As is well known (see for example [11], [2]) the following formula (I) gives by recurrence the number of "partitions" - the "partages" of French authors - of a positive integer \( n \); i.e., the number of non decreasing sequences of positive integers whose sum is \( n \).

Formula (I) is

\[
\pi_n = \sum_{i=1}^{n} (-1)^{i-1} \left( \pi_{\frac{n-i}{2}} + \pi_{\frac{n-i+1}{2}} \right)
\]

where we take \( \pi_n = 0 \) for \( n < 0 \), and \( \pi_0 = 1 \).

We present here a direct calculation - direct, in the sense that no generating function or Euler identities are used - Furthermore, for the proof we introduce a general lemma, which seems to merit some attention in itself.

Let \( \psi_{j,n} \) denote the number of such sequences whose first element is \( j \) (\( j \geq 1 \)). Then, \( \pi_{n-1} = \psi_{1,n} \) for \( n \geq 2 \); and \( \pi_n = \sum \psi_{j,n} \) for \( n \geq 1 \). Clearly, \( \psi_{j,n} = 1 \) whenever \( j = n \) or \( \lfloor n/3 \rfloor \leq j \leq \lfloor n/2 \rfloor \) and \( \psi_{j,n} = 0 \) when \( n \neq j > \lfloor n/2 \rfloor \) (\( \lfloor x \rfloor \) denotes "integral part of \( x \")).

The following array gives the non zero values of \( \psi_{j,n} \) and \( \pi_n \) for \( 1 \leq n \leq 14 \).

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<tr>
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| \( \pi_n \) | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 42 | 56 | 77 | 101 | 135 |
For convenience, we extend the function $\psi$ to all $n \in \mathbb{Z}$, setting
$\psi_{j,n} = 0$ whenever $j \geq 1$ and $n \leq 0$. We then have the equalities:

A) $\pi_{n-1} = \psi_{1,n}$
B) $\pi_n = \sum_{j \geq 1} \psi_{j,n}$ for $n \neq 0$
C) $\psi_{q,p} = \psi_{q-1,p-1} - \psi_{q-1,p-q}$

We now prove C. It is clear, when $j \neq p$, $\psi_{j,p} = \sum_{h \geq j} \psi_{h,p-j}$.

Then, for $p \neq q$ we have:

$$\psi_{q-1,p-1} = \sum_{h \geq q-1} \psi_{h,p-q} = \psi_{q-1,p-q} + \sum_{h \geq q} \psi_{h,p-q} = \psi_{q-1,p-q} + \psi_{q-p}$$

and for $p = q$, $\psi_{q,p} = \psi_{q-1,p-1} = 1$ and $\psi_{q-1,p-q} = 0$.

In passing, we note that Property C and the values $\psi_{j,n} = 0$ for $j \geq 1$, $n < 0$, and $\psi_{j,1} = \psi_{j,2j-1} = 1$ for all $j \geq 1$, determine uniquely the function $\psi$.

The following formula (II) is equivalent to (I), and will provide us with our basic approach.

$$\pi_n = 2\pi_{n-1} - \pi_{n-3} + \sum_{i \geq 2} (-1)^{i-1} \left( \frac{\Delta_s}{n^2 - 1} + \frac{\Delta_{s+1}}{n^2 - 1} \right)$$

where $\Delta_s = \pi_s - \pi_{s-1}$, $\pi_n = 0$ for $n < 0$ and $\pi_0 = \pi_1 = 1$.

Clearly, evaluating $\pi_n - \pi_{n-1}$ from (I) provides (II). On the other hand, evaluating with the aide of (II) the sum of the values $\pi_n, \pi_{n-1}, \pi_{n-2}, \ldots$ we have (I).

Let us see now that the evaluation of $\pi_n$ ($n \neq 0$) - i.e., the sum of the elements of the $n$-th column of the array $\psi$ - can be reduced to the evaluation of a difference: subtract the sum of the values of $\psi$ on the set $L = \{(j,n-j-1) ; j \geq 1\}$ from twice the sum of the elements of the $(n-1)$-th column of the array $\psi$.

Specifically, for $n \neq 0$ and from A), B) and C) we get:

$$\pi_n = \sum_{j \geq 1} \psi_{j,n} = \pi_{n-1} + \sum_{j \geq 1} (\psi_{j,n-1} - \psi_{j,n-j-1})$$

Therefore, using again A), B) and C) we have:
\[ \pi_n = 0 \text{ for } n < 0, \quad \pi_0 = \pi_1 = 1, \text{ and} \]
\[ \pi_n = 2 \pi_{n-1} - \pi_{n-3} - (\pi_n - \pi_n - 6) - \sum_{j \geq 3} \psi_{j,n-j-1} \text{ for } n \geq 2 \]  

The use of Property D, see below, reduces the evaluation of \[ \sum_{j \geq 3} \psi_{j,n-j-1} \] to the consideration of the elements of the array for another set \( T \), which will be determined implicitly by the resulting equations.

First, it will be convenient to visualize in the following array the sets \( L \) and \( T \), whose elements are represented by means of "+" and "0" respectively.

\[
\begin{array}{cccccccccccccccc}
\cdots & n-13 & n-12 & n-11 & n-10 & n-9 & n-8 & n-7 & n-6 & n-5 & n-4 & n-3 & n-2 & n-1 & n \\
& & & & & & & & & + & & & & \\
& & & & & & & & & + & & & & \\
& & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & \\
\end{array}
\]

The above mentioned Property D is the following:

\[
D) \psi_{q,p} = \psi_{1,p-1} - \sum_{l \geq q-1} \psi_{l,p} \psi_{h,p-q}
\]

For the proof, it suffices to reiterate \((q-1)\) times Property C.

From D, we obtain:

\[
\sum_{j \geq 3} \psi_{j,n-j-1} = \sum_{j \geq 3} (\psi_{1,n-2j} - \psi_{1,n-2j-1}) - \sum_{h \geq 2} \sum_{j \geq h+1} \psi_{h,n-2j-1}
\]  

For convenience, we regroup the terms as follows:

\[
\psi_{1,m} = \sum_{s \geq 0} (\psi_{1,m-2s} - \psi_{1,m-2s-1})
\]

\[
\psi_{h,m} = \sum_{s \geq 0} \psi_{h,m-2s} \quad \text{for } h \geq 2
\]

From C, we obtain \( \varphi_{2,m} = \varphi_{1,m-1} \), and more generally, from D

\[
\varphi_{r,m} = \varphi_{1,m-r+1} - \sum_{2 \leq h \leq r-1} \varphi_{h,m-r} \quad \text{for } r \geq 2
\]

Also, from A), C), and (3) we get:

\[
\varphi_{1,m-1} - \varphi_{2,m-1} = \pi_m - \pi_{m-1} - \Delta_m
\]
The last equality and the following proposition will permit us to express $\varphi_{r,m}$ and thus $\sum_{j>3} \psi_{j,n-j-1}$, in terms of $\Delta_k$ ($k \leq n-7$).

**PROPOSITION.** For $r \geq 3$, we have:

$$\varphi_{r,m} = \Delta_{m-r} + \sum_{w \in \mathcal{W}_r} (-1)^k \Delta_{m-r-(w_1+w_2+w_3+\ldots+w_k)}$$

where $\mathcal{W}_r$ is the set of sequences of positive integers $w_i$ ($1 \leq i \leq k$) such that $w_i > w_{i+1}$, $r-1 > w_1$, $w_k \geq 3$.

**Proof.** For $r=3$, the set $\mathcal{W}_r$ is empty, and thus, from (4), (5)

$$\varphi_{3,m} = \varphi_{1,m-2} - \varphi_{2,m-3} = \Delta_{m-3}$$

For $r > 3$, the use of (4), (5) and an inductive reasoning gives

$$\varphi_{r,m} = \varphi_{1,m-r+1} - \varphi_{2,m-r} - \sum_{3 \leq h \leq r-1} \varphi_{h,m-r} =$$

$$\Delta_{m-r} - \sum_{3 \leq h \leq r-1} \Delta_{m-r-h} + \sum_{w \in \mathcal{W}_r} (-1)^k \Delta_{m-r-(w_1+w_2+\ldots+w_k)}$$

where, for each $h$, we have $h-1 > w_1 > w_2 > w_3 > \ldots > w_h \geq 3$.

Setting $w_1 = h$ and $w_i = u_{i-1}$ for $2 \leq i \leq k = t+1$, we can write:

$$\varphi_{r,m} = \Delta_{m-r} - \sum_{w \in \mathcal{W}_r} (-1)^k \Delta_{m-r-(w_1+w_2+\ldots+w_k)}$$

where the $w$-sequences satisfy the required conditions.

Having obtained this, we return to (2). If in this formula we substitute $j$ by $(3+s)$ in the indexes $(1,n-2j)$, $(1,n-2j-1)$ and $j$ by $(h+1+s)$ in the indexes $(h,n-2j-1)$ for $h \geq 2$, we obtain:

$$\sum_{j \geq 3} \psi_{j,n-j-1} = \sum_{s \geq 0} (\psi_{1,n-6-2s} - \psi_{1,n-7-2s}) - \sum_{h \geq 2} \sum_{s \geq 0} \psi_{h,n-2h-3-2s}$$

Finally, using (3), (5) and the proposition above:

$$\sum_{j \geq 3} \psi_{j,n-j-1} = \Delta_{n-7} - \sum_{h \geq 3} (\Delta_{n-3-3h} + \sum_{w \in \mathcal{W}_h} (-1)^k \Delta_{n-3-3h-(w_1+w_2+\ldots+w_k)})$$

For $h=3$, the set $\mathcal{W}_h$ is empty, then:

$$\sum_{j \geq 3} \psi_{j,n-j-1} = \Delta_{n-7} - \Delta_{n-12} - S$$

where
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\[ S = \sum_{h=4}^{\infty} \left( \Delta_{n-3-3h} + \sum_{w \in W_h} (-1)^k \Delta_{n-3-3h-(w_1+w_2+\ldots+w_k)} \right) \] (7)

with \( h-1 > w_1 > w_2 > w_3 > \ldots > w_k \geq 3 \) for any \( h \geq 4 \).

The only remaining task is now to evaluate \( S \). This task is simplified by using the following Lemma. Although we need it only for \( a=3, b=1 \), we will present it for any \( a > 0, b \geq 0 \).

**Lemma.** For given arbitrary integers \( a > 0, b \geq 0 \), and every integer \( s \geq a \), let \( W_s \) denote the set of subsets of the set \( \{a,a+1,\ldots,s\} \). Then, if \( r \) is any function for which there is some \( x \) such that \( x > x \) implies \( r(x) = 0 \), we will have:

\[ R = \sum_{s \geq a} \sum_{W \in W_s} (-1)^{|W|} r(a(s+b) + \sum_{w \in W} w_i) = r(a^2+ab) + \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{2}(3(k+a)^2 - (k+a) + a(2b-a-1)) \right) \] (8)

\[ + r \left( \frac{1}{2}(3(k+a)^2 + (k+a) + a(2b-a-1)) \right) \]

Proof. The sums in (8) are finite, as follows easily from \( a > 0 \) and the above conditions on the support of \( r \).

For fixed \( s \), we shall proceed to associate to each \( w \in W_s \) - with one exception - an element \( w' \in W_j \), where \( j \) satisfies either \( j = s+1 \) or \( j = s-1 \) and in such a way that \( a(s+b) + \sum_{w \in W} w_i = a(j+b) + \sum_{w' \in W'} w_i' \), with cardinals \(|w|\) and \(|w'|\) differing in one.

Therefore, as \( r(a(s+b)+\sum_{w} w_i) \) is equal to \( r(a(j+b)+\sum_{w'} w_i') \), these terms can be omitted in the evaluation of \( R \), since they appear with opposite signs. Hence, for fixed \( s \), the sum

\[ \sum_{w \in W_s} (-1)^{|w|} r(a(s+b) + \sum_{w \in W} w_i) \] will be reduced to a single term.

To accomplish this, we start by identifying each \( w \in W_s, w \neq \emptyset \), with the decreasing sequence of its elements. That is to say, if \( k \) is the number of elements in \( w \) (\( w \in W_s \)), we will have \( w = (w_1,w_2,w_3,\ldots,w_k) \), with \( s > w_1 > w_2 > w_3 > \ldots > w_k \geq a \).

Denoting by:

\[ W^1_s = \{ w / w = (w_1, w_2, \ldots, w_k), w_k = a \} \]
\[
W^2_s = \{ w / w = (w_1, w_2, \ldots, w_k), w_k > a, w_1 < s \} \\
W^3_s = \{ w / w = (w_1, w_2, \ldots, w_k), w_k > a, w_1 = s \}
\]
we get: \( W_s = W_s^1 \cup W_s^2 \cup W_s^3 \cup \{ \emptyset_s \} \), where \( \emptyset_s \) will denote the empty set as element of \( W_s \).

Clearly, \( W^2_a \), \( W^2_a \), and \( W^2_{a+1} \) are empty.

We now define the rule of association and, for the sake of clarity, we divide the problem in two parts.

**PART I.** For fixed \( s \geq a \), let \( \alpha_s : W^1_s \to W^2_{s+1} \cup \{ \emptyset_{s+1} \} \) be the mapping defined by:
\[
\alpha_s(w_1, w_2, \ldots, w_{k-1}, a) = (w_1, w_2, \ldots, w_{k-1}) \text{ when } k > 1, \quad \alpha_s(a) = \emptyset_{s+1}
\]
Obviously, \( \alpha_s \) is onto and one-to-one.

We will then have:
\[
R = r(a(a+b) + \sum_{s \geq a+1}^{s_1} w_i)^3 \sum_{w \in W_s^1} (-1)^{w_1} r(a(s+b) + \sum_{i \in w} w_i)
\]
because for \( s \geq a \) the terms in \( W_s^1 \) compensate with those in \( W_{s+1}^2 \cup \{ \emptyset_{s+1} \} \), and for \( s \geq a+1 \) the same occurs with \( W_s^2 \cup \{ \emptyset_s \} \).

Moreover, that in \( R \) appear, only the terms corresponding to \( \emptyset_a \) and \( W_s^3 \) (\( s > a+1 \)).

**PART II.** First, let \( f, 1 \leq f \leq k \), be the greatest integer such that for \( w \in W_s^3 \), \( w = (w_1, w_2, \ldots, w_k) \) we have \( w_i = s - (i-1) \) whenever \( 1 \leq i \leq f \).

For convenience, we set: \( w_k = a + g \) and, since it will be needed later, we single out the case \( f = k \). Thus, the additional assumption \( g > f \) clearly implies \( s = 2k + a - 1 \) and reciprocally; while \( g = f+1 \) implies \( s = 2k + a \) and reciprocally. That is to say, \( f = k = g \) if and only if \( s-a \) is odd, while \( f = k = g-1 \) if and only if \( s-a \) is even. Thus, for both we have \( k = \left[ \frac{s-a+1}{2} \right] \), where \( [ \ ] \) denotes the function "integral part of".

Next, we consider in \( W_s^2 \) two subsets: \( W_s^4 \), whose elements are the sequences such that \( w_k = a + g \leq a + f \) but excluding that one with
\[ f = k = g \text{ (if any)}; \text{ and } \mathcal{W}_{s}^{5}, \text{ where } w_{k} = a + g > a + f, \text{ but excluding that one with } f = k = g - 1 \text{ (if any)}. \]

Clearly, \[ \mathcal{W}_{a+1}^{4}, \mathcal{W}_{a+1}^{5}, \text{ and } \mathcal{W}_{a+2}^{5} \text{ are empty}. \]

The above considerations enable us to complete the rule of association by defining the following mapping:

For fixed \( s \geq a+1 \), let \( \beta_{s} : \mathcal{W}_{s}^{4} \rightarrow \mathcal{W}_{s+1}^{5} \) be given by:

\[
\beta_{s}(w_{1}, w_{2}, \ldots, w_{k}) = (w'_{1}, w'_{2}, \ldots, w'_{k}) \quad \text{where}
\]

\[
w'_{i} = \begin{cases} w_{i} + 1 & \text{when } 1 \leq i \leq g \\ w_{i} & \text{when } g + 1 \leq i \leq k - 1 \end{cases}
\]

Clearly, \( \beta_{s} \) is one to one. Moreover, it is onto, since for \( w' \in \mathcal{W}_{s+1}^{5} \), \( w' = (w'_{1}, w'_{2}, \ldots, w'_{k}) \), the sequence \( w = (w_{1}, w_{2}, \ldots, w_{k}, w_{k+1}) \) that has

\[
w_{i} = \begin{cases} w_{i} - 1 & \text{if } 1 \leq i \leq f \\ w_{i} & \text{if } f + 1 \leq i \leq k \\ a + f & \text{if } i = k + 1 \end{cases}
\]

gives \( \beta_{s}(w) = w' \).

By means of \( \beta_{s} \), we see that for \( s \geq a+1 \) the terms corresponding to \( \mathcal{W}_{s}^{4} \) are equal to the terms corresponding to \( \mathcal{W}_{s+1}^{5} \), and for \( s \geq a+2 \) the same thing happens with the terms of \( \mathcal{W}_{s}^{5} \) and \( \mathcal{W}_{s-1}^{5} \).

Therefore, in the evaluation of \( R \) they compensate and hence can be omitted. Thus, for each \( \mathcal{W}_{s}^{3} \), only one element will remain for evaluating \( R \). Using this, we have the following reduced expression:

\[
R = r(a^{2} + ab) + \sum_{s \geq a+1} (-1)^{k} r(a(s+b) + \sum_{i=1}^{k} w_{i})
\]

where \( w_{i} = s - (i-1), 1 \leq i \leq k \), and \( k = \lfloor \frac{s-a+1}{2} \rfloor \).

According to these restrictions, we have, for \( s = 2k + a - 1 \)

\[
a(s+b) + \sum_{i=1}^{k} w_{i} = \frac{1}{2}(3(k+a)^{2} - (k+a) + a(2b-a-1))
\]

and for \( s = 2k + a \)
\[ a(s+b) + \frac{k}{i=1} w_i = \frac{1}{2} (3(k+a)^2 + (k+a) + a(2b-a-1)) \]

Replacing these expressions in (9), we obtain the asserted equality (8).

Returning to our problem, we substitute in the Lemma the values \( a=3, s=h-1, b=1, r(j) = \Delta_{n-3-j} \). Then, the resulting value of \( R \) is precisely the value of \( S \) given by (7).

If moreover, we set \( k+3 = i \), we obtain:

\[ S = \Delta_{n-15} + \sum_{i=4}^{n} \frac{(-1)^{i-1}}{2} \left( \Delta_{n-3i-2-i} + \Delta_{n-3i^2+i} \right) \quad (10) \]

Finally, by replacing in (1) the results from (6) and (10), we obtain (11), as asserted.

REFERENCES


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