INNER FUNCTIONS UNDER UNIFORM TOPOLOGY. II.

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The set of all inner functions in the unit disc is a topological semigroup under pointwise multiplication in the open disc and the metric topology induced by the $H^\infty$-norm. In this second article it is shown that, given any non-empty closed subset $\Gamma$ of the unit circle $\partial D$, the set of components containing nothing but Blaschke products whose zeroes cluster exactly on $\Gamma$ (i.e., $\text{Sp}(b) \cap \partial D = \Gamma$, in the notation of [6],[7]) has the power of the continuum; moreover, the same result is true in $(F^\sim, \tau)$, thus answering in the strongest negative form a question raised by R.G. Douglas in [1].

The results also include a matricial version of the classical theorem of Otto Frostman ([2]) about the density of Blaschke products, which is used to extend some of the previous results to matrix-valued inner functions (i.e., the case when $\text{dim } K < \infty$).

This paper is a sequel of [6] and [7], and we shall continue using the notation introduced there without further reference.

1. COMPONENTS OF $(F^\sim, \tau)$.

The aim of this section is to prove the following

**Theorem 1.1.** Let $\Gamma$ be any non-empty closed subset of $\partial D$ and let $(\varphi_k)_{k=1}^\infty$ be any arbitrary sequence of arguments such that

$$\Gamma = \cap_{N=1}^\infty \text{clos}(e^{i\varphi_k}; k \gg N)$$

If the sequence $0 = r_1 < r_2 < \ldots < r_k < \ldots < 1$ converges to one rapidly enough, then there exist $c$ sub-products

$$b_\theta(z) = \prod_{k=1}^\infty b(z, r_ke^{i\varphi_k})$$

such that $b_\theta H^2 \in F^\sim \cap (\text{det})^\sim$; moreover, $b_\theta H^2$ and $b_\theta H^2$ belong to different components of $(F^\sim, \tau)$ if $\theta \neq \theta'$.

The idea of the proof is very simple and it is based on the following result of O. Frostman:
THEOREM 1.2. ([2, §58, thm. 2, p. 107]). Let \( q \in \mathbb{F} \) and assume that \( q \) does not admit zero as a radial limit; then \( q \) is a Blaschke product.

We are going to choose the sequence \( \{r_k\}_{k=1}^{\infty} \) in such a way that, for each \( k \), there is an \( R_k \), \( r_k < R_k < r_{k+1} \) such that \( |b(z)| = 1 \) for \( |z| = R_k \) and, moreover, for every element \( qH^2 \) of the component of \( bH^2 \) in \( (\mathbb{F}, r) \), \( |q(z)| = |b(z)| \) whenever \( R < |z| < 1 \), for some \( R \) depending on \( q \). Then, by thm. 1.2, \( q \) is also a Blaschke product.

For each pair of functions, \( f, g \in H^\infty \), define

\[
[f, g] = \sup \{ |f(z)| - |g(z)| : z \in D \}
\]

Similarly, we shall write \( \{[f, g] : z \in A \} \) when that supremum is taken on the subset \( A \subset D \).

It is apparent that for each non-denumerable \( A \subset D \), \( \{(pH^2, qH^2)_A = \{(p, q) : z \in A \} \subset \mathbb{F} \} \subset \mathbb{F} \) defines a metric in \( \mathbb{F} \).

Let \( p, q \in F \) satisfy \( \tau_o (pH^2, qH^2) < \varepsilon < 1 \), then there exists \( f \in (N)_{1-\varepsilon} \) such that \( \|p-qf\|_1 < \varepsilon \); hence

\[
[p, q] = [p, qf] + \|qf\|_1 [f, 1] < \|p-qf\|_1 + [f, 1] < \varepsilon + (1-(1-\varepsilon)^{-1}) = \varepsilon (2-\varepsilon)/(1-\varepsilon).
\]

Since \( 0 \leq [p, q] \leq 1 \), a simple computation shows that

LEMMA 1.3. Let \( p, q \in F \); then

\[
[p, q] \leq (27/10) \tau_o (pH^2, qH^2).
\]

We shall need some auxiliary results, which are contained in the following three lemmas:

LEMMA 1.4. For each \( a \in D \), define

\[
A_1(a) = \{ a' \in D : |b(z, a), b(z, a')| < 1/2 \}
\]

and (by induction)

\[
A_n(a) = \{ a'' \in D : |b(z, a'), b(z, a'')| < 1/2 \text{, for some } a' \in A_{n-1}(a) \} , \quad n=2,3,\ldots .
\]

Then:

i) \( A_n(a) \) is an increasing family of open neighborhoods of \( a \) with a smooth boundary, such that \( \bigcup_n A_n(a) = D \) and \( \text{clos } [A_n(a)] \subset D \), for each \( n \);

ii) For fixed \( n \), if \( |a| = 1 \), then diameter \( [A_n(a)] \to 0 \).
LEMMA 1.5. The results of Lemma 1.4 are also true for finite Blaschke products up to a few minor modifications: instead of elementary Blaschke products, consider the elements of $(\det)_m$ (for each fixed $m$) and, instead of $A_n(a)$, define $A_n(a_1, \ldots, a_m)$ (where $a_1, \ldots, a_m$ are the zeroes of a given $p \in (\det)_m$), in the obvious way.

LEMMA 1.6. There exists an increasing sequence $0 < R_1 < R_2 < \ldots < R_n < R_{n+1} < \ldots < 1$, such that for any Blaschke product $b(z) = \prod_{k=n+1}^\infty b(z, a_k)$, where $R_k < a_k < R_{k+1}$,

$$|b(z)| > 9/10$$

on the closed disc $\{z: |z| \leq R_n\}$, for $n=1,2,3,\ldots$.

The proof of Lemmas 1.4 and 1.5 follow from some elementary arguments based on the continuity of finite Blaschke products in the closed unit disc and the properties of the inner functions in $\mathbb{C} \setminus \mathbb{F}$. Lemma 1.6 follows from the fact that $\lim b(z, a) = 1(|a|+1)$, uniformly on compact subsets of $D$.

The definition of the sequence $\{r_n\}$ proceeds as follows: By induction, we shall construct a sequence of 4-tuples $(r_n, s_n, t_n, u_n)_{n=1}^\infty$ so that

1) $r_1 < s_1 < t_1 < u_1 < r_2 < \ldots < r_n < s_n < t_n < u_n < r_{n+1} < \ldots < 1$.

2) if $q_n(z) = \prod_{k=1}^n b(z, a_k)$, where $|a_k| < R_n$ ($k=1,\ldots,n$), and $q_1, \ldots, q_n \in (\det)_n$ satisfy $\max|q_j| < 1/2$, $j=1,\ldots,n$, then the zeroes of $q_n$ lie on $\{z: |z| < s_n\}$.

3) let $q_n$ be as above; then $|q_n(z)| > 9/10$ for $|z| > t_n$.

4) for all $p \in (\det)_n$ having no zeroes in $\{z: |z| < u_n\}$ and no more than one zero in the annulus $\{z: u_m < |z| < u_{m+1}\}$, for $m=n,n+1,\ldots$, $|p(z)| > 9/10$ for $|z| < t_n$.

5) $A_{n+1}(r_{n+1}) \cap \{z: |z| < u_n\} = \emptyset$, for all $n=1,2,\ldots$.

Now, the 4-tuples are chosen as follows:

Set $r_1 = 0$ and $s_1 = 1/2$. Let $t_1$ be the greatest real zero of $|b(t, 1/2)| = 9/10$. If $k_1$ is the first index (in Lemma 1.6) such that $t_1 < R_{k_1}$, then set $u_1 = R_{k_1+1}$; clearly, $r_1 < s_1 < t_1 < u_1$ and 2), 3), 4) can be easily checked for $n=1$.

Assume that $r_1 < s_1 < \ldots < r_n < s_n < t_n < u_n < 1$ have been chosen.
so that they satisfy 2), 3), 4) and 5). Now use Lemma 1.4 to find an \( t_{n+1} < 1 \) satisfying 5); \( r_{n+1} > u_n \). Next, use Lemma 1.6 to find an \( s_{n+1} < 1 \) so that 2) is verified; then \( s_{n+1} > r_{n+1} \). Let \( t_{n+1} \) be the greatest real zero of \( |b(t, s_{n+1})|^{n+1} = 9/10 \). Finally, if \( k_{n+1} \) is the first index (in Lemma 1.6) such that \( t_{n+1} \leq R_{k_{n+1}} \), choose \( u_{n+1} = R_{k_{n+1}+1} \).

It is not difficult to check that the sequence of 4-tuples thus constructed satisfies 1) to 5).

CLAIM. If \( b(z) = \prod_{k=1}^{\infty} b(z, r_k e^{i\theta_k}) \) and \( q \in \mathcal{F} \) satisfies \( \tau_o(bq^{1/2}, q^{1/2}) < 1/54 \), then

\[
q(z) = \lambda \prod_{k=1}^{\infty} b(z, a_k)
\]

where \( \lambda \in \mathbb{D} \) and \( a_k \in A_1(r_k e^{i\theta_k}), k=1,2,\ldots \).

By 3) and 4), \( |\prod_{k=1}^{n} b_k| = |b_n'| > 9/10 \) for \( |z| > t_{n-1} \) (a fortiori the same result is true for any subproduct) and

\[
|\prod_{k=n+1}^{\infty} b_k| = |b_n'| > 9/10 \text{ for } |z| < t_n; \text{ hence } |b| > (9/10)^2 > 4/5 \text{ for } |z| = t_n \text{ and } |b/b_n| > 4/5 \text{ on the annulus }
\]

\( \Lambda_n = \{z: t_{n-1} < |z| < t_n\} \), for all \( n=1,2,\ldots \) \((t_0 = -1)\).

By Lemma 1.3, \( [b, q] < 1/20 \) and therefore \( |q| > 4/5 - 1/20 = 3/4 \) for \( |z| = t_n \) and \( |q(re^{i\theta})| < 1/20 < 3/4, \) \( n=1,2,\ldots \). Using thm. 1.2 we infer that \( q \) is a Blaschke product; moreover, \( q(z) \neq 0 \) for \( |z| = t_n \) (for all \( n \)) and \( q \) has at least one zero in \( \Lambda_n \), for each \( n \).

It follows from the inequality \( \tau_o(bq^{1/2}, q^{1/2}) < 1/54 \) that \( \|b-qf\|_m < 1/54 \) for some suitable function \( f \in \mathcal{N}_{53/54} \). The last inequality shows, in particular, that \( |b-qf| < |b| \) for \( |z| = t_n \) \((n=1,2,\ldots)\).

Thus, applying Rouche's theorem and using the fact that \( f \) is invertible in \( \mathbb{H}^n \), we conclude that \( q \) has exactly one zero in \( \Lambda_n \), for each \( n=1,2,\ldots \). Hence,

\[
q(z) = \lambda \prod_{k=1}^{\infty} b(z, a_k) = \lambda \prod_{k=1}^{\infty} q_k(z) = \lambda q'_n q''_n
\]

where \( \lambda \in \mathbb{D}, a_n \in \Lambda_n \) \((n=1,2,\ldots)\) and \( q_n', q_n \) and \( q_n'' \) denote the sub-products corresponding to those zeroes lying in the regions

\[
|z| < t_{n-1}, \quad t_{n-1} < |z| < t_n \text{ and } |z| > t_n, \quad \text{respectively.}
\]

Then the previous estimates and the modulus maximum theorem imply that \( |q'_n q''_n| > 3/4 \) for \( z \in \Lambda_n \) and
\[ \{ (b_n, q_n) : z \in D \Delta A_n \} \leq \{ (b_n, 1) : z \in D \Delta A_n \} + \{ (1, q_n) : z \in D \Delta A_n \} \leq 1/5 + 1/5 + 1/20 < 1/2 \]

(to see this, observe that \(|b| > 4/5\) for \(|z| = t_n\) and \(|z| = t_{n-1}\) implies that \(|q_n| > 4/5 \cdot 1/20\) for all \(z \in D \Delta A_n\).

Assume that \(\{ (b_n, q_n) : z \in A_n \} > 1/2\); then

\[ [b, q] > \{ (b, q) : z \in A_n \} > \{ (b, 1, q) : z \in A_n \} - \{ (b, 1, q) : z \in A_n \} > (\min \{ (b, 1, q) : z \in A_n \}) \{ (b, q) : z \in A_n \} - \{ (b, 1, q) : z \in A_n \} > (4/5)(1/2) - 1/4 = 3/20 > [b, q] \]

a contradiction.

Therefore \([b_n, q_n] < 1/2\) for all values of \(n\) and it follows that \(q_n \in A_n(r_n e^{i \theta_n})\) for \(n=1, 2, \ldots\). Similarly, it can be proven that \(b_n(q_n^n) < 1/\ell\). Thus, by (2), the zeroes of \(q_n^n\) lie in \(|z| < s_{n-1}\), \(n=2, 3, \ldots\).

Let \(p H^2(p \in F)\) belong to the component of \(bH^2\) in \((F^-, \tau)\); then, by [6, Thms. 5.6 and 6.1] there exists a finite family \(P_\circ = b, p_1, p_2, \ldots, p_m = p\) of inner functions such that

\[ \max \{ r_\circ(p_{j-1} H^2, p_j H^2) : 1 \leq j \leq m \} < 1/54 \]

By Lemma 1.3, \(\max \{ |p_{j-1} H^2, p_j H^2| : 1 \leq j \leq m \} < 1/20\). Now by induction on \(j\), it follows that \(p\) is a Blaschke product; moreover

\[ p(z) = \lambda \prod_{k=1}^{\infty} b(z, a_k) \]

where \(\lambda \in \partial D\), \(|a_h| < t_n\) for \(h=1, 2, \ldots, n\), and \(a_n \in \Lambda_m(r_n e^{i \theta_n})\), for all \(n > m\).

Finally, choose a subsequence \((\theta_{k_j})_{j=1}^\infty\) of \((\theta_k)\) such that \(e^{i \theta_{k_j}} \rightarrow \lambda_0 \in \Gamma\) as \(j \rightarrow \infty\), and write

\[ b(z) = ( \prod_{j=1}^{\infty} b(z, e^{i \theta_{k_j}}) ) \Pi(b(z, e^{i \theta_k}) : k \neq j, \text{for all } j) = b^+(z) b^-(z). \]

Then, if \(b_0(z) = b_0^+(z) b_0^-(z)\), where \(b_0^+(z)\) is defined as in [9] for \(0 < \theta < \pi/4\), then the \(b_0^+\)'s inherit all the properties that we
already proved for b. In particular:
The components of \( (F^\sim, \tau) \) containing the \( b_\theta H^2 \)'s are disjoint for different values of \( \theta, 0 < \theta < \pi/4 \) and, moreover, all these components are contained in \( (\det)^\sim \cap F^\sim \).

The proof of theorem 1.1 is complete now.

REMARKS. a) The only criterion that we know to decide whether or not \( \phi H^2 \) and \( \psi H^2 \) belong to the same component of \( (F, d) \) (and the one that we already used in [7] and here is this: the distance from the component of \( \phi H^2 \) to the component of \( \psi H^2 \) is larger than some positive number. According to [6, Thm. 6.1] this is enough for \( (F^\sim, \tau) \); however, we do not even know whether or not all the components of \( (F, d) \) are arcwise connected or open subsets. We conjecture that the results of [6, Thm. 6.1] are also true for \( (F^\sim, d) \), but we were unable to prove or disprove it.

b) The topology induced on \( F^\sim \) by the metric \( (\cdot,\cdot)_d \) is strictly weaker than the \( \tau \)-topology. Moreover, as it follows from Remark a) to [7, Thm. 3.6], if \( d(z) = d(z; 0,1) \) and \( 0 < t < t' \), then
\[
\theta(d^t H^2, d^{t'} H^2) = \theta(H^2, d^{t'-t} H^2) = 1.
\]
On the other hand, the map \( t \rightarrow d^t H^2 \) from the positive reals into \( (F^\sim, (\cdot,\cdot)_d) \) is continuous!

2. EXTENSIONS OF THE RESULTS TO THE CASE WHEN \( K \) IS FINITE DIMENSIONAL.

Throughout this section we shall assume that \( 2 \leq \dim K = N < \infty \).

As for \( \dim K = 1 \), we have \( (\det)_R = F_R \) (where \( R \) denotes the sub-index of a given subclass). If \( U \in F \), then \( U(z) \) can be represented by an \( N \times N \) matrix with entries in \( H^w \) such that the non-tangential limits \( U(\lambda) \) are unitary matrices for almost every \( \lambda \in \partial D \) and \( \det U(z) \) is an inner function with the same kind of singularities as \( U(z) \) (see [4; 5; 10]); in particular
\[
(\det U) H^2_K \subset U H^2_K \quad \text{and} \quad \text{Sp} (\det U) = \text{Sp} (U).
\]

The canonical form of an element of \( F \) or \( FN \) can be found in [3; 8].

The mappings \( U \rightarrow \det U \) (from \( H^w(K) \) onto \( H^w \)) and \( U H^2_K \rightarrow (\det U) H^2 \) (from \( F^w(K) \) onto \( F^w(C) \) with \( d \)-topologies or \( \tau \)-topologies in both sides) are continuous.
Part of the results of section 1 and [7] can be extended to the case when \( 2 \leq \dim K = N < \infty \) with minor changes; for example, if \( \{ b_\theta: 0 < \theta < \pi/4 \} \) is defined as in the proof of thm. 1.1, and

\[
(*) \quad B_\theta = \begin{pmatrix} b_\theta & 0 \\ 0 & I_{N-1} \end{pmatrix}
\]

then the component of \( B_\theta \) in \((F^-, \tau)\) is contained in \( F^- \cap (\det)_B \)
and \( B_\theta, B_{\theta'} \) belong to different components of \((F^-, \tau)\) for \( \theta \neq \theta' \).

The proof follows from [6, Thms. 5.5, 6.1], Thm. 1.1 and the following

**Lemma 2.1.** If \( U, V \in F \) and \( \tau_0(\det U)^2, (\det V)^2 < \varepsilon < 1/2 \), then

\[
\tau_0[(\det U)^2, (\det V)^2] < c(N)\varepsilon,
\]

where \( N = \dim K \) and \( c(N) \) is a positive constant depending on \( N \).

**Proof.** If \( \tau_0(\det U)^2, (\det V)^2 < \varepsilon \), then there exists a \( C(1-\varepsilon) \) such that \( \|U-V\| < \varepsilon \); i.e.,

\[
\text{ess sup } \{ |I-V(\lambda)C(\lambda)U^*(\lambda)|: \lambda \in \partial D \} < \varepsilon < 1/2
\]

and therefore \( \det C \in \mathcal{N}^{1-\varepsilon} \), where \( \mathcal{N}^\delta \) is the distinguished boundary of \( F^- \).

\[
\|\det U - \det V \det C\| = \|1 - \det(VCU^*)\|_\infty < c'(N)\varepsilon
\]

for a suitable constant \( c'(N) > 0 \).

Hence

\[
\tau_0[(\det U)^2, (\det V)^2] < \tau_0[(\det U)^2, (\det V)^2] < \tau_0[(\det U)^2, (\det V)^2] < c'(N)2c'(N)\varepsilon = c(N)\varepsilon. \quad \text{qed.}
\]

Our first partial extension of the results of [7] is a generalization of Frostman's theorem. The following theorem is contained in the author's thesis (see [5, p. 85]); its proof is based on a result due to W. Rudin ([8; 9]).

Since the proofs are the same for one or for several complex variables, we shall state the results for the most general setting. Some definitions are needed:

**Definition.** Let \( f(z_1, \ldots, z_n) \) be an analytic function defined in the unit polydisc

\[
D^n = \{ z=(z_1, \ldots, z_n): z_j \in D, j=1, \ldots, n \}
\]

the \( n \)-dimensional torus \( (\partial D)^n \) is the distinguished boundary of \( D^n \) and \( dm_n \) is the normalized Lebesgue measure on \( (\partial D)^n \). \( f \) belongs to
$N_n(D^n)$ (the Universal Hardy Class) if given any $\epsilon > 0$, there exists $\delta > 0$ such that if $m_n(A) < \delta$ ($A$ a measurable subset of $(\partial D)^n$) then
\[
\int_A \log^+|f(w)| \, dm_n < \epsilon, \quad w \in (\partial D)^n
\]
uniformly with respect to $r$, $0 \leq r < 1$.

For $f \in N_n(D^n)$, the radial limits $f(w)$ do exist a.e. $(dm_n)$ on $(\partial D)^n$. The inequalities
\[
\log^+|a+b| \leq \log^+|a| + \log^+|b| + \log 2
\]
\[
\log^+|ab| \leq \log^+|a| + \log^+|b| \quad (a,b \in \mathbb{C})
\]
show that $N_n(D^n)$ is actually an algebra under pointwise operations ([8]).

**DEFINITION.** ([8]). $f(z) \in H^n(D^n)$ is an inner function if $|f(w)| = 1$ a.e. $(dm_n)$ on $(\partial D)^n$. If, in addition, the minimal $n$-harmonic function $u[f]$ such that $\log |f(z)| \leq u[f](z)$ is identically zero on $D^n$, $f$ is said to be a good inner function.

According to this definition, we shall say that the analytic $N \times N$ matrix of functions $F(z) = (F_{jk}(z))_{j,k=1}^N$ is an inner (good inner, resp.) matrix if $|F(z)|_k \leq 1$, for $z \in D^n$ and det $F(z)$ is an inner (good inner, resp.) function.

Observe that if $F(z)$ is an inner matrix, then $F(w)$ is unitary a.e.

For $n=1$, the good inner functions are just the Blaschke products (see [2], [8], [9]).

**THEOREM 2.2.** Let $f(z,a) = a^N + \sum_{k=0}^{N-1} f_k(z) a^k$, $f_k(z) \in N_n(D^n)$, $k = 0, \ldots, N-1$, and let $\Lambda \subset \mathbb{C}$ be a compact set such that cont log cap $\Lambda$ is positive (see definition in [8], [9]). Then there exists $\alpha \in \Lambda$ such that
\[
u[f(z,a)] = \int_{(\partial D)^n} f(e^{i\phi_1}, \ldots, e^{i\phi_n}, a) \prod_{j=1}^n P(r_j, \theta_j, \phi_j) \, dm_n(\phi_1, \ldots, \phi_n)
\]
where $z = (r_1 e^{i\phi_1}, \ldots, r_n e^{i\phi_n})$, $0 \leq r_j < 1$, and
\[
P(r, \theta, \phi) = (1-r^2)/(1+r^2 - 2r \cos(\theta - \phi)) \text{ is the Poisson kernel for } D.
\]
THEOREM 2.3. Let \( F(z) = (F_{jk}(z))_{j,k=1}^N, z \in \mathbb{D}, \) be an \( N \times N \) inner matrix. Then for all \( \alpha \in \mathbb{D} \) except a subset of continuous logarithmic capacity zero,

\[
F_{\alpha}(z) = [F(z)-\alpha I][1-\alpha F(z)]^{-1}
\]

is a good inner matrix.

For \( N=n=1 \), this is Frostman's theorem. For \( N=1 \) and arbitrary finite \( n \), both results are due to W. Rudin.

LEMMA 2.4. Let \( A \) be a compact set of positive continuous logarithmic capacity and let \( \mu \) be a positive Borel measure with \( \text{supp}(\mu) \subseteq A \), such that

\[
g(u) = \int_{A} \log^+ (|\alpha-u|^{-1}) d\mu(\alpha)
\]
is continuous on \( C \). Then

\[
G(s_0,s_1,\ldots,s_{N-1}) = \int_{A} \log^+ (|\alpha^N + \sum_{k=0}^{N-1} s_k \alpha^k|^{-1}) d\mu(\alpha)
\]
is a bounded uniformly continuous function on \( C^N \).

Proof. Given \( (s_0,s_1,\ldots,s_{N-1}) \in C^N \) we can find \( u_1,u_2,\ldots,u_N \in C \) (the roots of the polynomial) such that

\[
\alpha^N + \sum_{k=0}^{N-1} s_k \alpha^k = \prod_{j=1}^{N} (\alpha-u_j).
\]

The function \( g(u) \) has compact support in \( C \). Therefore \( g(u) \) attains its maximum in \( C \). Now, the continuity of \( G \) follows from the continuity of \( g \) and the inequality

\[
\log^+ |\prod_{j=1}^{N} c_j| \leq \sum_{j=1}^{N} \log^+ |c_j|, \quad c_j \in C.
\]

This also shows that

\[
\max \{G(s_0,s_1,\ldots,s_{N-1}):(s_0,\ldots,s_{N-1}) \in C^N\} = N \cdot \max (g(u):u \in C).
\]
The proof will be completed by showing that

\[
\lim_{k \to \infty} G(s_0,\ldots,s_{N-1}) = 0, \quad \text{as} \quad \sum_{k=0}^{N-1} |s_k| \to \infty.
\]

Actually, if the sum of the modulus of the \( s_k \)'s tends to infinity then (by an elementary algebraic fact) at least one of the roots \( u_j \) must tend to infinity. Let \( R > 0 \), large enough so that \(|\alpha| < R\),
for all $\alpha \in A$; then
\[
\lim \left\{ G(s_0, \ldots, s_{N-1}); \sum_{k=0}^{N-1} |s_k| \to \infty \right\} =
\]
\[
= \lim \left\{ \int A \log^+ \left( \prod_{j=1}^{N} \frac{1}{|\alpha - u_j|} \right) \text{d}\mu(\alpha); \sum_{j=1}^{N} |u_j| \to \infty \right\} \leqslant \lim \left\{ \int A \log^+ \left( \prod_{j=2}^{N} \frac{1}{|u_j - R|} \right) \text{d}\mu(\alpha) = 0 \right\}. \]
\]
\[
\text{Let } f(z, \alpha) \text{ be as in thm. 2.2; then defining}
\]
\[
B_\varepsilon(\alpha) = \int_{(\mathcal{D})^n} \log^+ \left( |f(w, \alpha)|^{-1} \right) \text{d}m_n(w); B(\alpha) = \lim \inf B_\varepsilon(\alpha), (r+1)
\]
\[
\text{we have (following Rudin's proof)}
\]
\[
\int_A B(\alpha) \text{d}\mu(\alpha) \leqslant \lim \inf \int_A B_\varepsilon(\alpha) \text{d}\mu(\alpha) = \lim \inf \int_{(\mathcal{D})^n} G(f_0(rw), \ldots, f_{N-1}(rw)) \text{d}m_n(w)
\]
\[
= \int_{(\mathcal{D})^n} G(f_0(w), \ldots, f_{N-1}(w)) \text{d}m_n(w)
\]
\[
= \int_A \int_{(\mathcal{D})^n} \log^+ \left( |f(w, \alpha)|^{-1} \right) \text{d}m_n(w) \text{d}\mu(\alpha) \leqslant \int_A B(\alpha) \text{d}\mu(\alpha)
\]
\]
\[
(1) \text{ and } (5) \text{ are justified by Fatou's lemma; } (2) \text{ and } (4) \text{ are consequences of Fubini's theorem; } (3) \text{ follows from Lebesgue's theorem on majorized convergence and lemma 2.4.}
\]
\[
\text{We conclude that}
\]
\[
\lim \inf \int_{(\mathcal{D})^n} \log^+ \left( |f(rw, \alpha)|^{-1} \right) \text{d}m_n = \int_{(\mathcal{D})^n} \log^+ \left( |f(w, \alpha)|^{-1} \right) \text{d}m_n
\]
\[
\text{for almost every } \alpha \in A, \text{ with respect to } \text{d}\mu.
\]
\[
\text{Now the proof of thm. 2.2 follows as the proof of thm. 3.6.2 in [8].}
\]
\[
\text{If } U(z) \text{ is an inner function-operator and } \alpha \in D, \text{ then it is easy to see that}
\]
\[
U_\alpha(z) = (U(z) - \alpha I)(I - \overline{\alpha} U(z))^{-1}
\]
\[
is also an inner function-operator and
Let $F(z)$ be as in thm. 2.3. Then $(-1)^N \det(F(z) - aI)$ has the same form as $f(z, a)$ in thm. 2.2, with $f_0, \ldots, f_{N-1}$ bounded functions.

On the other hand, $\log |\det(I - \alpha F(z))|$ is bounded and $\alpha$-harmonic in $D^n$. Hence thm. 2.3 follows from thm. 2.2 as thm. 5.3.3 in [8].

The above acotation gives the following

COROLLARY 2.5. The family of all good (Blasschke, for $n=1$) matrices forms a dense subset in the family of all inner matrices, in the norm-topology of $H^n(K)$.

Now we are in a position to "translate" the results of [7] to the case when $2 < \dim K = N < \infty$. All these results are included in the following theorem; the details of the proof are left to the reader.

THEOREM 2.6. i) Let $U_r \in F$ be defined by (*) with $d^r(z; 0, 1)$ instead of $b_\theta$. Then the component of $U_1$ is isometric to the component of $U_r$, even though $U_1$ and $U_r$ belong to different components for every $r \neq 1$. If $0 < r < 1$, the subsets $\{Q U_{1-r} \in \text{component of } U_1\}$ and $\{U_{1-r}Q \in \text{component of } U_1\}$ are isometric to the whole component of $U_1$.

ii) For each $r > 0$, and for each $Q \in \text{(det)}_B$ in the component of $U_r$, $\det Q(z) = \lambda d^r(z; 0, 1)$, for some $\lambda \in \partial D$. Moreover, if $V \in \text{(det)}_B$ and $\text{Sp}(V) \cap \partial D$ is countable, then $\det Q = \lambda \det V$, for all $Q \in \text{(det)}_B$ in the component of $V$.

iii) There exists a component of $F$ containing $c$ elements of $(\text{det})_B$ whose determinants are pairwise coprime. If for some $\alpha \neq 0$, $Q = PS$ and $Q_\alpha = PR$, where $P, S, R \in F$, then $P \in \text{(det)}_B$.

iv) $(\text{det})_B$ is closed and nowhere dense in $F$. $(\text{det})_B$ is dense, but not open in $F$.

v) The "$r$-diameter" of each component of $(F^\sim, \tau)(H^\sim_K)$ is 2 and no component of $(F^\sim, \tau)(H^\sim_K)$ (or $F(\text{det})_0$) is compact.

vi) If $B_t (B_\theta$, resp.) $\in \text{(det)}_B$ and $B_t = b_t$, as defined in [7, thm. 2.1] (det $B_\theta = b_\theta$, as defined in [7, thm. 2.2], resp.), then the component of $B_t (B_\theta$, resp.) in $F$ is contained in $(\text{det})_B$. Moreover, if $0 < t < \theta' < \infty (0 < \theta < \theta' < \pi/4$, resp.), then $B_t$ and $B_t' (B_\theta$ and $B_\theta'$, resp.) belong to different components.

However, the direct analog of [7, thm. 1.3] is false; in fact we have:
PROPOSITION 2.7. If $U \in (\det)_0$ is non-constant, then the component of $UH_K^2$ in $(F, d)$ contains elements of $(\det)_0$.

Proof. If $U$ does not have the form $qX$ for some inner function $q$ and $X \in (\det)_0$, then for each $Y \in (\det)_0$ we write $U_Y(z) = YU(z)Y^*$. Clearly, $U_Y$ is arcwise connected with $U$ in $F$ and therefore $U_YU^2_K$ is arcwise connected with $UH_K^2$ in $(F, d)$; moreover $\det U = \det U_Y$.

Thus, it only remains to show that $Y + U_YH_K^2$ is not a constant mapping from $(\det)_0$ into $F$.

The set of all $\lambda \in \partial D$ such that the non-tangential limits $U(\lambda)$ are well-defined and unitary has Lebesgue measure $(dm)$ one; let $\Gamma = \Gamma(U)$ denote such set. Clearly, $\Gamma(U_Y) = \Gamma(U)$, for all $Y \in (\det)_0$.

Assume that $U_YH_K^2 = UH_K^2$ for all $Y \in (\det)_0$. Since $UH_K^2 = UXH_K^2$ for any $X \in (\det)_0$, without loss of generality we can assume that $U(\lambda_o) = I$, for some $\lambda_o \in \Gamma$. Then $U_YH_K^2 = UH_K^2$ implies that, for all $\lambda \in \Gamma$, $YU(\lambda) = U(\lambda)_X$, for some $X = X(Y) \in (\det)_0$, independent of $\lambda \in \Gamma$. For $\lambda = \lambda_o$, the above equality implies: $X(Y) = Y^* = Y^{-1}$, i.e., $U(\lambda)$ commutes with $Y$ for all $Y \in (\det)_0$. Therefore, $U(\lambda) = u(\lambda)I$ for some inner function $u$ and for $\lambda \in \Gamma$, contradicting our assumption about $U$. We conclude that $Y + U_YH_K^2$ is a non-constant map.

Finally, if $U = qX$, then $U = U_1U_2X$, where

$$U_1 = \begin{pmatrix} q & 0 \\ 0 & I_{N-1} \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} I_1 & 0 \\ 0 & qI_{N-1} \end{pmatrix} \quad (N = \dim K > 1)$$

and the result follows by applying the previous argument to $U_Y = (YU_1Y^*)U_2X$, $Y \in (\det)_0$. qed.

Replacing "Blaschke products" by "good inner functions", many of our results are still true for inner functions (or inner matrices) of several complex variables in the unit polydisc of $\mathbb{C}^n$. 
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Recibido en diciembre de 1974.