In this short Note we intend to give a slightly different proof, from the known ones, of the simplicity of the alternating group $A_n$, $n \geq 5$. The proof results by a direct application of Sylow theorems. In the literature on the subject this is always done before proving Sylow theorems. Although, perhaps, this should be (or not) the case, we found very instructive and natural proceed "via" the Sylow theorems. The proof resulted in a course taught at the I.M.A.F. of the Universidad Nacional de Córdoba, in trying to give a simple proof of the non-existence in $A_4$ of subgroups or order 6. In 1. we recall this proof and using same ideas we give in 2. the proof of the simplicity of $A_n$, $n \geq 5$.

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1. NON EXISTENCE IN $A_4$ OF SUBGROUPS OF ORDER 6.

In fact, let $H$ be a subgroup of $A_4$ of order 6. Then $H$ is an invariant subgroup. Let $H_3$ be a 3-sylow subgroup of $H$. Since $A_4$ has order $12 = 3 \times 4$, $H_3$ is also a 3-sylow subgroup of $A_4$. Notice that every tricycle $(abc)$ in $A_4$ generates a 3-sylow subgroup. By the conjugacy of sylow subgroups it follows that every tri-cycle is conjugated to an element of $H_3$ and "a fortiori" of $H$. But since $H$ is invariant in $A_4$ we conclude at once, that $H$ contains all the tricycles in $A_4$. These are 8, so $H$ has order greater than 8, a contradiction. This proves our claim.

2. SIMPLICITY OF $A_n$, $n \geq 5$.

We shall proceed inductively in $n$. We first prove it for $n = 5$ and next we prove for $n = 6$. The main ideas of the proof shall be taken from the case $n = 6$.

a) Simplicity of $A_5$.

Let $H$ be an invariant subgroup of $A_5$. Let $h$ be the order of $H$, $1 < h$.

We distinguish the following situations:

1) $5$ divides $h$. Therefore $H$ contains a 5-sylow subgroup which is also a sylow subgroup of $A_5$. Thus by the argument used in 1. it follows that $H$ contains all the elements of $A_5$ of order 5. These are $4! = 24$. 

By divisibility reasons H must have order 30 or 60. H contains therefore a 3-sylow subgroup (which is also a 3-sylow subgroup of $A_5$) and hence H contains the $\binom{5}{3}.2! = 20$ tricycles. This clearly implies that $H = A_5$.

ii) 3 divides h. By the same argument above, H contains the 20 tricycles of $A_5$, so its order might be 20, 30 or 60. But then 5 divides the order of H, and we are consequently in situation i). So again $H = A_5$.

iii) H has order a power of 2. That is to say, H has order 2 or 2^2. In the first case it follows that $A_5$ has center # (1). It is easy to see that this is not so. So, H has order 4. This implies that H is a 2-sylow subgroup. Since any element of $A_5$ of order 2 is contained in a 2-sylow subgroup H must contain the $\frac{1}{2} \binom{5}{3} \cdot \binom{3}{2} = 15$ elements of order 2 in $A_5$, a nonsense. This concludes the proof of the simplicity of $A_5$.

b) Simplicity of $A_6$.

Assume the simplicity of $A_5$. Let us fix some notation. Let for any index $i$, $i = 1, 2, 3, 4, 5, 6$, $A^i_5$ denote the alternanting group in the letters $1, \ldots, i, \ldots, 6$ where 1 means that the index i should be excluded. We identify the $A^i_5$ 's to the corresponding subgroups of $A_6$. Let H be an invariant subgroup of $A_6$, $H \neq (1)$. Then H behaves respect to each $A^i_5$ as follows:

$$ (1) = H \cap A^i_5 \text{ or } A^i_5 \subset H. $$

This clearly is consequence of the simplicity of $A_5 = A_5$. Assume that for some index i, $A^i_5$ is contained in H. So 5 divides the order of H and by the usual argument, H contains all the 5-sylow subgroups of $A_6$ or the same, H contains all the elements of $A_6$ of order 5. These are $\binom{6}{3}.4! = 144$ elements. But as $A_6$ has order $\frac{5}{2}.6! = 360$, we conclude, by looking the divisors of 360, that H should have order 180 or 360. In any case this would imply that 5-sylow subgroups of $A_6$ would be in H. In particular all the elements of order 5 should be in H. These are as many as $\binom{6}{3}.2! + \frac{1}{2} \binom{6}{3}.2!.2! = 80$. In conclusion H contains at least $144 + 80 = 224$. H = $A_6$ is the only possible case.

Therefore $$ A^i_5 \cap H = (1) \text{ for all } i, i = 1, 2, \ldots, 6 \text{ holds.} $$

Call $A = \cup_{i=1}^6 A^i_5$. Then $H \cap A = (1)$. It follows that the elements of H different from the identity, must be representable as product of disjoint cycles involving all the letters 1, 2, 3, 4, 5, 6. Let $x \in H$ be an element of prime order p. Assume $p = 2$. Then x has the following representation

$$ x = (ab).cd)(ef) $$

with all distinct letters. But then x is odd, so $x \notin A_6$ a contradiction.
So $p \neq 2$. Notice that $p = 5$ is impossible, since any element of $A_6$ of order 5 is a cycle (abcde) omitting one letter, so cannot be in $H$. We are therefore reduced to study the case $p = 3$. More precisely we have to consider the case where $H$ has order a power of 3. According with the order of $A_6$, the order of $H$ can be 3 or $3^2$. In case $3^2$, $H$ would be a Sylow subgroup, so $H$ would contain all the elements of $A_6$ of order 3, so as many as 80 elements. So $H$ might have order 3. Then would be generated by an element of the form

$$x = (abc).(efg)$$

with all distinct letters. But clearly $(abe).x.(abe)^{-1} = (bec).(afg)$ does not belong to $H$. We have proved the simplicity of $A_6$.

a) Simplicity of $A_n$, $n > 6$.

Let $i,j$ be two indices in the natural interval $I_n = \{1,2,3,\ldots,n\}$, $i \neq j$. We call $A_{n-2}^{i,j}$ the alternating group in the letters $1,\ldots,i,\ldots,j,\ldots,n$ with $i$ and $j$ omitted, included in $A_n$. Let $H$ be an invariant subgroup of $A_n$ and $H \neq (1)$. As in b) we have

$$(1) = H \cap A_{n-2}^{i,j} \text{ or } A_{n-2}^{i,j} \subset H.$$  

Assume that some $A_{n-2}^{i,j}$ is contained in $H$. We claim that all the $A_{n-2}^{r,s}$ are in $H$. In fact, let $r,s$ be a pair of indices in $I_n$, $r \neq s$.

If $r = i$, $s = j$, then $A_{n-2}^{i,j} \subset H$. If $r = i$, $s \neq j$ then

$$(jsr).A_{n-2}^{r,s}.(jsr)^{-1} = A_{n-2}^{i,j}$$

and therefore $A_{n-2}^{r,s} \subset H$. Assume $r \neq i,j$ and $s \neq i,j$. Then

$$(js)(ir).A_{n-2}^{r,s}.((js)(ir))^{-1} = A_{n-2}^{i,j}.$$  

So again $A_{n-2}^{r,s} \subset H$. Hence $H$ contains all the $A_{n-2}^{i,j}$, which implies that $H$ contains all the tri-cycles of $A_n$. Since $A_n$ is generated by tri-cycles, we conclude that $H = A_n$.

Let $A_{n-1}$ denote the alternating group in the letters $1,\ldots,i,\ldots,n$. Then by the simplicity of $A_{n-1} = A_{n-1}^i$, we have that $H \cap A_{n-1}^i = (1)$ or $A_{n-1}^i \subset H$. The latter is impossible since $A_{n-1}^i$ contains all the $A_{n-2}^{i,j}$. Consequently any element of $H$ is representable as a product of disjoint cycles involving all the letters $1,2,\ldots,n$. Let $x \in H$ be an element of prime order $p$. As in b) we can exclude the case $p = 2$. We have to analyze only two possible representations of $x$ as product of disjoint cycles. Namely

i) $x = (a_1\ldots a_p).(b_1\ldots b_p)\ldots(c_1\ldots c_p)$; $p < n$,

ii) $x = (a_1\ldots a_p)$

Assume i). In case $p > 3$ we can choose an element $y$, in the alternating group in the letters $a_1,\ldots,a_p$ such that $1 \neq [y,(a_1,\ldots,a_p)]$. Therefore

$$1 \neq [y,(a_1,\ldots,a_p)] = [y,x] \in H.$$
a contradiction, since \([y,x]\) involves at most \(p < n\) letters. Let \(p=3\). Since \(n > 6\), \(x\) contains at least 3 tricycles. We then repeat the previous argument by choosing an element \(y\) in the alternating group in the letters \(a_1, a_2, a_3, b_1, b_2, b_3\) such that \(1 \neq \{y, (a_1a_2a_3)(b_1b_2b_3)\}\). Assume ii). This means that \(p = n\) and that \(H\) is a \(p\)-group. Since \(p\) is the highest power of \(p\) dividing \(\frac{n!}{2}\), we have that \(H\) has order \(p\). Moreover it is a Sylow subgroup of \(A_n\), whence it contains all the elements of order \(p\), which are \((p-1)\!\!,\) a number clearly greater than \(p\).

The simplicity of \(A_n\) is completely proved. Pace e Bene.