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ON THE PARTITIONS OF AN INTEGER

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As is well known (see for example [1], [2]) the following formula (I) gives by recurrence the number of "partitions" - the "partages" of French authors - of a positive integer n; i.e., the number of non decreasing sequences of positive integers whose sum is n.

Formula (I) is

$$\pi_n = \sum_{i\geq 1} (-1)^{i-1} \left(\pi_{n-\frac{3i^2-i}{2}} + \pi_{n-\frac{3i^2+i}{2}}\right) \tag{I}$$

where we take $\pi_n = 0$ for $n < 0$, and $\pi_0 = 1$.

We present here a direct calculation - direct, in the sense that no generating function or Euler identities are used -. Furthermore, for the proof we introduce a general lemma, which seems to merit some attention in itself.

Let $\psi_{j,n}$ denote the number of such sequences whose first element is j ($j \geqslant 1$). Then, $\pi_{n-1} = \psi_{1,n}$ for $n \geqslant 2$; and $\pi_n = \sum_{j\geq 1} \psi_{j,n}$ for $n \geqslant 1$. Clearly, $\psi_{j,n} = 1$ whenever $j = n$ or $[n/3] < j \leqslant [n/2]$ and $\psi_{j,n} = 0$ when $n \neq j > [n/2]$ ([x] denotes "integral part of x").
The following array gives the non zero values of $\psi_{j,n}$ and π_n for $1 \leqslant n \leqslant 14$.

ψ j \ n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	2	3	5	7	11	15	22	30	42	56	77	101
2	.	1	.	1	1	2	2	4	4	7	8	12	14	21
3	.	.	1	.	.	1	1	1	2	2	3	4	5	6
4	.	.	.	1	.	.	.	1	1	1	1	2	2	3
5	.	.	.	.	1	.	.	.	.	1	1	1	1	1
6	.	.	.	.	.	1	.	.	.	.	.	1	1	1
7	.	.	.	.	.	.	1	.	.	.	.	.	.	1
π_n	1	2	3	5	7	11	15	22	30	42	56	77	101	135

For convenience, we extend the function ψ to all $n \in Z$, setting $\psi_{j,n} = 0$ whenever $j \geqslant 1$ and $n \leqslant 0$. We then have the equalities:

A) $\pi_{n-1} = \psi_{1,n}$

B) $\pi_n = \sum_{j\geq 1} \psi_{j,n}$ for $n \neq 0$

C) $\psi_{q,p} = \psi_{q-1,p-1} - \psi_{q-1,p-q}$

We now prove C. It is clear, when $j \neq p$, $\psi_{j,p} = \sum_{h\geq j} \psi_{h,p-j}$.

Then, for $p \neq q$ we have:

$$\psi_{q-1,p-1} = \sum_{h\geq q-1} \psi_{h,p-q} = \psi_{q-1,p-q} + \sum_{h\geq q} \psi_{h,p-q} = \psi_{q-1,p-q} + \psi_{q-p}$$

and for $p = q$, $\psi_{q,p} = \psi_{q-1,p-1} = 1$ and $\psi_{q-1,p-q} = 0$.

In passing, we note that Property C and the values $\psi_{j,n} = 0$ for $j \geqslant 1$, $n \leqslant 0$, and $\psi_{1,1} = \psi_{j,2j} = 1$ for all $j \geqslant 1$, determine uniquely the function ψ.

The following formula (II) is equivalent to (I), and will provide us with our basic approach.

$$\pi_n = 2\,\pi_{n-1} - \pi_{n-3} + \sum_{i\geq 2} (-1)^{i-1} \left(\Delta_{n-\frac{3i^2-i}{2}} + \Delta_{n-\frac{3i^2+i}{2}}\right) \quad \text{(II)}$$

where $\Delta_s = \pi_s - \pi_{s-1}$, $\pi_n = 0$ for $n < 0$ and $\pi_0 = \pi_1 = 1$.

Clearly, evaluating $\pi_n - \pi_{n-1}$ from (I) provides (II). On the other hand, evaluating with the aide of (II) the sum of the values $\pi_n, \pi_{n-1}, \pi_{n-2}, \ldots$ we have (I).

Let us see now that the evaluation of π_n ($n\neq 0$) - i.e., the sum of the elements of the n-th column of the array ψ - can be reduced to the evaluation of a difference: substract the sum of the values of ψ on the set $L = \{(j,n-j-1)\ ;\ j \geqslant 1\}$ from twice the sum of the elements of the (n-1)-th column of the array ψ.

Specifically, for $n\neq 0$ and from A), B) and C) we get:

$$\pi_n = \sum_{j\geq 1} \psi_{j,n} = \pi_{n-1} + \sum_{j\geq 1} (\psi_{j,n-1} - \psi_{j,n-j-1})$$

Therefore, using again A), B) and C) we have:

$$\pi_n = 0 \text{ for } n < 0, \quad \pi_0 = \pi_1 = 1, \quad \text{and}$$
$$\pi_n = 2\,\pi_{n-1} - \pi_{n-3} - (\pi_{n-5} - \pi_{n-6}) - \sum_{j\geq 3} \psi_{j,n-j-1} \quad \text{for } n \geqslant 2 \tag{1}$$

The use of Property D, see below, reduces the evaluation of $\sum_{j\geq 3} \psi_{j,n-j-1}$ to the consideration of the elements of the array for another set T, which will be determined implicitly by the resulting equations.

First, it will be convenient to visualize in the following array the sets L and T, whose elements are represented by means of "+" and "o" respectively.

...	n-13	n-12	n-11	n-10	n-9	n-8	n-7	n-6	n-5	n-4	n-3	n-2	n-1	n	
	o	o	o	o	o	o	o	o				+			j=1
	o		o		o		o				+				2
	o		o		o					+					3
	o		o						+						4
	o							+							5
							+								6
															⋮

The above mentioned Property D is the following:

$$\text{D)}\quad \psi_{q,p} = \psi_{1,p-(q-1)} - \sum_{1\leq h\leq q-1} \psi_{h,p-q}$$

For the proof, it suffices to reiterate (q-1) times Property C.

From D, we obtain:

$$\sum_{j\geq 3} \psi_{j,n-j-1} = \sum_{j\geq 3} (\psi_{1,n-2j} - \psi_{1,n-2j-1}) - \sum_{h\geq 2}\ \sum_{j\geq h+1} \psi_{h,n-2j-1} \tag{2}$$

For convenience, we regroup the terms as follows:

$$\begin{aligned} \varphi_{1,m} &= \sum_{s\geq 0} (\psi_{1,m-2s} - \psi_{1,m-2s-1}) \\ \varphi_{h,m} &= \sum_{s\geq 0} \psi_{h,m-2s} \quad \text{for } h \geqslant 2 \end{aligned} \tag{3}$$

From C, we obtain $\varphi_{2,m} = \varphi_{1,m-1}$, and more generally, from D

$$\varphi_{r,m} = \varphi_{1,m-r+1} - \sum_{2\leq h\leq r-1} \varphi_{h,m-r} \quad \text{for } r \geqslant 2 \tag{4}$$

Also, from A), C), and (3) we get:

$$\varphi_{1,m-i+1} - \varphi_{2,m-i} = \pi_{m-i} - \pi_{m-i-1} = \Delta_{m-i} \tag{5}$$

The last equality and the following proposition will permit us to express $\varphi_{r,m}$, and thus $\sum_{j\geq 3} \psi_{j,n-j-1}$, in terms of Δ_k $(k \leqslant n-7)$.

PROPOSITION. *For* $r \geqslant 3$, *we have:*

$$\varphi_{r,m} = \Delta_{m-r} + \sum_{w\in W_r} (-1)^k \, \Delta_{m-r-(w_1+w_2+w_3+\ldots+w_k)}$$

where W_r *is the set of sequences of positive integers* w_i $(1 \leqslant i \leqslant k)$ *such that* $w_i > w_{i+1}$, $r-1 \geqslant w_1$, $w_k \geqslant 3$.

Proof. For $r=3$, the set W_r is empty, and thus, from (4),(5)

$$\varphi_{3,m} = \varphi_{1,m-2} - \varphi_{2,m-3} = \Delta_{m-3}$$

For $r > 3$, the use of (4),(5) and an inductive reasoning gives

$$\varphi_{r,m} = \varphi_{1,m-r+1} - \varphi_{2,m-r} - \sum_{3\leq h\leq r-1} \varphi_{h,m-r} =$$

$$= \Delta_{m-r} - \sum_{3\leq h\leq r-1} (\Delta_{m-r-h} + \sum_{u\in U_h} (-1)^t \Delta_{m-r-h-(u_1+u_2+\ldots+u_t)})$$

where, for each h, we have $h-1 \geqslant u_1 > u_2 > u_3 > \ldots > u_t \geqslant 3$.

Setting $w_1 = h$ and $w_i = u_{i-1}$ for $2 \leqslant i \leqslant k = t+1$, we can write:

$$\varphi_{r,m} = \Delta_{m-r} + \sum_{w\in W_r} (-1)^k \, \Delta_{m-r-(w_1+w_2+\ldots+w_k)}$$

where the w-sequences satisfy the required conditions.

Having obtained this, we return to (2). If in this formula we substitute j by (3+s) in the indexes (1,n-2j), (1,n-2j-1) and j by (h+1+s) in the indexes (h,n-2j-1) for $h \geqslant 2$, we obtain:

$$\sum_{j\geq 3} \psi_{j,n-j-1} = \sum_{s\geq 0} (\psi_{1,n-6-2s} - \psi_{1,n-7-2s}) - \sum_{h\geq 2} \sum_{s\geq 0} \psi_{h,n-2h-3-2s}$$

Finally, using (3),(5) and the proposition above:

$$\sum_{j\geq 3} \psi_{j,n-j-1} = \Delta_{n-7} - \sum_{h\geq 3} (\Delta_{n-3-3h} + \sum_{w\in W_h} (-1)^k \Delta_{n-3-3h-(w_1+w_2+\ldots+w_k)})$$

For h=3, the set W_h is empty, then:

$$\sum_{j\geq 3} \psi_{j,n-j-1} = \Delta_{n-7} - \Delta_{n-12} - S \qquad (6)$$

where

$$S = \sum_{h\geq 4} (\Delta_{n-3-3h} + \sum_{w\varepsilon W_h} (-1)^k \Delta_{n-3-3h-(w_1+w_2+\ldots+w_k)}) \tag{7}$$

with $h-1 \geqslant w_1 > w_2 > w_3 > \ldots > w_k \geqslant 3$ for any $h \geqslant 4$.

The only remaining task is now to evaluate S. This task is simplified by using the following Lemma. Although we need it only for $a=3$, $b=1$, we will present it for any $a > 0$, $b \geqslant 0$.

LEMMA. *For given arbitrary integers* $a > 0$, $b \geqslant 0$, *and every integer* $s \geqslant a$, *let* W_s *denote the set of subsets of the set* $\{a,a+1,\ldots,s\}$. *Then, if* r *is any function for which there is some* $\underline{x}$ *such that* $x > \underline{x}$ *implies* $r(x) = 0$, *we will have:*

$$R = \sum_{s\geq a} \sum_{w\varepsilon W_s} (-1)^{|w|} r(a(s+b) + \sum_{w_i\varepsilon w} w_i) =$$

$$= r(a^2+ab) + \sum_{k\geq 1} (-1)^k\{r(\tfrac{1}{2}(3(k+a)^2 - (k+a) + a(2b-a-1))) + \tag{8}$$

$$+ r(\tfrac{1}{2}(3(k+a)^2 + (k+a) + a(2b-a-1)))\}$$

Proof. The sums in (8) are finite, as follows easily from $a > 0$ and the above conditions on the support of r.

For fixed s, we shall proceed to associate to each $w \in W_s$ - with one exception - an element $w' \in W_j$, where j satisfies either $j = s+1$ or $j = s-1$ and in such a way that $a(s+b) + \sum_{w_i\varepsilon w} w_i = a(j+b) + \sum_{w'_i\varepsilon w'} w'_i$, with cardinals $|w|$ and $|w'|$ differing in one.

Therefore, as $r(a(s+b)+\sum w_i)$ is equal to $r(a(j+b)+\sum w'_i)$, these terms can be omitted in the evaluation of R, since they appear with opposite signs. Hence, for fixed s, the sum

$\sum_{w\varepsilon W_s} (-1)^{|w|} r(a(s+b) + \sum_{w_i\varepsilon w} w_i)$ will be reduced to a single term.

To accomplish this, we start by identifying each $w \in W_s$, $w \neq \emptyset$, with the decreasing sequence of its elements. That is to say, if k is the number of elements in w ($w \in W_s$), we will have

$w = (w_1,w_2,w_3,\ldots,w_k)$, with $s \geqslant w_1 > w_2 > w_3 > \ldots > w_k \geqslant a$.

Denoting by:

$$W_s^1 = \{w \;/\; w = (w_1, w_2, \ldots , w_k) \;,\; w_k = a\}$$

$W_s^2 = \{w \ / \ w = (w_1, w_2, \ldots, w_k) \ , \ w_k > a \ , \ w_1 < s\}$

$W_s^3 = \{w \ / \ w = (w_1, w_2, \ldots, w_k) \ , \ w_k > a \ , \ w_1 = s\}$

we get: $W_s = W_s^1 \cup W_s^2 \cup W_s^3 \cup \{\emptyset_s\}$, where $\emptyset_s$ will denote the empty set as element of W_s.

Clearly, W_a^2, W_a^2, and W_{a+1}^2 are empty.

We now define the rule of association and, for the sake of clarity, we divide the problem in two parts.

PART I. For fixed $s \geqslant a$, let $\alpha_s : W_s^1 \longrightarrow W_{s+1}^2 \cup \{\emptyset_{s+1}\}$ be the mapping defined by:

$\alpha_s(w_1, w_2, \ldots, w_{k-1}, a) = (w_1, w_2, \ldots, w_{k-1})$ when $k > 1$, $\alpha_s(a) = \emptyset_{s+1}$

Obviously, α_s is onto and one-to-one.

We will then have:

$$R = r(a(a+b) + \sum_{s \geq a+1} \ \sum_{w \varepsilon W_s^3} (-1)^{|w|} \ r(a(s+b) + \sum_{w_i \varepsilon w} w_i)$$

because for $s \geqslant a$ the terms in W_s^1 compensate with those in $W_{s+1}^2 \cup \{\emptyset_{s+1}\}$, and for $s \geqslant a+1$ the same occurs with $W_s^2 \cup \{\emptyset_s\}$ and W_{s-1}^1. Moreover, we see that in R appear, now, only the terms corresponding to $\emptyset_a$ and W_s^3 $(s > a+1)$.

PART II. First, let f, $1 \leqslant f \leqslant k$, be the greatest integer such that for $w \in W_s^3$, $w = (w_1, w_2, \ldots, w_k)$ we have $w_i = s - (i-1)$ whenever $1 \leqslant i \leqslant f$.

For convenience, we set: $w_k = a+g$ and, since it will be needed later, we single out the case f=k. Thus, the additional assumption g=f clearly implies $s = 2k + a - 1$ and reciprocally; while $g = f+1$ implies $s = 2k + a$ and reciprocally. That is to say, f=k=g if and only if (s-a) is odd, while $f = k = g-1$ if and only if (s-a) is even. Thus, for both we have $k = [\frac{s-a+1}{2}]$, where [ ] denotes the function "integral part of".

Next, we consider in W_s^3 *two* subsets: W_s^4, whose elements are the sequences such that $w_k = a + g \leqslant a + f$ but excluding that one with

$f = k = g$ (if any); and W_s^5, where $w_k = a + g > a + f$, but excluding that one with $f = k = g-1$ (if any).

Clearly, W_{a+1}^4, W_{a+1}^5, and W_{a+2}^5 are empty.

The above considerations enable us to complete the rule of association by defining the following mapping:

For fixed $s \geqslant a+1$, let $\beta_s: W_s^4 \longrightarrow W_{s+1}^5$ be given by:

$$\beta_s\,(w_1, w_2, \ldots, w_k) = (w_1', w_2', \ldots, w_{k-1}') \qquad \text{where}$$

$$w_i' = \begin{cases} w_i + 1 & \text{when } 1 \leqslant i \leqslant g \\ w_i & \text{when } g+1 \leqslant i \leqslant k-1 \end{cases}$$

Clearly, β_s is one to one. Moreover, it is onto, since for $w' \in W_{s+1}^5$, $w' = (w_1', w_2', \ldots, w_k')$, the sequence $w = (w_1, w_2, \ldots, w_k, w_{k+1})$ that has

$$w_i = \begin{cases} w_i' - 1 & \text{if } \quad 1 \leqslant i \leqslant f \\ w_i' & \text{if } f+1 \leqslant i \leqslant k \\ a + f & \text{if } \quad i = k+1 \end{cases}$$

gives $\beta_s(w) = w'$.

By means of β_s, we see that for $s \geqslant a+1$ the terms corresponding to W_s^4 are equal to the terms corresponding to W_{s+1}^5, and for $s \geqslant a+2$ the same thing happens with the terms of W_s^5 and W_{s-1}^4. Therefore, in the evaluation of R they compensate and hence can be omitted. Thus, for each W_s^3, only one element will remain for evaluating R. Using this, we have the following reduced expression:

$$R = r(a^2 + ab) + \sum_{s \geq a+1} (-1)^k\, r(a(s+b) + \sum_{i=1}^{k} w_i) \tag{9}$$

where $\quad w_i = s - (i-1)$, $1 \leqslant i \leqslant k$, and $k = [\frac{s-a+1}{2}]$.

According to these restrictions, we have, for $s = 2k + a - 1$

$$a(s+b) + \sum_{i=1}^{k} w_i = \frac{1}{2}\,(3(k+a)^2 - (k+a) + a(2b-a-1))$$

and for $s = 2k + a$

$$a(s+b) + \sum_{i=1}^{k} w_i = \frac{1}{2}\,(3(k+a)^2 + (k+a) + a(2b-a-1))$$

Replacing these expressions in (9), we obtain the asserted equality (8).

Returning to our problem, we substitute in the Lemma the values $a=3$, $s=h-1$, $b=1$, $r(j) = \Delta_{n-3-j}$. Then, the resulting value of R is precisely the value of S given by (7).

If moreover, we set $k+3 = i$, we obtain:

$$S = \Delta_{n-15} + \sum_{i\geq 4} (-1)^{i-1} \{\Delta_{n-\frac{3i^2-i}{2}} + \Delta_{n-\frac{3i^2+i}{2}}\} \tag{10}$$

Finally, by replacing in (1) the results from (6) and (10), we obtain (II), as asserted.

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