \alpha, t_{\mid X - A} = \beta$, then $\Delta t = \theta = (\beta - \alpha)\xi$, with $\beta > \alpha$. Hence $\xi$ is an elementary tension.

Conversely, if $\varphi \neq 0$ is a flow, $s(\varphi)$ verifies that $(i,j) \in s(\varphi)$ implies $(j,k) \in s(\varphi)$ for some $k \neq i (\varphi_{ji} + \sum_{k \neq i} \varphi_{jk} = 0, \varphi_{ji} = -\varphi_{ij} < 0)$.

It follows that $s(\varphi)$ contains a cycle, and then also an elementary cycle $Z$. Hence if $\varphi$ is an elementary flow, $\xi = \lambda \varphi, \lambda > 0$. On the other hand, if $\theta = \Delta t \neq 0$ is a tension, taking $\alpha$: $\min t_i < \alpha < \max t_i$, the cocycle $Z$ defined by means of $A = \{i/t_i < \alpha\}$ is contained in $s(\theta)$. Hence, if $\theta$ is an elementary tension we have $\xi = \lambda \theta, \lambda > 0$. Z has to be an elementary cocycle, otherwise we could
take a connected component $A'$ of $A$ (alternatively of $X - A$) in $G_z$ and define $Z'$ in terms of $A'$. But this would imply $s(\mathcal{F}') \subseteq s(\mathcal{G})$, properly; which is a contradiction.

Resuming, the multiples $\lambda s, \lambda > 0, s = s(Z)$, for $Z$ elementary cycle (cocycle), are the elementary functions of $\Phi(\mathcal{G})$.

3. EXISTENCE THEOREM.

We consider a finite dimensional linear space with the scalar product $x \cdot y$. For a cone $Q = Q^+ Q \subseteq Q$, $\lambda Q \subseteq Q$ for every $\lambda > 0$ - the dual cone is defined by $Q^0 = \{x / x \cdot y \geq 0, \text{for any } y \in Q\}$.

We need the theorem of consistence of a system of linear inequalities under the following form:

"Given the polyhedral set $C$ and the polyhedral cone $Q$, $(C, Q \neq \emptyset)$ it holds: $Q \cap C \neq \emptyset$ if and only if, for every $x \in Q^0$ there is a $c \in C$ such that $x \cdot c \geq 0$".

In fact, if $Q \cap C = \emptyset$, we can separate the closed convex (polyhedral) set $Q - C$ from 0; i.e. there is $x$ such that $x \cdot (c - q) < 0$, for any $c \in C, q \in Q$. Taking $\lambda q, \lambda > 0$, instead of $q$, we conclude that $x \cdot q > 0$, i.e. $x \in Q^0$. For $q = 0$ we have $x \cdot c < 0$ for every $c \in C$. The converse is clear.

We will apply the theorem to a subspace $Q$. In this case $Q^0 = Q^\perp$.

THEOREM. Let $c_{ij} + c_{ji} > 0$, for any $(i, j) \in S$, and $F$ be a linear subspace of $E$. In order that there exists $f \leq c$, $f \in F$, it is necessary and sufficient that, for each elementary $g \in G = F^\perp$, $g^+ \cdot c > 0$.

REMARK. As it is usual: $g^+ = \max(g, 0)$. The condition $c_{ij} + c_{ji} > 0$ is obviously necessary for the existence of an anti-symmetric $f \leq c$.

Proof. Omitting the word "elementary", the equivalence follows from the theorem of consistence applied to $Q = F, C = \{x / x \leq c\}$ in the linear space $E$.

In fact, $Q \cap C \neq \emptyset$, i.e. there is $f \in F, f \leq c$, is equivalent to
assert that for any \( g \in G = Q_\lambda \) there is \( x \in E, x \leq c \) such that \( g.x \geq 0 \). This implies \( g^+.c \geq 0 \), since from \( (g^+-g^-)x \geq 0 \), \( c \geq x \), it follows \( g^+.c \geq g^+.x \geq g^-.x \), and then \( 2g^+.c \geq (g^++g^-)x = |g|x = 0 \) (\(|g| \) is symmetric).

And conversely, if \( g^+.c \geq 0 \), \( g \in G \), then defining, for a given \( g \in G \), \( x \) by \( x_{ij} = c_{ij}', -c_{ij}', -1/2(c_{ij}-c_{ji}) \), according to \( g_{ij} > 0 \), \( < 0 \) or \( = 0 \), we have \( g.x = 2g^+.c \geq 0 \).

Finally, if \( g^+.c \geq 0 \) for elementary functions \( g \in G \), the same holds for any \( g \in G \), since from the proposition we can write \( g^+ = \sum_a g_a^+ \), for elementary functions \( g_a \) of \( G \).

REFERENCES


Recibido en marzo de 1974