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A NOTE ON EMBEDDING OF FUZZY SETS IN A NORMED SPACE

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1. The idea of a fuzzy set was introduced in [1], with the intention of extending the concept of a set by introducing a membership function which is a generalization of the characteristic function.

It is natural to try to obtain some algebraic and topological structures for certain classes of fuzzy sets.

First of all, in this note we define a sum and a multiplication by scalars for some fuzzy sets, which are defined on a given linear space X. These operations are extensions of the usual sum and product by scalars for ordinary sets. For a class of fuzzy sets we introduce certain metrics which are pairwise uniformly equivalent. These metrics are a natural generalization of the Hausdorff metric. Finally, following an idea of Radstrom [2], it is possible to embed such a class in a cone of normed space.

2. Let X be a real linear space. We denote by A,B,C,... the *fuzzy sets* defined on X, which will be identified with their membership functions $A,B,C,\dots : X \longrightarrow [0,1]$. The extension of the usual operations between ordinary sets for fuzzy sets can be found in [1].

Let A and B be two fuzzy sets on X. Then we define their sum by

$$(A+B)(x) = \sup_{y\in X}\{\min[A(x-y),B(y)]\} \quad \text{for all } x \in X$$

and for any given real number λ we define λA

$$(\lambda A)(x) = \begin{cases} A(\frac{x}{\lambda}) & \text{if } \lambda \neq 0 \\ \chi_{\{\theta\}}(x) & \text{if } \lambda = 0 \end{cases} \quad \text{for all } x \in X$$

Here and in what is to follow χ denotes characteristic functions and θ is the zero element of the linear space X.

The following properties are immediate consequences of the above definitions:

a) $A + B = B + A$ b) $\lambda(A+B) = \lambda A + \lambda B$

c) $\mu(\lambda A) = (\mu\lambda)A$ d) $A + \chi_{\{\theta\}} = A$ e) $1.A = A$

From now on, we only consider a class of fuzzy sets such that the supremum in the definition of the sum of any two fuzzy sets in that class, is attained. Under such an assumption one can check easily that the following property holds:

f) $(A+B) + C = A + (B+C)$

In general, the equality $(\lambda+\mu)A = \lambda A + \mu A$ does not hold true even in the case of ordinary subsets of X. For ordinary subsets, it is well known that the equality remains to be true when λ and μ are real numbers having the same sign and when A is a convex set.

We say that A is a *quasi-concave* fuzzy set, if for each $\alpha \in (0,1]$ the ordinary sets $A_\alpha = \{x \in X: A(x) \geqslant \alpha\}$ is convex. If λ and μ have the same sign and A is a quasi-concave fuzzy set then the following expression holds true

g) $(\lambda+\mu)A = \lambda A + \mu A$

Proof. If $\mu.\lambda = 0$, it is immediate. Thus, we consider the case $\lambda.\mu > 0$. Because of the quasi-concavity for any pair $z, x \in X$

$$\min[A(\tfrac{z-x}{\mu}), A(\tfrac{x}{\lambda})] \leqslant A(\tfrac{z}{\lambda+\mu})$$

holds, since $\frac{z}{\lambda+\mu} = (\frac{\mu}{\lambda+\mu}) \cdot \frac{z-x}{\mu} + (\frac{\lambda}{\lambda+\mu}) \cdot \frac{x}{\lambda}$

Then $\lambda A + \mu A \leqslant (\lambda+\mu)A$.

On the other hand, for all $z, x \in X$

$$(\lambda A+\mu A)(z) \geqslant \min[A(\tfrac{z-x}{\mu}), A(\tfrac{x}{\lambda})].$$

If $x = \frac{\lambda z}{\lambda+\mu}$ we have the following inequality

$$(\lambda A+\mu A)(z) \geqslant A(\tfrac{z}{\lambda+\mu})$$

that is, $(\lambda A+\mu A) \geqslant (\lambda+\mu)A$, which guarantees the validity of g).

We note that the last inequality is true without any restriction on the fuzzy set A .

3. In order to prove the law of cancellation, we impose topological properties on the linear space X and the fuzzy sets defined on it. In what follows X will be a real normed space.

LEMMA 1. *Let* A,B,C *be fuzzy sets defined on* X *such that* B *is quasi-*

concave and continuous on the respective support. Let C *have bounded support,* $\max_{y \in X} A(y) \leqslant \max_{y \in X} C(y)$ *and* $A + C \leqslant B + C$. *Then,* $A \leqslant B$.

Proof. Let z and $\bar{z}$ be two points of X such that $A(z) > 0$ and $A(z) \leqslant C(\bar{z})$. From the following inequalities

$$A(z) \leqslant \sup_{y \in X} \{\min[A(z+\bar{z}-y), C(y)]\} \leqslant (B+C)(z+\bar{z})$$

we deduce that there exists an element $x_1 \in X$ such that

$$A(z) \leqslant B(x_1) \quad \text{and} \quad A(z) \leqslant C(z+\bar{z}-x_1)$$

By iteration, we can construct a sequence $\{x_n\}$ in X having the properties $A(z) \leqslant B(z_n)$ and $A(z) \leqslant C(nz+\bar{z} - \sum_{i=1} x_i)$.

Then, for all n, because of the quasi-concavity of the fuzzy set B it results

$$(*) \qquad A(z) \leqslant \min_{1 \leq i \leq n} B(x_i) \leqslant B(\frac{1}{n} \sum_{i=1} x_i)$$

Calling $y_n = nz + \bar{z} - \sum_{i=1}^{n} x_i$, since $C(y_n) > 0$, there exists a number $M > 0$ such that for all n, $\|y_n\| < M$. Thus, $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i = z$.

From the continuity of the fuzzy set B on its support, we have $A(z) \leqslant B(z)$. (Q.E.D.)

Let us denote by $\mathcal{D}$ a class of quasi-concave fuzzy sets contained in the functional space $[0,1]^X$, with bounded supports, continuous in their supports and having their maxima equal to 1. Then, from the previous result we obtain the following

COROLLARY. *If* $A, B \in \mathcal{D}$ *then* $A + B = A + C$ *implies* $B = C$.

Indeed, the condition that the maxima of these functions be 1 could be weakened by the assumption that they be equal to some $\alpha > 0$.

4. We will now introduce a metric on $\mathcal{D}$, in such a way that on $\mathcal{D} \cap 2^X$ it coincides with the well known Hausdorff metric [3]. Here 2^X denotes the space of all characteristic functions on X.

Consider $A, B \in \mathcal{D}$ and $\lambda \geqslant 0$. Let S be the closed unitary ball in X, and define the distance $d(A,B) = d_{\chi_S}(A,B)$ between A and B to be the infimum of those λ such that both $A + \lambda\chi_S \geqslant B$ and

$B + \lambda\chi_S \geqslant A$ hold.

One can easily check that, indeed this defines a metric on $\mathcal{D}$. The existence of a $\lambda \geqslant 0$ satisfying both requirements follows from the fact that the functions have bounded supports. In addition to χ_S there are other fuzzy sets $H \in \mathcal{D}$ for which the analogous construction yields a metric which we denote by d_H. All these metrics d_H are uniformly equivalent.

LEMMA 2. *If* A *and* B *are quasi-concave and* $\lambda \geqslant 0$, *then* $A + B$ *and* λA *are quasi-concave.*

Proof. Let α be an element of $(0,1]$. Consider $z, \overline{z} \in X$ such that

$$(A+B)(z) \geqslant \alpha \quad \text{and} \quad (A+B)(\overline{z}) \geqslant \alpha$$

Then there exist two elements x and $\overline{x}$ belonging to X with the properties

$$\min[A(z-x), B(x)] \geqslant \alpha \quad \text{and} \quad \min[A(\overline{z}-\overline{x}), B(\overline{x})] \geqslant \alpha$$

If we indicate by x_ρ the point $\rho x + (1-\rho)\overline{x}$ with $0 \leqslant \rho \leqslant 1$, we have

$$\min[A(\rho z+(1-\rho)\overline{z}-x_\rho),\ B(x_\rho)] \geqslant \alpha$$

which implies $(A+B)(\rho z+(1-\rho)\overline{z}) \geqslant \alpha$. The remaining proof for λA is immediate.

(Q.E.D.)

Along the lines of Lemma 3 in [2], one can derive the following

LEMMA 3. *Let* A,B,C *be elements of* $\mathcal{D}$ *such that* $A + C$ *and* $B + C$ *belong to* $\mathcal{D}$ *and the fuzzy sets* $A + \lambda\chi_S$ *and* $B + \lambda\chi_S$ *are continuous in their support, for all* $\lambda \geqslant 0$. *Then* $d(A,B) = d(\mu A,\mu B)$ *for all* $\mu \geqslant 0$.

We now present the following result relating to topological properties.

LEMMA 4. *The operations of sum and product by scalars are continuous on* $\mathcal{D}$ *with respect to the introduced metric.*

Proof. Clearly, $d(A_n,A) \to 0$ and $d(B_n,B) \to 0$ implies $d(A_n+B_n,A+B) \to 0$.

Let $\lambda_u \to \lambda$ and $A_u \to A$ be two convergent sequences. Let $\varepsilon > 0$ be arbitrary. Because the product by scalars is continuous in X one can immediately obtain for all n bigger than a certain n_0:

$$\lambda_n A + \varepsilon\chi_S \geqslant \lambda A \quad \text{and} \quad \lambda A + \varepsilon\chi_S \geqslant \lambda_u A$$

Since $d(A_n, A) \to 0$, for all $n \geqslant n_1$ we have $A_n + \varepsilon\chi_S \geqslant A$ and $A + \varepsilon\chi_S \geqslant A_n$, from which we derive the following expressions:

$$\lambda A \leqslant \varepsilon\chi_S + \lambda_n\varepsilon\chi_S + \lambda_n A_n \qquad \text{and}$$

$$\lambda_n A_n \leqslant \lambda_n(A+\varepsilon\chi_S) \leqslant \lambda A + \varepsilon\chi_S + \lambda_n\varepsilon\chi_S$$

Since M is a bound for all $\lambda_n; n \geqslant n_o, n_1$, we have the inequalities

$$\lambda_n A_n \leqslant \varepsilon(M+1)\chi_S + \lambda A \quad \text{and} \quad \lambda A \leqslant \varepsilon(M+1)\chi_S + \lambda_n A_n$$

which implies that $d(\lambda_n A_n, \lambda A) \to 0$. (Q.E.D.)

LEMMA 5. *Let* A *and* B *have compact supports and* $\lambda \geqslant 0$, *then* A + B *and* λA *have compact supports.*

Proof. This follows from the next following relations between the supports:

$$\operatorname{supp}(A+B) = \operatorname{supp} A + \operatorname{supp} B \quad \text{and} \quad \operatorname{supp}(\lambda A) = \lambda \operatorname{supp} A \qquad \text{(Q.E.D.)}$$

Finally we will need a closedness property.

LEMMA 6. *If* A *and* B *have compact supports and are continuous there on* $\lambda \geqslant 0$, *then* A + B *and* λA *are continuous on their supports.*

Proof. The result holds for λA since it is the composition of two continuous mappings on $\operatorname{supp}(\lambda A)$. On the other hand, the remaining conclusion follows from the fact that $(A+B)(z)$ can be obtained by considering the supremum on $x \in \operatorname{supp} B$ of the composition of the following continuous functions:

$$\operatorname{supp}(A+B) \times \operatorname{supp} B \to \operatorname{supp} A + \operatorname{supp} B$$

given by $$(x,z) \mapsto (z-x, x),$$

$$\operatorname{supp} A + \operatorname{supp} B \to [0,1]^2$$

given by $$(x,y) \mapsto (A(x), B(y)),$$

and $$[0,1]^2 \to [0,1]$$

given by $$(r,s) \mapsto \min(r,s).$$ (Q.E.D.)

Similarly one can obtain the following:

LEMMA 7. *If* A *has compact support and it is continuous there on, then* $A + \chi_S$ *is continuous on its support.*

All the above material together with the theorems 1 and 2 of [2] is condensed in the next result.

THEOREM. *If* X *is a real normed linear space and* $E \subset \mathcal{D}$ *is such that all its members have compact supports, then* E *can be embedded as a cone in a real normed linear space.*

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