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MULTIPLIERS FOR (C,κ)-BOUNDED FOURIER EXPANSIONS IN WEIGHTED LOCALLY CONVEX SPACES AND APPROXIMATION

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Dedicated to Professor Alberto González Domínguez on the occasion of his seventieth birthday.

I. INTRODUCTION.

In [11] some first extensions of the multiplier theory as developed in Banach spaces in [5] and [17] were presented for locally convex spaces. In view of the applications these considerations were essentially restricted to order-preserving operators. In the mean time, however, we observed that some of the given and other examples are also valid in a non-order-preserving setting. In this general frame a multiplier theory for arbitrary multiplier operators has interesting new applications, in particular to weighted locally convex spaces. Motivated by these viewpoints we therefore continue our investigations in [11], this time for general multipliers in locally convex spaces.

In the applications we treat projective and inductive limits, essentially of weighted locally convex Hausdorff spaces. The Fourier series are defined via classical orthogonal systems such as the trigonometric system, Laguerre-, Hermite- or ultraspherical polynomials.

After giving some definitions and general results in Section II, first of all in Section III multipliers are defined. Then some classical inequalities of approximation theory are extended to locally convex spaces and the saturation problem for approximation processes of multiplier operators is treated. In Section IV we derive a multiplier criterion via the (C^{κ})-condition (4.3).

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Finally Section V gives some nontrivial new applications of this criterion in (weighted) locally convex spaces for Fourier expansions by trigonometric and Laguerre polynomials. By similar considerations further examples concerning Hermite and ultraspherical expansions could be worked out.

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II. PRELIMINARIES.

Let $\mathbf{Z}$, $\mathbf{N}$, $\mathbf{P}$ denote the set of all, of all non-negative and of all positive integers, respectively. Furthermore, let $\mathbf{R}$ and $\mathbf{R}^+$ be the set of all real and of all positive real numbers. In the following $(X, \{p_r\}_{r\in J})$, J being an arbitrary index set, will always denote a locally convex Hausdorff space whose topology T is generated by a family of filtrating seminorms, or to be short, by a system of seminorms $\{p_r\}_{r\in J}$. Let $[X]$ be the class of all continuous linear operators of X into itself. A family $\{T(\rho)\}_{\rho>0} \subset [X]$ is called an (equicontinuous) approximation process on $(X, \{p_r\})$, if for each $r \in J$ there exists $t \in J$ and a constant $M(r,t) > 0$ such that

$$p_r(T(\rho)f) \leq M(r,t)p_t(f) \qquad (f \in X,\ \rho > 0) \tag{2.1}$$

$$\lim_{\rho\to\infty} p_r(T(\rho)f - f) = 0 \qquad (f \in X) \tag{2.2}$$

Let $(X,\{p\})$ and $(Y,\{q\})$ be two locally convex spaces such that Y is continuously embedded in X. Let X be complete. With $\mathcal{K} := \prod_{q\in\{q\}} \mathbf{R}^+$, the completion of Y relative to X is defined by (cf.[3])

$$Y^{\sim X} := \bigcup_{R\in\mathcal{K}} \overline{S(R)}^X , \qquad S(R) := \bigcap_{q\in\{q\}} S_q(0;R_q) ,$$

where $\overline{S(R)}^X$ denotes the closure of $S(R)$ in the X-topology and $S_q(0;\varepsilon) := \{h \in Y;\ q(h) < \varepsilon\}$. Let $\{f_\beta\}_{\beta\in\mathbf{D}}$ be a net in Y with directed domain $\mathbf{D}$ and $N_f(Y)$ the class of all nets in Y which converge to f in X. Then for every $q \in \{q\}$

$$q(f) := \inf \{\sup_{\beta\varepsilon D} q(f_\beta);\ \{f_\beta\}_{\beta\varepsilon D} \in N_f^b\} \qquad (f \in Y^{\sim X}) \tag{2.3}$$

is a seminorm on $Y^{\sim X}$ with

$$N_f^b := \{\{f_\beta\}_{\beta\varepsilon D} \in N_f(Y);\ \{f_\beta\}_{\beta\varepsilon D} \text{ is bounded in } Y\} \qquad (f \in X).$$

The locally convex spaces X to be considered in the applications are representable as projective or inductive limits (cf. [16]) of locally convex spaces or even Banach spaces. In treating projective limits, we always examine the special case $X \subset X_r$, $r \in J$, with locally convex spaces X_r, and the linear mappings $u_r: X \longrightarrow X_r$ are the identity mappings. Furthermore, the system of seminorms $\{p_r\}_{r\varepsilon J}$ on X is usually given by a countable system of norms $\{a_k\}_{k\varepsilon P}$ which are in concordance . If the spaces X_k are complete for all $k \in P$, we obtain the class of the complete, countably normed spaces (Fréchet spaces) (cf. [7]) as a special class of the projective limit. In the same way in our examples of inductive limits the linear mappings $u_r: X_r \longrightarrow X$ are always the restrictions of the identity map from X to the locally convex subspaces $X_r \subset X$. The topology of the inductive limit is then the finest locally convex topology on X which induces on each X_r a coarser topology than the initial topology. Particularly $X = \bigcup_{m=0}^{\infty} X_m$, the inductive limit of a monotone increasing sequence $\{(X_m, T_m)\}_{m\varepsilon P}$ of locally convex spaces, is called the countable inductive limit of the spaces (X_m, T_m), $m \in P$, or sometimes a countable union space.

Let $X = \bigcup_{m=0}^{\infty} X_m$ be the countable inductive limit of a sequence of metrisable, locally convex spaces (X_m, T_m) and let Y be the strict inductive limit of locally convex spaces (Y^k, T^k) with the additional property that each Y^k is closed in Y^{k+1}. Then a family $\{T(\rho)\}_{\rho>0}: X \longrightarrow Y$ of linear operators is equicontinuous iff to each $m \in P$ there exists a $k = k(m) \in P$ such that $\{T(\rho)\}: X_m \longrightarrow Y^k$ is equicontinuous (cf. [16; p. 89], [1]).

Furthermore, each closed linear operator T from a countable inductive limit of B-complete locally convex Baire spaces $\{X_k\}_{k\varepsilon P}$ into itself is known to be continuous (cf. [1]). A corresponding version of the closed graph theorem holds for a linear closed operator T of a barreled, B-complete space X into itself, especially for complete countably normed spaces X (cf.[16; p. 126]).

III. GENERAL THEORY.

III.1 MULTIPLIERS.

Let X be a locally convex (Hausdorff) space whose topology T is generated by a system $\{p_r\}_{r\varepsilon J}$ of seminorms. Furthermore, let $\{P_k\}_{k\varepsilon \mathbf{P}} \subset [X]$ be a total sequence of mutually orthogonal projections on X, in short a system $\{P_k\}$, i.e., (i) mutually orthogonal: $P_j P_k = \delta_{jk}P_k$, δ_{jk} being Kronecker's symbol, and (ii) total: $P_k f=0$ for all $k \in \mathbf{P}$ implies $f=0$. Then to each $f \in X$ one may associate its unique Fourier series expansion

$$f \sim \sum_{k=0}^{\infty} P_k f \qquad (f \in X) \tag{3.1}$$

The sequence $\{P_k\}_{k\varepsilon \mathbf{P}}$ is said to be fundamental if the set Π of all polynomials, i.e., the set of all finite linear combinations $\sum_{k=0}^{n} f_k$ with $f_k \in P_k(X)$, is dense in $(X, \{p_r\}_{r\varepsilon J})$.
With ω the set of all sequences $\tau = \{\tau_k\}_{k\varepsilon \mathbf{P}}$ of scalars, $\tau \in \omega$ is called a multiplier for X (with respect to $\{P_k\}$) if for each $f \in X$ there exists an element $f^\tau \in X$ such that

$$P_k f^\tau = \tau_k P_k f \qquad (k \in \mathbf{P}) \tag{3.2}$$

Since $\{P_k\}$ is total, f^τ is uniquely determined by f. The class of all multipliers τ for X with respect to $\{P_k\}$ is denoted by $M = M(X; \{P_k\})$.

To each multiplier $\tau \in M$ there corresponds a closed linear multiplier operator $T^\tau: X \longrightarrow X$, defined by $T^\tau f = f^\tau$. (In general we don't distinguish between multipliers and the corresponding multiplier operators). The set $M_C = M_C(X; \{P_k\})$ of all $\tau \in M$ for which the operator T^τ is continuous on X, can be identified with a closed subspace of $[X]$, denoted by $[X]_{M_C}$. In general $M_C \subset M$, but if the closed graph theorem holds on X, then $M_C = M$. In this case, to each $r \in J$ there exists $t \in J$ and a constant $B(r,t) > 0$ such that

$$p_r(T^\tau f) \leqslant B(r,t)p_t(f) \qquad (f \in X) \tag{3.3}$$

and we set

$$\|T^\tau\|_{r,t} := \inf \{B(r,t);\ p_r(T^\tau f) \leqslant B(r,t;\tau)p_t(f),\ f \in X\} := \|\tau\|_{M,r,t} \tag{3.4}$$

If the seminorms $\{p_r\}_{r\varepsilon J}$ on X are norms as in the case of countably normed spaces, then $\|T^\tau\|_{r,t} = \|T^\tau\|_{[X^r,X^t]}$, where the Banach spaces X^t and X^r are the completions of the locally convex space X under the ~~norms~~ p_t and p_r, respectively.

For an arbitrary $\psi \in \omega$ we define

$$(3.5) \qquad X^\psi := \{f \in X;\text{ there exists an } f^\psi \in X \text{ with } \psi_k P_k f = P_k f^\psi \text{ for all } k \in \mathbf{P}\}$$

Evidently $X^\psi \subset X$, and the linear operator $B^\psi: X^\psi \longrightarrow X$, defined by $B^\psi f = f^\psi$ for $f \in X^\psi$, is closed for each $\psi \in \omega$. Furthermore, $P_k(X) \subset X^\psi$ for each $k \in \mathbf{P}$, so that B^ψ is densely defined if $\{P_k\}$ is fundamental on X. The operators B^ψ are called operators of multiplier-type.
It is easy to see

LEMMA (3.6). (*a*) *Under the system of seminorms* $\{p_r^\psi\}_{r\varepsilon J}$, *defined by*
$$p_r^\psi(f) := p_r(f) + p_r(B^\psi f) \qquad (r \in J,\ f \in X^\psi)$$
X^ψ *becomes a locally convex subspace of* X; *the system* $\{p_r^\psi\}_{r\varepsilon J}$ *is filtrating and separating.*

(*b*) *If* $(X, \{p_r\}_{r\varepsilon J})$ *is a complete locally convex space, then* $(X^\psi, \{p_r^\psi\}_{r\varepsilon J})$ *is complete.*

In contrast to the Banach space theory in arbitrary locally convex spaces there here exist unbounded multipliers τ corresponding to a continuous operator T^τ. A simple example is the differential operator $\mathbf{R} = -i(d/dx)$ with eigenvalues $\{\lambda_k\}_{k\varepsilon\mathbf{P}} = \{k\}_{k\varepsilon\mathbf{P}}$ which is not a continuous multiplier operator on $C_{2\pi}$ (with respect to the system $\{e^{ikx}\}$) but a bounded one on $\mathcal{D}_{2\pi}$, the locally convex space of 2π-periodic infinitely differentiable test functions.

III.2 INEQUALITIES OF JACKSON-, BERNSTEIN- AND ZAMANSKY-TYPE AND SATURATION.

In the following some fundamental inequalities in approximation theory will be extended to locally convex spaces, and with these means the saturation problem for multiplier operators in locally convex spaces will be treated.

Let $\phi(\rho)$ be a positive, monotonely decreasing function on $(0,\infty)$

with $\lim_{\rho\to\infty} \phi(\rho) = 0$.

THEOREM (3.7). *Let* $\{T(\rho)\}_{\rho>0}$ *be a family of multiplier operators on* $(X,\{p_r\})$ *corresponding to* $\{\tau(\rho)\} \subset M$, *and let* $\psi \in \omega$. *Furthermore, let the family of multiplier operators* $\{L(\rho)\}_{\rho>0}$, *given via* $\{\lambda(\rho)\} \subset M_C$, *be equicontinuous on* X *with respect to* ρ. *Then the condition*

$$\phi^{-1}(\rho)\{\tau_k(\rho) - 1\} = \psi_k\lambda_k(\rho) \qquad (k \in P) \tag{3.8}$$

implies that to each $r \in J$ *there exists* $t \in J$ *and a constant* $B(r,t) > 0$ *such that the Jackson-type inequality*

$$\phi^{-1}(\rho)p_r(T(\rho)f - f) \leqslant B(r,t)p_t(B^{\psi}f) \qquad (f \in X^{\psi}) \tag{3.9}$$

holds. On the other hand, the condition

$$\psi_k\tau_k(\rho) = \phi^{-1}(\rho)\lambda_k(\rho) \qquad (k \in P) \tag{3.10}$$

implies the Bernstein-type inequality

$$p_r(B^{\psi}T(\rho)f) \leqslant B(r,t)\phi^{-1}(\rho)p_t(f) \qquad (f \in X) \tag{3.11}$$

and

$$\psi_k\tau_k(\rho) = \phi^{-1}(\rho)\lambda_k(\rho)\{\tau_k(\rho) - 1\} \qquad (k \in P) \tag{3.12}$$

the Zamansky-type inequality

$$p_r(B^{\psi}T(\rho)f) \leqslant B(r,t)\phi^{-1}(\rho)p_t(T(\rho)f - f) \qquad (f \in X) \tag{3.13}$$

The proofs are easy and follow analogously as those in Banach spaces (cf. [6]). Indeed, (3.8) immediately implies (3.9) since

$$\phi^{-1}(\rho)\{T(\rho)f - f\} = L(\rho)B^{\psi}f \qquad (f \in X^{\psi}, \rho > 0) \tag{3.14}$$

In a similar way one may treat Bohr-type inequalities (cf. [8]) and the comparison problem (cf. [11], [12]).
Let us briefly examine the saturation problem for approximation processes $\{T(\rho)\}_{\rho>0}$, defined via multipliers $\{\tau(\rho)\}_{\rho>0}$. Proceeding as in the Banach space frame (cf. [5;II] and [17]), we set for an approximation process $\{T(\rho)\}_{\rho>0} \subset [X]_{M_C}$

$$\bar{\mathbf{T}} := \{k \in \mathbf{P};\ \tau_k(\rho) = 1 \quad \text{for all } \rho > 0\}$$

Under the assumption $\bar{\mathbf{T}} \neq \mathbf{P}$ we postulate as a sufficient condition for the solution of the saturation problem:

(3.15) *Let* $\{T(\rho)\} \subset [X]_{M_C}$ *be an approximation process on* $(X, \{p_r\}_{r\in J})$ *with associated multipliers* $\{\tau(\rho)\}$. *Let there exist a family* $\{\eta(\rho)\} \subset M_C$, *whose associated multiplier operators* $\{E(\rho)\}$ *form an approximation process on* X, *and a sequence* $\psi \in \omega$ *with* $\psi_k \neq 0$ *if* $k \notin \bar{\mathbf{T}}$ *such that for all* $\rho > 0$ *and* $k \in \mathbf{P}$

$$\phi^{-1}(\rho)\{\tau_k(\rho) - 1\} = \psi_k \eta_k(\rho)$$

As $\{E(\rho)\}$ is an approximation process, there holds $\lim_{\rho\to\infty} \eta_k(\rho) = 1$ for all $k \in \mathbf{P}$ so that

$$\lim_{\rho\to\infty} \phi^{-1}(\rho)\{\tau_k(\rho) - 1\} = \psi_k \qquad (k \in \mathbf{P}) \tag{3.16}$$

THEOREM (3.17). *If* $p_r(T(\rho)f - f) = 0_r(\phi(\rho))$ *for each* $r \in J$ *then* $f \in \bigcup_{m\in\bar{\mathbf{T}}} P_m(X)$, *and* $T(\rho)f = f$ *for all* $\rho > 0$, *i.e.* f *is an invariant element.*

Proof. As $P_k \in [X]$ and

$$P_k(\phi^{-1}(\rho)\{T(\rho)f - f\}) = \phi^{-1}(\rho)\{\tau_k(\rho) - 1\}P_k f$$

for each $k \in \mathbf{P}$ and $r \in J$ there exists some $t \in J$ such that by (3.16)

$$\begin{aligned} p_r(\psi_k P_k f) &= \lim_{\rho\to\infty} p_r(\phi^{-1}(\rho)\{\tau_k(\rho) - 1\}P_k f) \leqslant \\ &\leqslant B(r,t;k) \lim_{\rho\to\infty} p_t(\phi^{-1}(\rho)\{T(\rho)f - f\}) = 0 \end{aligned}$$

Thus $\psi_k P_k f = 0$ for all $k \in \mathbf{P}$ which implies $P_k f = 0$ for $k \notin \bar{\mathbf{T}}$, while for $k \in \bar{\mathbf{T}}$ one has $P_k T(\rho)f = P_k f$. Hence $P_k T(\rho)f = P_k f$ for each $k \in \mathbf{P}$, and the theorem is proved.

If, in addition, the set

$$F[X;\, T(\rho)] := \{f \in X;\ p_r(\phi^{-1}(\rho)\{T(\rho)f - f\}) = 0_r(1) \quad \text{for } \rho\to\infty \text{ and each } r \in J\} \tag{3.18}$$

contains a noninvariant element, then the approximation process

$\{T(\rho)\}$ is saturated in X with order $\phi(\rho)$, and $F[X; T(\rho)]$ is called its Favard or saturation class. Such a noninvariant element always exists as for each $k \notin \overline{T}$, $r \in J$ and $0 \neq h \in P_k(X)$

$$p_r(T(\rho)h - h) = |\tau_k(\rho) - 1| p_r(h)$$

Let us observe that (3.15) implies for each $r \in J$

$$\text{(3.19)} \qquad \phi^{-1}(\rho) p_r(T(\rho)f - f) = p_r(B^\psi E(\rho)f) \qquad (f \in X, \rho > 0)$$

THEOREM (3.20). *Given* $(X, \{p_r\}_{r\varepsilon J})$ *such that the closed graph theorem holds on* X. *If* $\{T(\rho)\}$ *satisfies* (3.15), *then the Favard class of* $\{T(\rho)\}$ *is characterized as* $(X^\psi)^{\sim X}$ *and the following seminorms are equivalent on* $F[X; T(\rho)]$:

$$\text{(i)} \quad p_r(f) + \sup_{\rho>0} p_r(\phi^{-1}(\rho)\{T(\rho)f - f\}) \qquad (r \in J)$$

$$\text{(ii)} \quad \tilde{q}^\psi_r(f) \quad \textit{and} \quad \text{(iii)} \quad \sup_{\rho>0} p^\psi_r(S(\rho)f) \qquad (r \in J)$$

where $\{S(\rho)\}_{\rho>0} \subset [X]_{M_C}$ *is an approximation process with* $S(\rho)(X) \subset \subset X^\psi$.

Proof. (i) $\Leftrightarrow$ (iii): On account of (3.15) one may choose $\{S(\rho)\}_{\rho>0} = \{E(\rho)\}_{\rho>0}$, and the assertion follows immediately by (3.19) and

$$\begin{aligned} \sup_{\rho>0} p^\psi_r(E(\rho)f) &\leqslant B(r,t)p_t(f) + \sup_{\rho>0} p_r(\phi^{-1}(\rho)\{T(\rho)f - f\}) \leqslant \\ &\leqslant B(r,t)[p_s(f) + \sup_{\rho>0} p_s(\phi^{-1}(\rho)\{T(\rho)f - f\})] \leqslant \\ &\leqslant B(r,t) \sup_{\rho>0} p^\psi_s(E(\rho)f) \end{aligned}$$

as $\{p_r\}_{r\varepsilon J}$ is filtrating.

(ii) $\Rightarrow$ (iii): Given $f \in (X^\psi)^{\sim X}$, there exists a net $\{f_\beta\}_{\beta\varepsilon D} \subset X^\psi$ such that $p^\psi_r(f_\beta) \leqslant R_r$ $(r \in J)$ for some $R_r > 0$ and $\lim_{\beta\varepsilon D} p_r(f_\beta - f) = 0$. Obviously $B^\psi S(\rho)$ is defined and closed on X, so that $B^\psi S(\rho) \in [X]_{M_C}$.
On X^ψ we have $B^\psi S(\rho) = S(\rho)B^\psi$, and therefore

$$\begin{aligned} p^\psi_r(S(\rho)f) &= p_r(S(\rho)f) + \lim_{\beta\varepsilon D} p_r(B^\psi S(\rho)f_\beta) = \\ &= p_r(S(\rho)f) + \lim_{\beta\varepsilon D} p_r(S(\rho)B^\psi f_\beta) \leqslant \\ &\leqslant B(r,t)[p_t(f) + \sup_{\beta\varepsilon D} p_t(B^\psi f_\beta)] \leqslant \\ &\leqslant B(r,t) \sup_{\beta\varepsilon D} p^\psi_t(f_\beta) \end{aligned}$$

The left side is independent of the special choice of the net $\{f_\beta\}_{\beta\in D}$, and the right side is independent of ρ; therefore

$$\sup_{\rho>0} p_r^\psi(S(\rho)f) \leq B(r,t) \inf\{\sup_{\beta\in D} p_t^\psi(f_\beta);\ \{f_\beta\} \in N_f^b\} = B(r,t)\tilde{q}_t^\psi(f).$$

(iii) $\Rightarrow$ (ii): This direction is easily proved by examining the particular net $\{S(\beta)\}_{\beta\in\mathbb{R}^+} \subset X^\psi$.

IV. A MULTIPLIER CRITERION FOR CESÀRO BOUNDED FOURIER EXPANSIONS.

In the applications the problem arises whether a given sequence $\eta = \{\eta_k\}_{k\in P} \in \omega$ is a multiplier with respect to a given orthogonal system $\{P_k\}_{k\in P} \subset [X]$ in a locally convex space $(X,\{p_r\}_{r\in J})$. In this section we obtain a first criterion for subclasses of $M(X;\{P_k\})$ by the uniform boundedness of the (C,κ)-means; these are just the classes $bv_{\kappa+1}$, well known in the literature for some time, particularly in connection with the theory of divergent series.

In the locally convex space $(X,\{p_r\})$ with the system of projections $\{P_k\}_{k\in P}$ let the (C,κ)-means for $\kappa \geq 0$ be defined by

$$(C,\kappa)_n f := (A_n^\kappa)^{-1} \sum_{k=0}^{n} A_{n-k}^{\kappa} P_k f \qquad (f \in X, n \in P) \tag{4.1}$$

$$A_n^\kappa := \binom{n+\kappa}{n} = \frac{\Gamma(n+\kappa+1)}{\Gamma(n+1)\Gamma(\kappa+1)} \tag{4.2}$$

DEFINITION (4.3). ("*The* (C^κ)-*condition*"). *Let* $\kappa \geq 0$ *and* $(X,\{p_r\})$ *be complete. The pair* $(X,\{p_r\}_{r\in J})$, $\{P_k\}$ *satisfies the* (C^κ)-*condition, if for each* $r \in J$ *there exists* $t \in J$ *and a constant* $C(r,t;\kappa) > 0$ *such that*

$$p_r((C,\kappa)_n f) \leq C(r,t;\kappa)p_t(f) \qquad (f \in X, n \in P)$$

If (4.3) is satisfied for a fixed $\kappa \geq 0$, then it follows that for all $\beta > \kappa$

$$p_r((C,\beta)_n f) \leq C(r,t;\kappa)p_t(f) \qquad (f \in X, n \in P)$$

To derive an appropriate multiplier criterion we introduce the sequence spaces $bv_{\kappa+1}$ as subspaces of ℓ^∞, the set of all bounded sequences, by

(4.4) $bv_{\kappa+1} := \{\eta \in \ell^\infty; \|\eta\|_{bv_{\kappa+1}} = \sum_{k=0}^{\infty} A_k^\kappa |\Delta^{\kappa+1}\eta_k| + \lim_{k\to\infty} |\eta_k| < \infty\}$

where the (fractional) difference operator Δ^β is defined via

(4.5)
$$\Delta^\beta \eta_k = \sum_{l=0}^{\infty} A_l^{-\beta-1} \eta_{k+l}$$

With $\beta \geqslant 0$ and $\eta \in \ell^\infty$ the series (4.5) converges absolutely. We still remark that $\lim_{k\to\infty} \eta_k = \eta_\infty$ exists for $\eta \in bv_{\kappa+1}$ and

(4.6)
$$bv_{\kappa+1} \subset bv_{\gamma+1} \qquad 0 \leqslant \gamma < \kappa$$

Furthermore, for each $\eta \in bv_{\kappa+1}$

(4.7)
$$\eta_n - \eta_\infty = \sum_{k=0}^{\infty} A_k^\kappa \Delta^{\kappa+1} \eta_{k+n} \qquad (n \in \mathbf{P})$$

For these fundamentals see [17; Sec.3] and the literature cited there.

THEOREM (4.8). *Let* $(X, \{p_r\}_{r\in J})$, $\{P_k\}$ *satisfy the* (C^κ)*-condition* (4.3) *for some* $\kappa \geqslant 0$. *Then* $bv_{\kappa+1}$ *is continuously embedded in* $M_C(X;\{P_k\})$, *i.e., to each* $r \in J$ *there exists* $t \in J$ *and a constant* $C(r,t;\kappa) > 0$ *such that*

$$\|\eta\|_{M,r,t} \leqslant C(r,t;\kappa)\|\eta\|_{bv_{\kappa+1}} \qquad (\eta \in bv_{\kappa+1})$$

Proof. Analogously to [5; II] or [12] we set up for an arbitrary $f \in X$ and $\eta \in bv_{\kappa+1}$

$$f^\eta := \sum_{k=0}^{\infty} A_k^\kappa \Delta^{\kappa+1}\eta_k (C,\kappa)_k f + \eta_\infty f$$

Then $f^\eta \in X$ since by (4.3) and (4.4) ($\{p_r\}$ being filtrating)

$$p_r(f^\eta) \leqslant C(r,t;\kappa) p_t(f) \sum_{k=0}^{\infty} A_k^\kappa |\Delta^{\kappa+1}\eta_k| + |\eta_\infty| p_r(f) \leqslant$$
$$\leqslant C(r,t;\kappa)\|\eta\|_{bv_{\kappa+1}} p_s(f)$$

To prove that $P_n f^\eta = \eta_n P_n f$ for each $n \in \mathbf{P}$ we consider

$$P_n (C,\kappa)_k f = \begin{cases} 0 & \text{if } n > k \\ (A_{k-n}^\kappa / A_k^\kappa) P_n f & \text{if } n \leqslant k \end{cases}$$

and obtain by (4.7) that

$$P_n f^{\eta} = P_n f\{\sum_{k=n}^{\infty} A_k^{\kappa} \frac{A_{k-n}^{\kappa}}{A_k^{\kappa}} \Delta^{\kappa+1}\eta_k + \eta_\infty\} = \eta_n P_n f$$

Concerning sufficient conditions for $\eta \in bv_{\kappa+1}$ we refer to [5; II] and [17; Sec.3 ff].

To give an application, let $(X,\{p_r\})$, $\{P_k\}$ satisfy (4.3) for some $\kappa \geq 0$. Then one may consider the Abel-Cartwright and the Riesz means of (3.1), thus for $\sigma > o$ and $\lambda > 0$

$$(4.9) \qquad W_\sigma(\rho)f \sim \sum_{k=0}^{\infty} w((\tfrac{k}{\rho})^\sigma)P_k f \text{ and } R_{\sigma,\lambda}(\rho)f \sim \sum_{k=0}^{\infty} r_\lambda((\tfrac{k}{\rho})^\sigma)P_k f$$

$$w(x) = e^{-x} \quad \text{and} \quad r_\lambda(x) = \begin{cases} (1-x)^\lambda & 0 \leq x \leq 1 \\ 0 & 1 < x \end{cases}$$

For each $\sigma > 0$, $\lambda \geq \kappa$ one has (cf. [5; II], [17; Sec.4]) that

$$(4.10) \qquad \{w((\tfrac{k}{\rho})^\sigma)\} \quad , \quad \{r_\lambda((\tfrac{k}{\rho})^\sigma)\} \subset bv_{\kappa+1}$$

uniformly in $\rho > 0$. Furthermore,

$$\lim_{\rho\to\infty} w((\tfrac{k}{\rho})^\sigma) = 1 \quad \text{and} \quad \lim_{\rho\to\infty} r_\lambda((\tfrac{k}{\rho})^\sigma) = 1$$

so that convergence of the Abel-Cartwright and the Riesz means follows on Π (cf. Sec. III.1). If Π is dense in X and X barreled, then the families $\{W_\sigma(\rho)\}_{\rho>0}$ and $\{R_{\sigma,\lambda}(\rho)\}_{\rho>0}$ form approximation processes on X.

To determine the Favard class $F[X; W_\sigma(\rho)]$ we examine (3.15) and have for any $\sigma > 0$

$$\psi_k = -k^\sigma, \quad \phi(\rho) = \rho^{-\sigma}, \quad e(x) = -x^{-\sigma}[\exp(-x^\sigma)-1] \ ,$$

$$\eta_k(\rho) = e(k/\rho), \ \lim_{\rho\to\infty} \eta_k(\rho) = 1, \text{ and } \{\eta_k(\rho)\}_{k\in\mathbf{P}} \in bv_{j+1}$$

for each $j \in \mathbf{P}$ (cf. [5; II]).

By theorem (3.20) it therefore follows that

$$F[X; W_\sigma(\rho)] = (X^\psi)^{\sim X} \quad \text{with} \quad \psi = \{-k^\sigma\}_{k\in\mathbf{P}}$$

For further examples of processes, also in connection with theorem (3.7) see [4], [5], [6], [9], [17] and the literature cited there. In this direction one has the following Bernstein-type inequality

THEOREM (4.11). *Let* $\nu > 0$. *If* $(X,\{p_r\})$, $\{P_k\}$ *satisfy the* (C^{κ})*-condition* (4.3) *for some* $\kappa > 0$, *then to each* $r \in J$ *there exists* $t \in J$ *and a constant* $B(r,t;\kappa) > 0$ *such that for all polynomials* $f = \sum_{k=0}^{n} P_k f \in \Pi$ *it follows that*

$$p_r\Big(\sum_{k=0}^{n} k^{\nu} P_k f\Big) \leqslant B(r,t;\kappa)C\, n^{\nu}\, p_t\Big(\sum_{k=0}^{n} P_k f\Big)$$

Proof. We reduce the proof to (3.10) with $\rho\to\infty$ replaced by $n\to\infty$. Let $\nu > 0$ and $e(x) \in C_{oo}^{\infty}([0,\infty))$, the class of infinitely differentiable functions with compact support on $[0,\infty)$, such that $e(x) = x^{\nu}$ if $0 \leqslant x \leqslant 1$ and $e(x) = 0$ if $x \geqslant 2$. Evidently the sequence $\lambda(n)$ with $\lambda_k(n) = e(k/n)$ belongs to bv_{j+1} for each $j \in \mathbf{P}$ uniformly in $n \in \mathbf{P}$ (cf. [5; I, II]; the dependence of the parameter $n \in \mathbf{P}$ being of Fejér-type). Thus we choose $j = [\kappa] + 1$ with $[\kappa]$ the greatest integer less than or equal to κ. Now we identify $\psi_k = k^{\nu}$, $\phi^{-1}(n) = n^{\nu}$; finally, $\tau \in \omega$ with $\tau_k(n) = (n/k)^{\nu}\lambda_k(n)$ is a multiplier on X. Particularly, for $f = \sum_{k=0}^{n} P_k f$ we have by (3.11) as $\tau_k(n) = 1$ if $k \leqslant n$

$$p_r\Big(\sum_{k=0}^{n} (k/n)^{\nu} P_k f\Big) = p_r\Big(\sum_{k=0}^{\infty} \lambda_k(n) P_k f\Big) \leqslant B(r,t;\kappa)\|\lambda\|_{bv_{j+1}}\, p_t(f) ,$$

and the theorem is proved with $C = \|\lambda\|_{bv_{j+1}}$.

V. APPLICATIONS TO CESÀRO BOUNDED ORTHOGONAL SYSTEMS.

V.1. TRIGONOMETRIC SERIES.

As a first example we treat trigonometric expansions in weighted locally convex spaces of 2π-periodic functions. Here theorem (5.2) (cf. [14]) gives necessary and sufficient conditions upon the weight functions $U_r(x)$ such that the (C,1)-means of the Fourier series expansion satisfy condition (4.3). This in turn determines examples of locally convex spaces $X^p_{D,J}$ and $X^p_{V,J}$ for which the (C^{κ})-condition is satisfied for $\kappa = 1$ but not for $\kappa = 0$.

Let the system $\{P_k\}_{k\in\mathbf{P}}$ be defined by

(5.1) $\quad P_0 f(x) = f^\wedge(0), \quad P_k f(x) = f^\wedge(k)e^{ikx} + f^\wedge(-k)e^{-ikx} \quad (k \in \mathbb{N})$

$f^\wedge(k)$ denoting the usual Fourier coefficients

$$f^\wedge(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-iku}du \qquad (k \in \mathbb{Z})$$

THEOREM (5.2). [14; p. 223/224]: *Assume that* $1 \leqslant p < \infty$, $f(x)$ *is integrable on* $[0,2\pi]$, $U_r(x) \geqslant 0$, $f(x)$ *and* $U_r(x)$ *have period* 2π. *Then the following are equivalent:*

(5.3) $$\lim_{n\to\infty} \int_0^{2\pi} |(C,1)_n f(x) - f(x)|^p U_r(x)dx = 0$$

for every function f *satisfying* $a_r^p(f) := (\int_0^{2\pi} |f(x)|^p U_r(x)dx)^{1/p} < \infty$.

(5.4) $$\int_0^{2\pi} |(C,1)_n f(x)|^p U_r(x)dx \leqslant C_p \int_0^{2\pi} |f(x)|^p U_r(x)dx \ ,$$

the constant C_p *being independent of* f *and* n.

(5.5) *For every interval* I *with* $|I| \leqslant 2\pi$ ($|I|$ *the length of* I) *one has with a constant* K, *independent of* I,

$$A_p := \int_I U_r(x)dx \left(\int_I [U_r(y)]^{-1/(p-1)}dy\right)^{p-1} \leqslant K|I|^p \qquad (1 < p < \infty)$$

$$A_1 := \int_I U_r(x)dx \ \operatorname*{ess\,sup}_{y\in I}[U_r(y)]^{-1} \leqslant K|I| \qquad (p=1)$$

REMARK. *In the case* $1 < p < \infty$ *one may replace the* (C,1)*-means in theorem* (5.2) *by the usual partial sums* $S_n(f;x)$ (cf. [10]).

LEMMA (5.6). *For every subinterval* $I \subset \mathbb{R}$ *the weights* $U_r(x) = |x|^r$ *satisfy the condition*

$$A_p \leqslant K|I|^p \quad \textit{if} \quad -1 < r < p-1 \qquad (1 < p < \infty)$$

$$A_1 \leqslant K|I| \quad \textit{if} \quad -1 < r \leqslant 0 \qquad (p=1)$$

with a constant $K = K(r,p)$ *independent of* I.

In view of the estimate

(5.7) $$\frac{2}{\pi} \leqslant \frac{\sin x/2}{x/2} \leqslant 1 \qquad x \in [0,\pi]$$

Lemma (5.6) immediately implies

LEMMA (5.8). *The weight functions*

$$U_r(x) = |2 \sin x/2|^r \qquad (x \in \mathbb{R})$$

satisfy (5.5) *for*

$$-1 < r < p-1 \quad \text{if} \quad 1 < p < \infty ,$$
$$-1 < r \leq 0 \quad \text{if} \quad p = 1 .$$

Let us observe that on the fundamental interval $[-\pi,\pi]$ the functions $U_r(x)$ have to be defined here such that they are symmetric in some neighborhood of its singularities and zeros.

Via the weights $U_r(x)$ of Lemma (5.8) we define the Banach spaces

$$(5.9) \qquad X_{2\pi}^{r,p} := \{f \in L_{2\pi}^1;\ a_r^p(f) := $$
$$:= \left(\int_{-\pi}^{\pi} |f(x)|^p U_r(x)dx\right)^{1/p} < \infty\} \qquad (1 \leq p < \infty)$$

as subspaces of $L_{2\pi}^1$. We have in the sense of continuous embedding

$$(5.10) \qquad L_{2\pi}^{\infty} \subset X_{2\pi}^{r,p} \subset L_{2\pi}^1 \qquad r \in (-1,p-1),\ 1 \leq p < \infty$$

and $L_{2\pi}^2$ is dense in $X_{2\pi}^{r,p}$. Therefore the projections P_k, $k \in \mathbb{P}$, defined in (5.1), are continuous, total and fundamental on $X_{2\pi}^{r,p}$ for all $r \in (-1,p-1)$.

Given some open interval $J \subset (-1,p-1)$, by

$$(5.11) \qquad X_{D,J}^p := \bigcap_{r\varepsilon J} X_{2\pi}^{r,p} \quad \text{and} \quad X_{V,J}^p := \bigcup_{r\varepsilon J} X_{2\pi}^{r,p} \qquad (1 \leq p < \infty)$$

there are defined locally convex spaces in which the (C^{κ})-condition (4.3) holds with $\kappa=1$ for $p=1$ and by [10] even with $\kappa=0$ for $1 < p < \infty$.

Evidently, for $p=1$ the $(C,0)$-means are not equicontinuous on $X_{D,J}^1$ or $X_{V,J}^1$ as the example

$$f_n(x) = \operatorname{sgn} D_n(x), \quad D_n(x) = 1/2 + \sum_{k=1}^{n} \cos kx$$

shows. Indeed, one has $(n\to\infty)$

$$a_r^1(S_n(f_n;x)) \geq (1/2)\|S_n(\operatorname{sgn} D_n;x)\|_{L_{2\pi}^1} = O(\log n) .$$

Under the system of norms $\{a_r^p\}_{r\in J}$ which are in concordance, $X_{D,J}^p$ $1 \leq p < \infty$, is a countably normed, barreled and B-complete Hausdorff space, metrisable but not normable. As the closed graph theorem holds we have $M_C(X_{D,J}^p ; \{P_k\}) = M(X_{D,J}^p ; \{P_k\})$. On each $X_{2\pi}^{r,p} \subset L_{2\pi}^1$, $r \in (-1,p-1)$, and therefore on $X_{D,J}^p$ and $X_{V,J}^p$ the system $\{P_k\}_{k\in P}$ is continuous, fundamental, total and mutually orthogonal.

$X_{V,J}^p$, $1 \leq p < \infty$, is a countable strict inductive limit, complete, barreled and Hausdorff but not metrisable and not necessary B-complete; however by the closed graph theorem we have $M_C = M$.

Let us conclude with an application of Theorem (4.11) in connection with the spaces (5.11) and weights (5.8).

COROLLARY (5.12). *Let* $\nu > 0$. *Then to each* $r \in (-1,p-1)$ *there exists a* $t \in (-1,p-1)$ *such that for each trigonometric polynomial* $\sum_{k=-n}^{n} c_k e^{ikx}$ *with some constant* $D(r,t;\kappa,\nu) > 0$

$$\left(\int_{-\pi}^{\pi} \left|\sum_{k=-n}^{n} |k|^\nu c_k e^{ikx}\right|^p |2 \sin x/2|^r dx\right)^{1/p} \leq$$

$$\leq D(r,t;\kappa,\nu) n^\nu \left(\int_{-\pi}^{\pi} \left|\sum_{k=-n}^{n} c_k e^{ikx}\right|^p |2 \sin x/2|^t dx\right)^{1/p}$$

V.2. LAGUERRE SERIES.

Let $L^p(0,\infty)$, $1 \leq p < \infty$, denote the usual Lebesgue spaces (with respect to ordinary Lebesgue measure) of functions for which the norms

$$\|f\|_p := \left(\int_0^\infty |f(x)|^p dx\right)^{1/p} \qquad (1 \leq p < \infty)$$

are finite. $L_{loc}^p(0,\infty)$ denotes the set of functions which belong locally to $L^p(0,\infty)$, i.e. on every compact subset of $(0,\infty)$.

With the weight functions, defined on $(0,\infty)$ for some (fixed) $\alpha > -1$

$$(5.13) \qquad U_{b,r}(x) := x^{\alpha/2} x^b (1+x)^{r-b} \exp(-x/2) \qquad (b,r \in \mathbb{R})$$

let us introduce the Banach spaces

(5.14) $X^p_{b,r} := \{f \in L^p_{loc}(0,\infty);\ \pi^p_{b,r}(f) := \|f(x)U_{b,r}(x)\|_p < \infty\}$

Then for every open subset $J \subset \mathbf{R}$, by

(5.15) $X^p_{D,J} := \bigcap_{r\varepsilon J} X^p_{b,r}$ and $X^p_{V,J} := \bigcup_{r\varepsilon J} X^p_{b,r}$ $\qquad (1 \leqslant p < \infty)$

locally convex spaces are defined in which considerations analogous to V.1 are valid. Therefore the closed graph theorem holds so that each multiplier τ is continuous. Obviously these and the following considerations are also true for a variation of the parameter b in an open set $J \subset \mathbf{R}$ or of the pair (b,r) in an open $J \subset \mathbf{R}_2$.

Let $L^\alpha_k(x)$, $\alpha > -1$, denote the kth Laguerre polynomial given via

$$\sum_{k=0}^{\infty} L^\alpha_k(x)s^k = (1-s)^{-\alpha-1} \exp(-\tfrac{sx}{1-s})$$

Then to each $f \in X^p_{D,J}$ or $f \in X^p_{V,J}$, respectively, one may associate its (well-defined) (cf. [15; p.17]) Laguerre series expansion

(5.16) $$f \sim \sum_{k=0}^{\infty} P^\alpha_k f$$

where

(5.17) $$(P^\alpha_k f)(x) := (\frac{k!}{\Gamma(k+\alpha+1)} \int_0^\infty f(y)e^{-y}y^\alpha L^\alpha_k(y)dy)L^\alpha_k(x)$$

These projections $\{P^\alpha_k\}_{k\varepsilon\mathbf{P}}$ form a total, fundamental and mutually orthogonal system on $X^p_{D,J}$ and $X^p_{V,J}$. With the results in [13] and [15] we determine now open intervals $J = J(\kappa,p) \subset \mathbf{R}$ such that on $X^p_{D,J}$ and $X^p_{V,J}$ the (C^κ)-condition (4.3) holds with $r \leqslant t$, $r,t \in J$ for $\kappa=1$; but (4.3) isn't valid with $\kappa=0$ for all $r \in J$ and any choice of $t \in J$. This is an immediate consequence of the fact that the conditions about the parameters b and r in [13] concerning (C,0)-summability are sharp. However, it was not Muckenhoupt's aim to study summability conditions in a locally convex frame. In particular for $\alpha > -1/2$ and

(5.18) $$1/4 - 1/p < b < 3/4 - 1/p \qquad (1 \leqslant p < \infty)$$

one has the following intervals $J(\kappa,p)$ for the parameter r:

$$(5.19)\quad \text{for } \kappa=0 \quad J(0,p) = \begin{cases} (1/4 - 1/p,\ 3/4 - 1/p), & p \in [4/3,4] \\ \emptyset & , p \notin [4/3,4] \end{cases}$$

In case $\kappa=0$ the restriction $r < t$, $r,t \in J$, is necessary for $p=4/3$ and $p=4$. Thus, if $p \notin [4/3,4]$, then the (C,0)-means are not equicontinuous on $X^p_{D,J}$ or $X^p_{V,J}$ respectively. On the other hand,

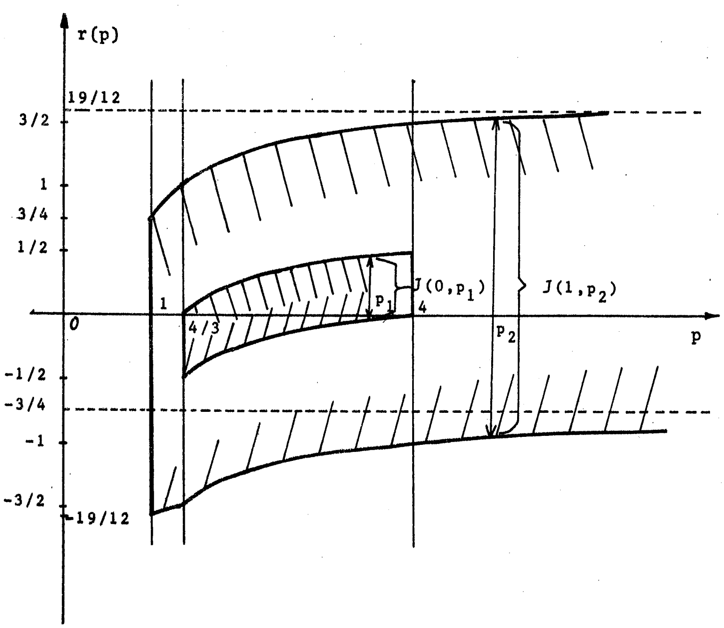
$$(5.20)\quad \text{for } \kappa=1 \quad J(1,p) = \begin{cases} (-1/(3p) - 5/4,\ 7/4 - 1/p), & 1 \leqslant p < 4/3 \\ (-1/p - 3/4\ ,\ 7/4 - 1/p), & 4/3 \leqslant p \leqslant 4 \\ (-1/p - 3/4\ ,\ 19/12 - 1/(3p)), & 4 < p < \infty \end{cases}$$

Furthermore, for $b=r$, $\alpha > -1$, in (5.13), i.e.
$U_{r,r}(x) = x^{\alpha/2}x^r \exp(-x/2)$ there are other open intervals $J(1,p)$ (cf. [15, p. 11]) such that the (C,1)-means, but not the (C,0)-means are equicontinuous on $X^p_{D,J}$ or $X^p_{V,J}$, $1 \leqslant p < \infty$, respectively, namely

$$(5.21)\quad \left\{ \begin{aligned} &-1/p - \min(\alpha/2,1/4) < r < 1 - 1/p + \min(\alpha/2,1/4) \\ &-2/(3p) - 1/2 < r < 7/6 - 2/(3p) \end{aligned} \right\} \quad (1 \leqslant p < \infty)$$

So (5.20) means, in the case $1 \leqslant p < 4/3$, for example, that to each $r \in (-1/(3p) - 5/4,\ 7/4 - 1/p)$ and each $t \in [r,\ 7/4 - 1/p)$ there exists a constant $C(r,t) > 0$ such that

$$\pi^p_{b,r}((C,1)_n f) \leqslant C(r,t)\pi^p_{b,t}(f) \qquad (f \in X^p_{V,J} \text{ or } f \in X^p_{D,J})$$



The figure gives the upper and lower bounds of the parameter $r = r(p)$ of conditions (5.19), (5.20) which determine the allowable intervals $J(0,p)$ and $J(1,p)$.

Obviously $J(0,p) \subset J(1,p)$, and the restriction $r \leqslant t$ in (5.18) - (5.20) may not be omitted. In the case $\kappa=1$ the parameter r generally still depends on the parameter $b = b(\alpha,p)$.

Further examples of weighted locally convex spaces with the more general weight functions $U_{b,r}(x)$ of (5.13) may be derived from the results in [13] and [15] by variation of the parameters b and r under the given restrictions.

Correspondingly one may treat Hermite expansions in suitable weighted functions spaces (cf. [11], [12]) using results of [13] and [15] or ultraspherical expansions by taking inductive and projective limits with respect to the parameter $p \in [1,\infty)$ (cf. [2], [12]). Some examples of domains J valid for Hermite expansions are given in [11; cf. (3.5) and (3.6)] in the order-preserving case, but they are valid on spaces analogous to (5.15) also in the general case. One can obtain further examples of locally convex spaces satisfying the condition (4.3) from the examples in V.1 and V.2 by forming countably normed spaces and inductive limits with respect to the free parameters in an open set A, for example in V.2 with $p \in A \subset [1,\infty)$

$$X_{D,A} := \bigcap_{p\varepsilon A} X^p_{D,J} \quad \text{and} \quad X_{V,A} := \bigcup_{p\varepsilon A} X^p_{D,J} \quad \text{or}$$

$$X^I_{V,A} := \bigcup_{p\varepsilon A} X^p_{V,J}$$

In the last case the closed graph theorem may fail on $X^I_{V,A}$ as this inductive limit is not countable, and hence one may then get $M_C \subset M$ with proper inclusion.

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