ABSTRACT. Meromorphic differential operators on a reduced locally irreducible complex analytic space are studied in this paper. Conditions are established for regularity of such operators.

1. INTRODUCTION.

This section contains some basic definitions and facts about meromorphic functions on complex analytic spaces.

Let $X$ be a connected complex analytic space and $\mathcal{O}_X$ be the sheaf of germs of analytic functions on $X$. It will be assumed that $X$ is reduced and locally irreducible. This means that each $x \in X$ has arbitrarily small neighborhoods $U$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to a ringed space $(V, \mathcal{O}_W/d_V)$ where $W$ is the unit polydisc in $\mathbb{C}^N$ for some $N$, $V$ is a closed irreducible subvariety of $W$, and $d_V$ is the sheaf of ideals of analytic functions vanishing on $V$. In this case the stalks $\mathcal{O}_{X,x}$ are integral domains and $\mathcal{M}_X$, the sheaf of quotients of $\mathcal{O}_X$, is a sheaf of fields. $\mathcal{M}_X$ is the sheaf of germs of meromorphic functions on $X$. $\mathcal{O}_X$ can be considered a subsheaf of $\mathcal{M}_X$.

The results in this paper are of a local nature and the proofs are most conveniently carried out for germs. If $X$ is an analytic space or a subvariety of an analytic space and $x \in X$, then $X_x$ denotes the germ of $X$ at $x$. For $f \in \mathcal{O}_{X,x}$, let $f(x)$ denote the value of $f$ at $x$, that is, the residue class modulo the maximal ideal. For a function $m \in \mathcal{M}(U,\mathcal{M}_X)$ and a point $x \in U$, let $m_x$ denote the germ of $m$ at $x$.

Let $U$ be an open subset of $X$ and $m \in \mathcal{M}(U,\mathcal{M}_X)$. The singular set of $m$, $\text{sing}(m)$, is defined to be $\{x \in U: m_x \notin \mathcal{O}_{X,x}\}$. Since $m_x \in \mathcal{M}_{X,x}$ can be represented (not necessarily uniquely) by $m_x = f/g$ where $f, g \in \mathcal{O}_{X,x}$, $x$ will be in $\text{sing}(m)$ if and only if $g(x) = 0$ for all possible representations $m_x = f/g$. Because of this, $\text{sing}(m)$ is an analytic subvariety of $U$. A germ $m_x \in \mathcal{M}_{X,x}$ can always be represented by a function $m$ in a neighborhood of $x$, so $\text{sing}(m_x)$
can be defined as \( \text{sing}(m)_x \).

A more thorough discussion of meromorphic functions can be found in Narasimhan [2], p. 88.

Certain facts about extensions and restrictions of holomorphic differential operators will be assumed; these can be found in [1].

2. MEROMORPHIC DIFFERENTIAL OPERATORS.

Let \( \text{Diff}^n_x \) be the sheaf of germs of holomorphic differential operators of order \( n \) on \( X \). \( \text{Diff}^n_x \) is a coherent sheaf of \( \mathcal{O}_x \)-modules.

A holomorphic differential operator \( D \in \Gamma(X, \text{Diff}^n_x) \) can be considered to be a \( \mathbb{C} \)-homomorphism \( D: \mathcal{O}_X \rightarrow \mathcal{O}_X \) or a \( \mathbb{C} \)-linear sheaf homomorphism \( D: \mathcal{O}_X \rightarrow \mathcal{O}_X \). A germ of a differential operator, \( D_x \in \text{Diff}^n_{x,x} \), defines a \( \mathbb{C} \)-homomorphism \( D: \mathcal{O}_{x,x} \rightarrow \mathcal{O}_{x,x} \).

The sheaf of germs of \( n \)-th order meromorphic differential operators on \( X \) is defined to be \( \mathcal{M}_x \otimes \mathcal{O}_x \text{Diff}^n_x \). An \( n \)-th order meromorphic differential operator on \( X \) is a section \( D \in \Gamma(X, \mathcal{M}_x \otimes \mathcal{O}_x \text{Diff}^n_x) \) and can be considered to be a \( \mathbb{C} \)-homomorphism \( D: \Gamma(X, \mathcal{O}_x) \rightarrow \Gamma(X, \mathcal{M}_x) \) or a \( \mathbb{C} \)-linear sheaf homomorphism \( D: \mathcal{O}_X \rightarrow \mathcal{M}_x \). A germ \( D_x \in (\mathcal{M}_x \otimes \mathcal{O}_x \text{Diff}^n_x)_{x,x} \) defines a \( \mathbb{C} \)-homomorphism \( D: \mathcal{O}_{x,x} \rightarrow \mathcal{M}_{x,x} \).

\( \text{Diff}^n_x \) can be considered a subsheaf of \( \mathcal{M}_x \otimes \mathcal{O}_x \text{Diff}^n_x \).

A meromorphic differential operator \( D \) on \( X \) is singular at \( x \in X \) if its germ at \( x \) is not in \( \text{Diff}^n_{x,x} \). We denote the set of such singular points by \( \text{sing}(D) \). The set \( \text{sing}(D) \) is a subvariety of \( X \). For a germ \( D_x \in (\mathcal{M}_x \otimes \mathcal{O}_x \text{Diff}^n_x)_{x,x} \), \( \text{sing}(D_x) \) can be defined as was done for meromorphic functions.

**Proposition 2.1.** Let \( X \) be an analytic space and \( D \) be a meromorphic differential operator on \( X \). If \( D(\mathcal{O}_x) \subset \mathcal{O}_x \) then \( D \) is holomorphic.

**Proof.** Let \( n \) be the order of \( D \). It suffices to prove this proposition locally, so it can assumed that \( X \subset \mathbb{C}^N \) as a closed analytic subvariety. For \( x \in X \), \( \text{Diff}^n_{x,x} \) is finitely generated as an \( \mathcal{O}_{x,x} \)-module since \( \text{Diff}^n_X \) is coherent.

Let \( D \) denote the germ of \( D \) at \( x \). Then

\[ D = m_1 D_1 + \ldots + m_k D_k \]

where \( D_1, \ldots, D_k \in \text{Diff}^n_{x,x} \) are the generators at \( x \) and \( m_1, \ldots, m_k \in \mathcal{M}_{x,x} \). Each \( m_i \) can be represented by \( m_i = f_i/g_i \) with \( f_i, g_i \in \mathcal{O}_{x,x} \). Let \( g = g_1 \ldots g_k \in \mathcal{O}_{x,x} \). Then the operator

\[ D' = g D = f_1 D_1 + \ldots + f_k D_k \in \text{Diff}^n_{x,x} \]
and \( D = 1/g \, D' \). Let \( m = 1/g \in M_{x,x} \).

\( g \) can be extended to \( \tilde{g} \in C^{N,x} \) and thus \( m \) can be extended to \( \tilde{m} = 1/\tilde{g} \in C^{N,x} \).

Furthermore, \( D' \) can be extended to \( \tilde{D}' \in \text{Diff}^n \)

and the operator \( \tilde{D} = \tilde{m} \, \tilde{D}' \) is \( (M_{x,x} \circ C^{N})_x \).

It is clear that \( \text{sing}(\tilde{D}) \subseteq \text{sing}(\tilde{m}) \) and since \( \tilde{m} \) is an extension of \( m \in M_{x,x} \), \( \text{sing}(\tilde{m}) \subseteq X_x' \).

Let \( \tilde{\varphi} \in 0^{C_N} \) and let \( \varphi = \tilde{\varphi}|x \in 0_{x,x} \).

By hypothesis \( D(\varphi) = m \, D'(\varphi) \in 0_{x,x} \).

Furthermore, \( \tilde{D}(\tilde{\varphi}) \in M_{x,x} \)

and \( \tilde{D}(\tilde{\varphi})|x \in M_{x,x} \) because \( \text{sing}(\tilde{D}(\tilde{\varphi})) \subseteq \text{sing}(\tilde{D}) \) and \( \text{sing}(\tilde{D}) \subseteq X_x \).

Now, \( \tilde{D}(\tilde{\varphi})|x = [\tilde{m} \, \tilde{D}'(\tilde{\varphi})]|x = \tilde{m}|x \, \tilde{D}'(\tilde{\varphi})|x = m \, D'(\varphi) = D(\varphi) \) so if \( \tilde{\varphi} \in 0^{C_N}_x \), \( \tilde{D}(\tilde{\varphi})|x \in 0_{x,x} \).

\( \tilde{D}' \) can be expressed in terms of the generators of \( \text{Diff}^n \) as

\[
\tilde{D}' = \sum a_i \frac{i_1 + \ldots + i_N}{\partial i_1 \ldots \partial i_N}.
\]

with \( a_i \in 0^{C_N,x} \), so

\[
\tilde{D} = \sum \tilde{m} \, a_i \frac{i_1 + \ldots + i_N}{\partial i_1 \ldots \partial i_N}.
\]

The functions \( z_1 \ldots z_N \) are all in \( 0^{C_N,x} \), so

\[
\tilde{D}(z_1 \ldots z_N)|x \in 0_{x,x}. \]

Since all the \( \tilde{m} \, a_i \ldots i_N \) are linear combinations of the \( \tilde{D}(z_1 \ldots z_N) \), \( \tilde{m} \, a_i \ldots i_N |x \in 0_{x,x} \)

Let \( b_i \ldots i_N = (\tilde{m} \, a_i \ldots i_N)|x \in 0_{x,x} \).

Let \( \tilde{b}_i \ldots i_N \) be an extension of \( b_i \ldots i_N \) to \( 0^{C_N,x} \).

Let

\[
\tilde{D} = \sum \tilde{b}_i \ldots i_N \frac{i_1 + \ldots + i_N}{\partial i_1 \ldots \partial i_N}.
\]

The operators \( \tilde{D} \) and \( \tilde{D} \) agree on \( X_x \), so for \( \psi \in 0^{C_N,x} \)

\[
\tilde{D}(\psi)|x = \tilde{D}(\psi)|x.
\]

If \( \psi \in d_{x,x} \) then \( \psi|x = 0 \in 0_{x,x} \) so \( \tilde{D}(\psi)|x = 0 \).
Thus $\overline{D}(\psi)|_{X} = 0$ and since $\overline{D}$ is a holomorphic operator, 
$\overline{D}(d_{x,x}) \subset d_{x,x}$ and $\overline{D}$ induces a holomorphic differential operator 
on $X$. Since $\overline{D}$ and $\overline{D}$ agree on $X$, the induced operator must coincide with $D$ and thus $D$ is holomorphic.

3. MEROMORPHIC OPERATORS WITH STABLE IDEALS.

Let $D$ be a differential operator on an analytic space $X$ and let $V$ be an analytic subvariety of $X$ with associated sheaf of ideals $d_{\nu}$.

**DEFINITION 3.1.** $d_{\nu}$ is stable under $D$ if for all $x \in V$,
$D(d_{\nu,x}) \subset d_{\nu,x}$.

Let $V$ and $W$ be analytic subvarieties of $X$.

**DEFINITION 3.2.** $V$ is transversal to $W$ if there is an $x \in V \cap W$ 
such that $V_{x} \not\subset W_{x}$ and $W_{x} \not\subset V_{x}$ (in which case $V$ is transversal to
$W$ at $x$).

The proposition and corollary that follow deal with varieties transversal to $\text{sing}(D)$.

**PROPOSITION 3.3.** Let $D$ be a meromorphic differential operator of order $n$ on $X$. Let $x \in X$. Suppose that there is a germ $f \in O_{x}$, $f \neq 0$, such that for all $g \in O_{x}$, $D(gf) \in O_{x}$.

Then for all $g \in O_{x}$, there is a representation $D(g) = \frac{\varphi}{f^{n+1}}$ where $\varphi \in O_{x}$.

**Proof.** The proof is by induction on $n$, the order of $D$.

For $n=1$, $D = D' + h$ where $D'$ is a meromorphic derivation and $h \in M_{x,x}$. For $g \in O_{x,x}$
$D(g) = D'(g) + h(g)$.

But $g D'(fg) \neq g D(fg) \in O_{x}$, so $f D'(g) = k \in O_{x}$, and $D'(g) = k/f$.

Now $D(f^{2}) = 2f D'(f) + hf^{2} \in O_{x,x}$.

But $2f D'(f) \in O_{x}$ as above, so $hf^{2} = k' \in O_{x,x}$ and $h = \frac{k'}{f^{2}}$.

Therefore $D(g) = D'(g) + hg = \frac{k}{f} + \frac{k'g}{f^{2}} = \frac{\varphi}{f^{2}}$ where $\varphi \in O_{x,x}$.

Now suppose that order $D = n$ and that the proposition is proved for differential operators of order $< n - 1$. 
Let \( g \in O_{x,x} \). The operator \( D_f \) defined by
\[
D_f(g) = D(fg) - f D(g)
\]
has order \( \leq n-1 \) and satisfies the hypothesis \( D_f(gf) \in O_{x,x} \) for all \( g \in O_{x,x} \). Therefore
\[
D_f(g) = \frac{\psi}{f^{n+1}} \text{ with } \psi \in O_{x,x}.
\]

Now \( D_f(g) = D(fg) - f D(g) \) with \( D(fg) = k \in O_{x,x} \) so
\[
f D(g) = k - \frac{\psi}{f^n}.
\]
Thus \( D(g) = \frac{\varphi}{f^{n+1}} \) where \( \varphi = kf^n - \psi \in O_{x,x} \).

**COROLLARY 3.4.** Let \( D \) be a meromorphic differential operator on \( X \). Let \( V \) be an analytic subvariety with sheaf of ideals \( d_V \). If \( d_V \) is stable under \( D \) then \( \text{sing}(D) \) is not transversal to \( V \).

**Proof.** By hypothesis, for all \( x \in V \), \( D(d_V,x) \subset d_V,x \). Then for \( f \in d_V,x \) and \( g \in O_{x,x} \), \( D(gf) \in O_{x,x} \) so by Proposition 3.3 \( D(g) = \frac{\varphi f^{-n}}{f^{n+1}} \) with \( \varphi \in O_{x,x} \). Thus \( d_V,x \subset d_{\text{sing}(D),x} \) so \( V_x \supset \text{sing}(D_x) \).

Therefore \( \text{sing}(D) \) is not transversal to \( V \) at \( x \), and this holds for all \( x \in V \).

**REFERENCES**


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