INTERPOLATION BETWEEN TWO PUTNAM'S INEQUALITIES

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1. Let $T = H + iJ$ be the cartesian decomposition ($H = \text{Re } T$, $J = \text{Im } T$) of the (bounded linear) operator $T$ acting on a complex separable Hilbert space $H$. $T$ is called \textit{hyponormal} if its self-commutator

$$D = T^*T - TT^* = 2i(HJ - JH)$$

is a positive semi-definite hermitian operator.

Let $\Lambda(T)$ denote the spectrum of $T$ and let $m_1$ and $m_2$ denote the linear Lebesgue measure (on a given line) and planar Lebesgue measure on the complex plane $C$, respectively. C.R. Putnam ([5]) proved the following

(1) $\pi |D| \leq m_2[\Lambda(T)]$ \hspace{1em} (Putnam's inequality).

If $H$ has simple spectrum (in the sense of [3]), then (1) can be sharpened to (see [6])

(2) $\pi |D|_1 \leq m_2[\Lambda(T)]$

where $|D|_p$, $1 \leq p \leq \infty$, denotes the $p$-norm of an element of the Schatten's bilateral ideal $C_p$ of compact operators on $H$ ($C_1$ denotes \textit{trace class} operators, $C_2$ denotes the class of Hilbert-Schmidt operators, $C_\infty$ is the ideal of all \textit{compact} operators, etc., The reader is referred to [8] for the definition and properties of these ideals).

Actually, Putnam has also obtained the following improvement of (2): Let $H = \int \lambda \ dE_\lambda = \bigoplus \int_{B_n} \lambda \ dE_\lambda$ be the spectral decomposition of $H$, where $\Lambda(H) = \bigcup \{B_n : 1 \leq n \leq n_0\}$ is the (essentially unique) decomposition of the spectrum of $H$ into pairwise disjoint Borel subsets such that $H_n = \int_{B_n} \lambda \ dE_\lambda$ is an hermitian operator of \textit{uniform} spectral multiplicity $n$ on the space $E(B_n)H$ (see [3]) and let $F(t) = m_1[\Lambda(T) \cap \{z : \text{Re } z = t\}]$; then

(3) $\pi |D|_1 \leq \sum (1 \leq n \leq n_0) \int_{B_n} F(t) \ dt.$
Clearly, we can restrict our attention to the case when $B_{N_0} = \phi$ (see [7]).

In this note, the following two results will be proven:

**THEOREM 1.** Let $T = H + iJ$ be a hyponormal operator such that $B_{N_0} = \phi$. Then $D \in C_\infty$.

**THEOREM 2.** (Interpolation theorem) Let $T$ be as above. Then

\[ \pi|D|^p_p \leq \sum_{n=1}^\infty \left( \int_{B_n} F(t) dt \right)^{1/p} \left( \int_{B_n} F(t) dt \right)^{1/q} \]

for all $p$, $1 < p < \infty$, where $q = p/(p-1)$.

(ii) Let $0 < p < 1$. If $|D|^p_p$ denotes the invariant metric of the ideal $C_p$, then

\[ \pi^p|D|^p_p \leq \sum_{n=1}^\infty \left( \int_{B_n} F(t) dt \right)^p \leq \sum_{n=1}^\infty \left( \int_{B_n} F(t) dt \right)^p \]

**COROLLARY.** Let $T$ be as above and assume that the left spectrum coincides with the right one. Then there exist a normal operator $N$ and a sequence $(F_n)_{n=1}^\infty$ of finite rank operators such that $A(N) = A(T)$, $-1 \notin A(F_n)$ (for $n = 1, 2, \ldots$) and

\[ \lim_{n \to \infty} \|T - (I + F_n)N(I + F_n)^{-1}\| = 0. \]

The proof follows from Theorem 1 and [4, §2].

**REMARKS.** (a) The condition $B_{N_0} = \phi$ is sufficient, but not necessary. Indeed, a concrete example of a completely hyponormal operator $T$ (i.e., there is no non-zero reductive subspace $M$ such that the restriction $T|M$ is normal in $M$) such that $B_{N_0} = \Lambda(H)$ and $B_n = \phi$ (for $n = 1, 2, \ldots$), but $D \in C_1$ can be found in [1]. Theorem 1 affirmatively answers the author's Conjecture (b) of [4].

(b) Example. Let $J$ be an arbitrary bilateral ideal properly contained in $C_\infty$. Then there exists a sequence $(r_n)_{n=1}^\infty$ decreasing to 0 such that no (necessarily compact!) hermitian operator of the form $L = \sum_{n=1}^\infty r_n \varphi_n \otimes \varphi_n$ (where $(\varphi_n)$ is a suitable orthonormal system of $H$ and the operator $\varphi \otimes \varphi$ is defined by $\varphi \otimes \varphi(\psi) = (\psi, \varphi)\varphi$; see [8]) belongs to $J$. Let $c_n$ be the positive square root of $r_n$ and set $T = \sum_{n=1}^\infty c_n S$, where $S$ denotes the unilateral shift; then
\[ D = T^*T - TT^* = \sum_n c_n^2 \{ S^*S - SS^* \} = \sum_n r_n \varphi_n \otimes \varphi_n \]
does not belong to \( J \). In this example, \( B_n = [-c_n, c_n] \setminus [-c_{n+1}, c_{n+1}] \) \((n = 1, 2, \ldots)\).

2. PROOF OF THEOREM 1.

Let \( E_N = E( \cup B_n) \). Following \([7]\), we have
\[
D = D^{1/2}E D^{1/2} = D^{1/2}E_N D^{1/2} + D^{1/2}(I - E_N) D^{1/2}
\]
and
\[
|D^{1/2}E_N D^{1/2}|_1 = \sum_n |D^{1/2}E(B_n) D^{1/2}|_1 = \sum_n E(B_n) DE(B_n)|_1 \leq \sum_n (n/\pi) \int_{B_n} F(t) \, dt < \infty
\]
and therefore \( D^{1/2}E_N D^{1/2} \in C_1 \subset C_\omega \).

On the other hand, \([5; 6; 7]\)
\[
\|D^{1/2}(I - E_N) D^{1/2}\| = \| (I - E_N) D(I - E_N) \| < (1/\pi) m_2 (\Lambda(T)) \sum_{n=1}^N \| B_n \| + 0
\]
as \( N \to \infty \), because \( m_2 (\Lambda(T)) < \infty \) and \( m_2 (B_n) = 0 \).

We conclude that \( D \) is the norm limit of a sequence \( \{D_n\} \) of compact operators and therefore, \( D \) itself is compact.

3. PROOF OF THEOREM 2.

(i) As in \([7]\), \( D = \sum_{n=1}^{\infty} D^{1/2} E(B_n) D^{1/2} = \sum_n D_n \) where
\[
|D_n|_1 = |E(B_n) DE(B_n)|_1 \leq \frac{n}{\pi} \int_{B_n} F(t) \, dt.
\]
On the other hand, Putnam's inequality \((1)\) implies that
\[
|D_n| = |E(B_n) DE(B_n)| \leq \frac{1}{\pi} \int_{B_n} F(t) \, dt
\]
so that we can interpolate the \( C_p \)-norm between \((4)\) and \((5)\) in order to obtain
\[
\pi |D_n|_p \leq \pi |D_n|_1^{1/p} |D_n|_{1/d}^{1/d} \leq n^{1/p} \int_{B_n} F(t) \, dt \leq n^{1/p} \int_{B_n} F(t)^p \, dt.
\]
\[ .[m_1(B_n)]^{1/q} , \quad \text{for all} \ p, 1 < p < \infty \ (\text{see} \ [8]). \]
Therefore
\[ \pi |D|_p \leq \pi \sum_n |D_n|_p \leq \sum_n n^{1/p} \int_{B_n} F(t) \, dt \leq \sum_n \left( \int_{B_n} n F(t)^p \, dt \right)^{1/p} \left( m_1(B_n) \right)^{1/q} \leq (m_1(\Lambda(H)))^{1/q} \sum_n \int_{B_n} n F(t)^p \, dt^{1/p}. \]

(ii) For \( 0 < p < 1 \), we have
\[ \pi^p |D|_p^p \leq \pi^p \sum_n |D_n|_p^p \leq \pi^p \sum_n |D_n|_p \leq \sum_n n^p \left( \int_{B_n} F(t) \, dt \right)^p \leq \sum_n n^p \int_{B_n} F(t)^p \, dt. \]

The author suggests that analogous results should be true for the trace estimates of C.A. Berger and B.I. Shaw ([1;2]).

REFERENCES


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