We dedicate this paper to the memory of Prof. Dr. Carlota Szabó, nee Nagy, an exceptional teacher of our science.

ABSTRACT. Let \( \{V_k ; k = 1,2,3,\ldots\} \) be the system of eigenfunctions and associated functions of Bessel’s equation of order \( \nu \geq 1 \), \( y'' + (\lambda-q)y = 0 \), with boundary conditions depending polynomially on the parameter \( \lambda \): \( P(\lambda) y(1) + Q(\lambda) y'(1) = 0, y \in L^2(0,1) \). It is shown that the structure of \( \{V_k\} \) is similar to that of the case where \( q \in C([0,1]) \) and the boundary conditions depend polynomially on \( \lambda \) at both ends.

This system is not a basis, although any function in \( L^2 \) can be expanded into a series of the form: \( f = \sum c_k(f) V_k \).

0. INTRODUCTION. A solid sphere with an initial distribution of temperature symmetrical about the z-axis (for example, a linear distribution \( C_r \cos \theta \)), is cooled by immersion in a mass of a well-stirred liquid which has at each instant a uniform temperature throughout it. Assume that the sphere has radius one and \( u(r,\theta,t) \) denotes its temperature at the instant \( t \), while the initial distribution is of the form

\[ I) \quad f(r) \cdot \cos \theta. \]

The coefficients of the expansion of \( u \) obtained by separation of variables are determined by those of the following expansion of the radial part of the initial distribution of temperature

\[ II) \quad g(r) = r f(r) = \sum_{j=1}^{\infty} B_j \cdot \gamma_j(r), \quad \gamma_j(r) = r^{1/2} \cdot J_{1/2}(\lambda_j^{1/2} r), \text{ where} \]

\[ III) \quad (a-b \lambda-1) \gamma(1) + \gamma'(1) = 0, a \text{ and } b \text{ constants.} \]

The boundary condition is of the form

\[ IV) \quad P(\lambda) \gamma(1) + Q(\lambda) \gamma'(1) = 0, P \text{ and } Q \text{ polynomials.} \]

We shall not enter into more details in relation with this particular example (cf. [L] and [S]). In this paper our main objective is to study expansions into series of cylindrical functions as in II) satisfying a boundary condition of the type IV), but we shall restrict ourselves to Bessel functions of order \( \nu \geq 1 \).
The problems that we consider here and the objectives we pursue are in nature similar to those considered by R.E. Langer, [L], and C. Miranda, [M]. However our approach follows the same lines as in [B]. There, the following problem was studied:

\[
\begin{align*}
(P) & \quad y'' - (\lambda + q(x))y = 0, \quad 0 < x < 1 \\
& \quad P(\lambda) y(0) + Q(\lambda) y'(0) = 0 \\
& \quad \tilde{P}(\lambda) y(1) + \tilde{Q}(\lambda) y'(1) = 0
\end{align*}
\]

with \( q = 0 \), \( \tilde{P}, \tilde{Q} \) polynomials with real coefficients and \( P, Q \) constants not both zero. The more general situation in (P) when \( q(x) \) is real and continuous in \([0,1]\) and the four real polynomials \( P, Q, \tilde{P}, \tilde{Q} \) verify \( \text{G.C.D.}(P,Q) = 1 \), \( \text{G.C.D.}(\tilde{P},\tilde{Q}) = 1 \), was studied by E. Guichal in his doctoral thesis (cfr. [G]).

The boundary problem that we treat in this paper could become an introductory work to a more general theory of anomalous systems with \( q \in C([0,1]) \).

We shall finish this introduction with an alternative form for the boundary condition IV).

Let us consider the differential equation:

\[
y'' - f(x)y + \lambda y = 0, \quad a < x < b, \quad f \in C^m((a,b)).
\]

Then

\[
\lambda y^{(h)} = -y^{(h+2)} + \sum_{j=0}^{h} \binom{h}{j} f^{(j)} y^{(h-j)} \quad \text{if} \quad h = 0, 1, \ldots, m.
\]

It follows by induction that

\[
V) \quad \lambda^k y(x) = (-1)^k y^{(2k)}(x) + \sum_{s=0}^{2k-2} \binom{2k}{s} f_{k,s}(x) y^{(s)}(x), \quad 2k \leq m,
\]

where \( f_{k,s} \in C^1 \). Differentiating V) we get

\[
VI) \quad \lambda y^{(h+1)}(x) = (-1)^{k+1} y^{(2k+1)}(x) + \sum_{s=0}^{2k-1} g_{k,s}(x) y^{(s)}(x), \quad 2k+1 \leq m,
\]

where \( g_{k,s} \in C \).

Assume that the degrees of \( P \) and \( Q \) are \( p \) and \( q \) respectively, and that \( m = 2p + (2q+1) \). Then there exist constants \( c_0, c_1, \ldots, c_m \), independent of \( y(x) \), such that

\[
VII) \quad P(\lambda) y(b) + Q(\lambda) y'(b) = \sum_{j=0}^{m} c_j y^{(j)}(b).
\]

Conversely, given \( c_0, \ldots, c_m \), and using V) we get

\[
\begin{align*}
\sum_{j=0}^{m} c_j y^{(j)}(b) &= a_m \lambda^m y(b) + \sum_{j=0}^{m-1} c_j y^{(j)}(b), \quad \text{if} \quad m = 2p, \\
\sum_{j=0}^{m} c_j y^{(j)}(b) &= a_m \lambda^m y'(b) + \sum_{j=0}^{m-1} c_j y^{(j)}(b), \quad \text{if} \quad m = 2q+1.
\end{align*}
\]

So, step by step we see that

\[
\sum_{j=0}^{m} c_j y^{(j)}(b) = P(\lambda) y(b) + Q(\lambda) y'(b).
\]
where the polynomials P and Q are of degrees p and q respectively and 
\( m = 2p + (2q+1) \). These polynomials are independent of the solution 
\( y(x) \). This means in particular that the boundary condition IV), which 
is the same used in [0], pp. 241-242, can be replaced by a condition of 
the form \( \sum_{j=0}^{m} y^{(j)}(1) = 0 \) for solutions of a Bessel differential equation

VIII) \( y''(x) + (\lambda - (r^2 - 1/4)/x^2) y(x) = 0 \), \( 0 < x < 1 \).

1. ON EXPANSIONS OF AN \( L^2 \)-FUNCTION WITH RESPECT TO CERTAIN NON-ORTOGONAL SYSTEMS.

The following general results apply to several situations. They are 
an assembling of results that can essentially be found in [B].

THEOREM 1. Let \( \{V_s; s=1,2,\ldots\} \) be a system of normalized complex functions in \( L^2 \), verifying i),ii),iii), and iv).

i) \( s \neq t \) implies

\[
(V_s, V_t) = \frac{O(1)}{\sigma(s) \sigma(t)} \quad \text{where} \quad \sum_{n=1}^{\infty} \frac{1}{|\sigma(n)|^2} < \infty,
\]

ii) if \( f \in L^2 \) its Fourier products \( b_s(f) = (f, V_s) \) verify

\[
\|b(f)\|^2 = (\sum_s |b_s(f)|^2)^{1/2} < K, \quad K \text{ independent of } f,
\]

iii) for each \( s \) there exists a continuous linear functional on \( L^2 \), 
\( c_s(f) \neq 0 \), and a set \( D \) dense in \( L^2 \) such that \( f \in D \) implies that 
\( \sum c_s(f) V_s \) converges in the mean to \( f \), and

iv) if \( s \) is great enough, say \( s > s_0 \),

\[
c_s(f) = b_s(f) \eta_s, \quad \eta_s \text{ a constant}.
\]

Then

\[
f \in L^2 \text{ implies } f = \sum c_s(f) V_s \text{ (L}^2\text{), and } \eta_s \longrightarrow 1
\]

when \( s \longrightarrow \infty \); also

\[
\|f\|_2 < K \|c(f)\|_2 < M \|f\|_2,
\]

\( c(f) = (c_1(f), c_2(f), \ldots) \), and

\[
b_t(f) = c_t(f) + O(1) \|c(f)\|/\sigma(t).
\]

Proof. Let \( X_{st} = (V_t, V_s) = \overline{X_{ts}} \). Then if \( N > M \)

\[
\int_a^N \left| \sum_{s=1}^{N} e_s V_s \right|^2 dx = \sum_{s=M}^{N} |e_s|^2 + \sum_{s \neq t} A_{st} e_s \overline{e_t}
\]
From (1) it follows that the last sum is not greater than
\[ C \sum_{M} |e_{f}(s)|^{2} \leq C \sum_{M} \sum_{M} |e_{f}(s)|^{2} \leq e(M) \|\varepsilon\|_{2}^{2} \]
where \( e(M) = o(1) \) for \( M \to \infty \). Applying this to (7) we get
\[ \sum_{M} |e_{f}(s)|^{2} = (1 + o(1)) \sum_{M} |e_{f}(s)|^{2} \]

The equality (8) implies the following propositions j) and jj):

j) \( \sum e_{f} V_{s} \) converges in \( L^{2} \) if and only if \( \{e_{f}\} \in L^{2} \),

jj) \( \|\sum e_{f} V_{s}\|_{2} < K \|\varepsilon\|_{2} \), \( K \) independent of \( e = \{e_{f}\} \).

It also holds:

jjj) \( \{e_{f}\} \in L^{2} \) implies \( \|\sum e_{f} V_{s}, V_{t}\| = e_{t} + O(1) \|\varepsilon\|_{2}/\alpha(t) \).

In fact, making use of j) we obtain
\[ \|\sum e_{f} V_{s}, V_{t}\| = \sum e_{f} A_{ts} = e_{t} + \sum_{s \neq t} 0(1) e_{f}/\alpha(t) \alpha(s) \]
and jjj) follows from Schartz inequality. Let us see now that \( \eta_{s} \to 1 \).

Assume that \( f \in \mathcal{D} \) and \( t > s_{0} \). From j)-jjj) we obtain for \( f = \sum c_{f}(f) V_{s} \):
\[ b_{f}(f) = c_{f}(f) + O(1) \|\varepsilon\|_{2}/\alpha(t) = \eta_{t}, b_{f}(f) = O(1) \|\varepsilon\|_{2}/\alpha(t) . \]

But, if \( M \) is great enough and fixed
\[ \|c_{f}\|_{2}^{2} = \sum_{M+1} |c_{f}(f)|^{2} + \sum_{M+1} |c_{f}(f)|^{2} < \sum_{M+1} |c_{f}(f)|^{2} + 2\|\sum_{M+1} c_{f} V_{s}\|^{2} = \sum_{M+1} |c_{f}|^{2} + 2\|f\| + \|c_{f}\|^{2} \]
\[ < K_{0} \|f\|_{2}^{2} . \]

Then from (9) we obtain for any \( f \) belonging to \( \mathcal{D} \) \( 1 - \eta_{t} = \]
\[ = 0(1)K_{0} \|f\|_{2}/\alpha(t) |b_{f}(f)| , \]
and therefore
\[ 1 - \eta_{t} = \frac{O(1)}{\alpha(t)} \inf_{D} \|f\|_{2}/|b_{f}(f)| . \]

Since \( D = L^{2} \), taking \( f \)'s near to \( V_{t} \) one sees that the inf is equal to one. Since \( \alpha(t) \to \infty \) with \( t \), it follows \( 1 - \eta_{t} = o(1) \).

Let us prove now (4). Assume that \( g_{m} \in D \) converges to \( g \) in \( L^{2} \). Then

ii) implies that \( b(g_{m}) \to b(g) \) in \( L^{2} \). From the hypothesis on the \( c \)'s and the fact that \( \eta_{s} \to 1 \), it follows
\[ c(g_{m}) \to c(g) \text{ in } L^{2} . \]

In consequence, making use of jjj),
\[ \| g - \sum c_s(g)v_s \|_2 = \| (g - g_m) - \sum c_s(g - g_m)v_s \|_2 \leq C \| c(g) - c(g_m) + g - g_m \|. \]

Since the right-hand side tends to zero as \( m \) tends to infinity, the left-hand side is null, and (4) follows. The first inequality in (5) is a consequence of \( jjj \) and the second one follows from (10) for \( f \in D \) and in general, from an approximation argument and (11).

(6) is a consequence of \( jjj \) and (4). Q.E.D.

REMARKS. Theorem 1 assures that the transformation \( T: L^2 \ni e \rightarrow f = \sum e_s v_s \in L^2 \) is continuous and onto, since \( f \rightarrow c(f) \) is a right inverse of \( T \).

Let us call \( r = \{ c(f); f \in L^2 \} \). \( r \) is a subspace (closed) of \( L^2 \) (cf.(5)).

\( r = L^2 \) if and only if each function has a unique expansion.

In fact, each function has a unique expansion iff \( T \) is one-to-one, and this happens iff \( f \rightarrow c(f) \) is a left inverse of \( T \). That is, iff \( r = L^2 \).

THEOREM 2. Assume that the hypothesis of theorem 1 hold.

a) If \( A \) is the gramian of the system \( \{ V_s \} \): \( A_{ij} = (V_j, V_i) \), then \( A = I + T \) where \( T \) is defined by a matrix of finite Hilbert-Schmidt norm.

b) Assume that \( A_{ki} \neq 0 \) for a pair of different subscripts \( i, k \), both greater than \( s_0 \). Then no function of \( L^2 \) has a unique expansion with respect to the system \( \{ V_j \} \).

c) Let \( B = \{ b(f); f \in L^2 \} \) be the space of all Fourier products. \( B \) is a subspace of \( L^2 \) and \( \| f \|_2 \) is equivalent to \( \| b(f) \|_2 \).

d) \( G = L^2 \Theta B \) is of finite dimension \( g \). Besides, \( G \) and \( r \) form a pair of complementary manifolds.

Proof. a) \( T_{ij} = (1 - \delta_{ij})A_{ij} \). Then, \( \sum_{i,j} |T_{ij}|^2 = \sum_{i,j} 0(1) / (\sigma(i)\sigma(j))^2 = 0(1) \sum_{i} \sigma(i)^{-2} < \infty \).

b) Take \( f = V_i \). Then \( c_k(f) = \eta_k b_k(f) = \eta_k A_{ki} \neq 0 \), and therefore \( V_i - \sum c_s(f)v_s \) is equal to \( 0 \) but not all its coefficients vanish.

c) Observe that if \( f = \sum e_s v_s \) then \( b(f) = A.e = (I+T).e \) and therefore \( B \) is the range of \( I+T \) with \( T \) completely continuous. This implies \( B = B \), (cf.[A], [RS]), and since \( f \rightarrow b(f) \) is a one-to-one continuous transformation from \( L^2 \) onto \( B \), its inverse is also bounded, i.e., \( \| b(f) \| \sim \| b(f) \| \).

d) \( A \) defines a transformation whose range is \( B \). \( G \) is the null space of \( A^* = A \), i.e., the eigenspace of \( T \) corresponding to the eigenvalue \( -1 \).
Then \( \dim G = g < \infty \). On the other hand, given \( e \in l^2 \), if \( f = \sum e_g V_g \), we have \( f = \sum (e_g - c_g(f))V_g + \sum c_g(f)V_g \). Then, the first summand is equal to 0 and also \( A(e - c(f)) = 0 \). So, \( e = c + (e - c) \) with \( c \in \Gamma \) and \( e - c \in G = A^{-1}(0) \). Q.E.D.

Next we apply the preceding general results to non-orthogonal expansions in series of Bessel functions.

### 2. EIGENVALUE EQUATION.

Consider Bessel's equation VIII), \( \nu \geq 1 \).\( (\sqrt{\lambda} J_{\nu}(x/\lambda), \sqrt{\lambda} Y_{\nu}(x/\lambda)) \) is a set of linearly independent solutions. The only solutions in \( L^2(0,1) \) are of the form \( \sqrt{\lambda} J_{\nu}(x/\lambda) \), (when \( \nu \geq 1 \), Bessel's equation belongs to H. Weyl's "Limit point case"). Let us call

\[
\Delta(x, \lambda) = J'_{\nu}(x/\lambda) Y_{\nu}(x/\lambda) - Y'_{\nu}(x/\lambda) J_{\nu}(x/\lambda) = (2i)^{-1}(H_{\nu}^{(1)}(x/\lambda) H_{\nu}^{(2)}(x/\lambda) - H_{\nu}^{(2)}(x/\lambda) H_{\nu}^{(1)}(x/\lambda)),
\]

\[
D(x, \lambda) = J_{\nu}(x/\lambda) Y'_{\nu}(x/\lambda) - Y_{\nu}(x/\lambda) J'_{\nu}(x/\lambda) = (2i)^{-1}(H_{\nu}^{(1)}(x/\lambda) H_{\nu}^{(2)}(x/\lambda) - H_{\nu}^{(2)}(x/\lambda) H_{\nu}^{(1)}(x/\lambda)).
\]

Then,

\[
(12) \quad \phi(x, \lambda) = (\pi/2) \sqrt{\lambda} D(x, \lambda) \quad ; \quad \theta(x, \lambda) = \phi/2 + (\pi/2) \sqrt{\lambda} \Delta(x, \lambda),
\]

are solutions of Bessel's equation verifying

\[
\phi(1, \lambda) = 0 \quad , \quad \phi'(1, \lambda) = -1 \quad ; \quad \theta(1, \lambda) = 1 \quad , \quad \theta'(1, \lambda) = 0.
\]

Therefore: \( (\theta \phi')(x, \lambda) - (\theta' \phi)(x, \lambda) = \theta \cdot \phi'(1, \lambda) - \theta' \cdot \phi(1, \lambda) = -1 \).

The characteristic values \( \lambda_n \) are the zeros of \( J_{\nu}(x/\lambda) \) for the boundary conditions \( y(1) = 0, y \in L^2 \). An \( L^2 \)-solution for \( \lambda \neq \lambda_n \) is of the form \( \psi = \theta + m(\lambda) \phi \) where (cf. [T]),

\[
(13) \quad m(\lambda) = -\sqrt{\lambda} \cdot J'_{\nu}(x/\lambda) / J_{\nu}(x/\lambda) - 1/2
\]

Since the wronskian \( W(J_{\nu}(z), Y_{\nu}(z)) = 2/(\pi z) \), it holds

\[
(14) \quad \psi(x, \lambda) = \sqrt{\lambda} \cdot J_{\nu}(x/\lambda) / J_{\nu}(x/\lambda)
\]

Let \( P \) and \( Q \) be polynomials in \( \lambda \). Consider the function

\[
(15) \quad \Phi(x, \lambda) = -Q(\lambda) + P(\lambda)
\]

This is a solution of Bessel's equation satisfying

\[
\Phi(1, \lambda) = -Q(\lambda) \quad , \quad \Phi'(1, \lambda) = P(\lambda).
\]

Taking into account (12), we obtain
\[ (16) \ \Phi(x,\lambda) = \frac{-Q(\lambda)\sqrt{x} \Delta(x,\lambda)}{2} - \frac{(P+Q/2)\sqrt{x}}{2} D(x,\lambda) = \]
\[ = (\pi/2)\sqrt{x} \{ J_v(xs) [-Q(s^2)s Y'_v(s) - (P+Q/2)(s^2) Y_v(s)] + + Y_v(xs) [Q(s^2)s J'_v(s) + (P+Q/2)(s^2) J_v(s)] \} = \]
\[ = (\pi/2)\sqrt{x} \{ J_v(xs) \ldots \} + Y_v(xs) s^2 \omega(s^2) \}.
\]

Now we assume that \( P \) and \( Q \) are real polynomials such that \( \text{GCD}(P,Q)=1 \), and \( p \leq q \geq 1 \), where \( p \) and \( q \) are the degrees of \( P \) and \( Q \) respectively. In particular we have excluded the case where some of them is identically zero. Next we define two other polynomials, also real, \( \rho(\lambda) \), \( \rho(\lambda) \), by the following conditions: \( \deg \pi < p, \deg \phi < q \) and
\[ (17) \ \pi(\lambda) Q(\lambda) + \rho(\lambda) P(\lambda) = -1. \]

Let us define now the function \( \Theta \):
\[ (18) \ \Theta(x,\lambda) = \rho(\lambda) \theta(x,\lambda) - \pi(\lambda) \phi(x,\lambda). \]

This function verifies: \( \Theta(1,\lambda) = \rho(\lambda), \ \Theta'(1,\lambda) = \pi(\lambda) \), and is a solution of Bessel's equation. Therefore,
\[ (19) \ \Theta, \Phi'(x,\lambda) - \Theta', \Phi(x,\lambda) = \Theta, \Phi'(1,\lambda) - \Theta', \Phi(1,\lambda) = \]
\[ = \pi Q + \rho P = -1. \]

Let us call
\[ (20) \ \Psi(x,\lambda) = \Theta(x,\lambda) + M(\lambda) \Phi(x,\lambda), \]

where \( M \) is so chosen that \( \Psi \in L^2 \). Then, except at the poles of \( m(\lambda) \):
\[ \Psi = C(\Theta + m\phi). \]

From (15), (18) and (20) it follows that \( C = \rho M Q, \)
\[ - (\pi + MP) = mC; \]
that is,
\[ (21) \ M(\lambda) = \frac{\pi + \lambda M}{-P + \lambda Q} \]
\[ m(\lambda) = - \frac{\pi + \lambda P}{\rho - \lambda Q}. \]

\( M \) and \( m \) are meromorphic functions of \( \lambda \). From the first formula in (21) it follows that at a pole of \( M \), \( m = P/Q \), (eventually equal to \( \infty \) if \( Q = 0 \)). Conversely, if \( m = P/Q \) at \( \lambda \) then if \( Q \neq 0, m \neq \infty \), it follows from (17): \( \pi + MP \neq 0 \). In consequence, \( M \) has a pole there. If \( Q = 0 \) and \( m = P/Q \) then \( P \neq 0, m = \infty \). Therefore, \( M (-\rho/Q) \) has a pole at \( \lambda \). We have then

**PROPOSITION 1.** \( M(\lambda) = \infty \) if and only if \( m(\lambda) = P(\lambda)/Q(\lambda) \), and the poles of \( M \) are the roots of the equation
\[ (22) \ \frac{P(\lambda)}{Q(\lambda)} = \frac{1}{2} \sqrt{x} J'_v (\lambda). \]

The poles of \( M(\lambda) \) are exactly those \( \lambda \) for which \( \Phi(x,\lambda) \in L^2 \). That is, those \( \lambda \) for which the boundary problem: \( y \) solution of Bessel's equation
VIII) verifying \( P(\lambda) y(1) + Q(\lambda) y'(1) = 0, y \in L^2, \) has a non-trivial solution.

So (22) is the equation for the eigenvalues of this problem.

3. ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES.

Call \( \lambda = s^2 \). Then (22) is equal to

\[
\frac{P(s^2)}{sQ(s^2)} = \frac{(d/ds)(\sqrt{2} J_v(s))}{\sqrt{2} J_v(s)}.
\]

Taking into account the asymptotic expansions of \( \sqrt{2} J_v(z) \) and its derivative, which hold in \(|\arg z| < \pi - \epsilon\), we have:

\[
\frac{-\sin(s-\pi/2-\pi/4) (1+O(1/s^2)) + \cos(...) (A/s+O(1/s^3))}{sQ(s^2)}
\]
\[
+ \frac{\cos(s-\pi/2-\pi/4) (1+O(1/s^2)) + \sin(...) (A/s+O(1/s^3))}{sQ(s^2)}
\]

where \( A = (1 - 4v^2)/8 \). Then (23) is the equation for the poles of \( M(s^2) \) and the \( O(1/s^3) \)'s that appear there, are real for \( s \) real and have asymptotic expansions with real coefficients.

Let us write \( s - \pi/2 - \pi/4 = \alpha + i\beta \). From (23) it follows:

\[
0 = P(s^2) + AQ(s^2) + [sQ(s^2) - P(s^2)A/s] \tan(\alpha + i\beta) + [P(s^2).O(1/s^3) + sQ(s^2).O(1/s^3)] \tan(\alpha + i\beta) + P(s^2).O(s^{-2}) + Q(s^2).O(s^{-2}).
\]

PROPOSITION 2. a) If \( q > p \) there exists a real constant \( C \) such that the poles of \( M(s^2) \) with \(|\arg s| < \pi/2\) verify

\[
\tan(s - \pi/2 - \pi/4) = C/s + O(s^{-3});
\]

b) if \( p > q \) they verify, also with a real constant \( C \):

\[
\cot(s - \pi/2 - \pi/4) = C/s + O(s^{-3}).
\]

The \( O's \) are real if \( s > 0 \).

Proof. Assume that \( Q(\lambda) = a_\lambda^q + \ldots, P(\lambda) = \lambda^p + \ldots \). In case a) it follows from (24) that

\[
s^{2p} + Aq s^{2q} + O(s^{2q-2}) - (s^{2q+1} a_q + O(s^{2q-1})) \tan(\alpha + i\beta) = 0.
\]

Then if \( q > p \):

\[
\tan(\alpha + i\beta) = \frac{A a_q + O(s^{-2})}{s a_q + O(s^{-1})} = \frac{A}{s} + O(s^{-3}).
\]

If \( q = p \), instead of \( A \) we must put: \( a_q^{-1} + A \).
In case b), (24) implies:
\[ s^2 p + O(s^{2p-2}) + (As^{2p-1} - aq^{2q+1} + O(s^{2p-3})) \tan(\alpha + i\beta) = 0. \]
Therefore,
\[ \cot(\alpha + i\beta) = \frac{-A + aq^{2(q-p+1)} + O(s^{-2})}{s(1 + O(s^{-2}))} = \frac{C}{s} + O(s^{-3}). \] Q.E.D.

Proposition 2 will be used to prove that for \(|\lambda|\) great enough, \(M(\lambda)\) has a pole only if \(\lambda\) is real and positive. This is the content of next theorem. But before let us prove an auxiliary result.

**Lemma 1.** Assume that \(F(z)\) is an analytic function defined in the square \(S\) with sides parallel to the axes of length 2 with center at the origin. If \(F(2) = F(-2)\) then
\[ |\text{Im} F(i\beta)| < 3\beta \sup_{\overline{S}} |F(z)|, \text{ for } 0 < \beta < 1/3. \]

**Proof.** \(F(s) - F(\overline{s}) = \frac{i}{\pi} \int_{\overline{S}} \frac{F(t)}{(t-s)(t-\overline{s})} \, dt.\) But, from
\[ |\int_{\overline{S}} ...| < \sup_{\overline{S}} |F(z)| \cdot \beta/(2/3)^2 \text{ it follows } |\text{Im} F(s)| < \frac{18\beta}{2\pi} \sup_{\overline{S}} |F(z)|. \] Q.E.D.

**Theorem 3.** Let us suppose that \(|\arg s| < \pi/2.\) Then
i) the poles of \(M(s^2)\) are real if \(|s|\) is great enough, and
ii) they are simple with negative residues, for \(s\) positive great enough.
iii) consecutive real poles of \(M(s^2),\) like those of \(m(s^2),\) tend asymptotically to be at distance \(\pi;\) besides from a moment on the two sets interlace, i.e., the poles of each occur alternately.

**Proof.** i) It follows easily that
\[ |\text{Im} \tan(\alpha + i\beta)| = |(\tan \beta)/(\cos^2\alpha + \sin^2\alpha \cdot \tanh^2\beta)| > |\tanh \beta|, \]
\[ |\text{Im} \cot(\alpha + i\beta)| > |\tanh \beta|. \] Then from (25) or (26) we obtain
\[ (28) \quad |\text{Im} (C + O(s^{-3}))| > \tanh |\text{Im} s| \text{ if } s \text{ is a pole of } M(s^2). \]

In consequence if \(|s|\) is great enough: \(|\text{Im} s| < 1/3.\) Let us call \(H(z)\) the meromorphic function denoted by \(O(z^{-3})\) in the right-hand side of (25) (or (26)), and \(G(z),\) the function defined by the left-hand side minus \(C/z.\) Then \(s\) is a pole of \(M(s^2)\) if and only if \(G(s) = H(s);\) besides if \(s\) is a pole, \(S\) is also one (cf. (22)), and,
\[ (29) \quad G(s) = (G(s) + G(\overline{s}))/2 = (H(s) + H(\overline{s}))/2. \]

Define \(F(z) = (H(z) + H(\overline{z}))/2.\) (29) implies that
\[ \tanh |\text{Im} s| < |\text{Im} (C/s + F(s))| < |C \text{ Im } \overline{s}| \cdot |s|^{-2} + |\text{Im} F(s)|. \]
A straightforward application of lemma 1 to $F(z)$ if $\beta = \text{Im } s$, $0 < |\beta| < 1/3$ gives:

$$
(30) \quad \tanh |\beta| \ll |C s^{-2} \beta| + |\beta| O(s^{-3}).
$$

For these values of $\beta$, there exists a number $d$, positive, such that $d < \tanh \beta/\beta$. Then, (30) can be verified only for $s$ bounded, and this proves i).

ii) $M(\lambda)$ has a simple pole at $s_0 = s_0^2 \neq 0$ if and only if $M(s^2)$ has a simple pole at $s_0$. Assume now that $s_0 > 0$ is a pole for $M(z^2)$; then $m(s_0^2) = P(s_0^2)/Q(s_0^2)$. From (13) it follows

$$
(31) \quad \frac{d}{ds} m(s^2) = -\left(\frac{s J_1'(s)}{J_0(s)} + s \cdot \frac{J_1^2(s)}{J_0^2(s)}\right) = (s - \frac{1}{2}) + \frac{1}{2} (m(s^2) + 1/2)^2.
$$

Therefore

$$
(32) \quad \left. \frac{d}{ds} m(s^2) \right|_{s = s_0} = \left( s_0 - \frac{1}{2} \right) + \frac{1}{2} \left( \frac{P(s_0^2)}{Q(s_0^2)} \right)^2
$$

But $p > q$ implies $(d/ds)(P/Q)(s^2) \sim \text{constant} \cdot s^{-1}(P/Q)(s^2)$ and $p \leq q$ implies $(d/ds)(P/Q)(s^2) = O(1)$. In consequence from (32) it follows that at $s = s_0$, $(d/ds)m(s^2) > (d/ds)(P/Q)(s^2)$ if $s_0$ is great enough. This proves that $s_0$ is a simple pole. Its residue is

$$
\frac{\mu(s_0^2) + m(s_0^2) \rho(s_0^2)}{(d/ds)(m - P)(s_0^2)} = \frac{1/2}{Q(s_0^2)}.(d/ds)(m - P/Q)(s_0^2) < 0,
$$

as it follows taking into account (17) and (21).

iii) The result for $m$ is immediate and that for $M$ follows from the asymptotic formulae of Proposition 2 if one remembers that the $O(s^{-3})$ that appear are real for $s$ real. It remains only to see that the poles of $M$ and $m$ interlace. Since $m(s^2)$ is real and has simple poles with negative residues on the positive real axis, the values of $m(s^2)$ run from $-\infty$ to $\infty$ when $s$ runs from one pole to the next one. Then for a certain $s_0$ in between, $m(s_0^2) = P(s_0^2)/Q(s_0^2)$. For the same reason between two consecutive poles of $M(s^2)$ there exists a point $s_1$ such that $M(s_1^2) = \rho(s_1^2)/Q(s_1^2)$, i.e., a pole of $m$ (cf. (21) and (17)). Q.E.D.

4. SIMPLE POLES AND RESIDUES OF GREEN'S KERNEL.

First we shall introduce some notation and auxiliary formulae. Recall that $\psi = \theta + m \psi = \sqrt{2} \ J_\nu(xs)/J_\nu(s)$ if $J_\nu(s) \neq 0$, i.e. when $m(s^2) \neq \infty,$
and that $\Psi$ defined by (20) is, at least where $M(s^2) \neq \infty$, $J_\nu(s) \neq 0$, equal to

\[ (33) \quad \Psi = (\rho - MQ) \sqrt{x} \frac{J_\nu(xs)}{J_\nu(s)} = (mQ-P)^{-1} \sqrt{x} \frac{J_\nu(xs)}{J_\nu(s)} = -\sqrt{x} \frac{J_\nu(xs)}{s^\nu \omega(s^2)}, \]

where $\omega(s^2) = (sJ_\nu'(s)Q(s^2) + [P(s^2) + Q(s^2)/2] J_\nu(s)) / s^\nu$, is an even entire function.

On the other hand, $s$ is a pole of $M(z^2)$ if and only if

\[ (34) \quad \omega(s^2) = (mQ-P) [s^{-\nu} J_\nu(s)] = s^{-\nu} (sJ_\nu'(s))Q(s^2) + [P(s^2) + Q(s^2)/2] J_\nu(s) = 0. \]

(The factor $s^{-\nu}$ has been introduced to include the point $s=0$). In fact, from the definition of $\Psi$ (cf. (20)) it follows that the poles of $M(s^2)$ are exactly the poles of $\Psi(x,s^2)$, which, because of (33), are the zeros of $\omega(s^2)$. ((34) can also be proved using proposition 1). For those points, we have (cf. (15), (16)):

\[ \Phi(x,s^2) = -\sqrt{x} J_\nu(xs) (Q(s^2)s Y_\nu'(s) + [P(s^2) + Q(s^2)/2] Y_\nu(s)). \]

This expression must be understood as a limit when $M(0) = \infty$. Then if $s_n$ is a pole of $M$, from (34) and that $W(J_\nu,Y_\nu) = 2/(\pi s)$, we get

\[ (35) \quad \Phi_n = -\sqrt{x} \frac{J_\nu(xs_n)Q(s_n^2)/J_\nu(s_n)}{\sqrt{x} J_\nu(xs_n)\{P(s_n^2) + Q(s_n^2)/2\}/J_\nu(s_n) s_n} \]

Since $M(s^2) = 0$ implies that $s^{-\nu} J_\nu(s) = 0$ is equivalent to $Q(s^2) = 0$, if the expression in the middle of (35) is indeterminate then its right-hand side must be used. If $s=0$ is a pole of $M(s^2)$, formula (35) is still valid but in its limit form

\[ \Phi_0(x) = x^{-\nu+1/2} (P(0) + Q(0)/2) = -x^{-\nu+1/2} Q(0). \]

Making use of Lommel's formula we obtain for real $s_n^2 \neq 0$:

\[ (36) \quad \Phi_n = (1 - \nu^2/s_n^2) Q^2(s_n^2) + [P(s_n^2) + Q(s_n^2)/2] / s_n^2, \]

and then, the normalized functions:

\[ (37) \quad \hat{\psi}_n = \Phi_n / \Phi_n, \quad \hat{\psi}_n = -\sqrt{2 \pi} \frac{J_\nu(xs_n)}{J_\nu(s_n)} \frac{D_n}{J_\nu(s_n)} = \sqrt{2 \pi} \frac{J_\nu(xs_n)}{J_\nu(s_n)} \frac{\hat{D}_n}{J_\nu(s_n)}, \]

where

\[ D_n = \left[ 1 - \frac{\nu^2}{s_n^2} + \frac{(P(s_n^2) + Q(s_n^2)/2)^2}{s_n^2} \right]^{-1/2}, \quad \text{sgn } D_n = \text{sgn } Q(s_n^2); \]

\[ \hat{D}_n = \left[ 1 + \frac{(s_n^2 - \nu^2) Q^2(s_n^2)}{(P + Q/2)^2 (s_n^2)} \right]^{-1/2}, \quad \text{sgn } \hat{D}_n = \text{sgn } (P+Q/2)/s_n. \]

Observe that $D_n / s_n Q(s_n^2) = \hat{D}_n / (P+Q/2)(s_n^2)$ if both denominators are different from zero; on the other hand, both denominators cannot be simul-
The functions $\psi_n$ are uniformly bounded. Precisely

**PROPOSITION 3.** $\psi_n(x) = \sqrt{x s_n} J_\nu(xs_n) O(1)$.

**Proof.** If $q \geq p$ then $(P+Q/2)/s_n Q = o(1)$ and from (34) it follows that $J'_\nu(s_n)/J_\nu(s_n) = o(1)$. In consequence, $s_n$ approaches zeros of $J'_\nu$, and $|J_\nu(s_n)| > c n^{-1/2}$. This together with (37) imply the thesis.

Analogously, if $q < p$ then $Q/s_n (P+Q/2) = o(1)$ and from (34):

$J'_\nu(s_n)/J_\nu(s_n) = o(1)$. Then $s_n$ approaches zeros of $J'_\nu$, and $|J'_\nu(s_n)| > c n^{-1/2}$.

Again from (37) the thesis follows. Q.E.D.

The Green's kernel $G(x,y,\lambda)$ of the differential system we are considering is equal to $-\Psi(x,\lambda) \Phi(y,\lambda)$ if $x \leq y$ and equal to $-\Psi(y,\lambda) \Phi(x,\lambda)$ if $x > y$.

By definition, $G_\lambda(f)(x) = \int_0^1 G(x,y,\lambda) f(y) dy$. Since $\Theta$ and $\Phi$ are entire functions of $\lambda$ for fixed $x$, it follows from the definition of $\Psi$ that

(38) $G(x,y,\lambda) = \text{entire function of } \lambda - M(\lambda) \Phi(x,\lambda) \Phi(y,\lambda)$.

Therefore, the poles of $G(x,y,\lambda)$ are the same as those of $M(\lambda)$. It holds,

**PROPOSITION 4.** If $\lambda = \lambda_n$ is a simple pole of $M$, then, for $f \in L^2(0,1)$ we have

$$\lim_{n \to \infty} G_\lambda(f) = \lim_{n \to \infty} \psi_n(x) \int_0^1 \psi_n(y) f(y) dy = 1$$

The proof of the proposition makes use of the following lemma whose detailed proof we leave to the reader, since it is a simple application of Fubini's theorem.

**LEMMA 2.** Let $F(y,\lambda)$ be a measurable function of $(y,\lambda)$, holomorphic in $|\lambda - \lambda_0| < d$, for each $y \in (0,1)$. Assume also the existence of a constant $A$ such that $\int_0^1 |F(y,\lambda)| dy < A$ for each $\lambda$. Then if for each $y$,

$$F(y,\lambda) = \sum_{n=0}^\infty a_n(y) (\lambda - \lambda_0)^n$$

holds, then

$$\int_0^1 F(y,\lambda) dy = \sum_{n=0}^\infty \int_0^1 a_n(y) dy (\lambda - \lambda_0)^n$$

whenever $d > |\lambda - \lambda_0|$.

**Proof of proposition 4.** Applying lemma 2 to the function

$F(y,\lambda) = G(x,y,\lambda) f(y) (\lambda_n - \lambda) r$ where $x \in (0,1)$ is fixed and $r$ is the
order of the pole of \( M(\lambda) \) at \( \lambda_n \), we obtain

\[
\text{res} \ G_\lambda(f) = \left\{ \begin{array}{ll}
\frac{1}{0} \text{res} \ G(x,y,\lambda) \ f(y) \ dy & \text{if } \lambda_n \\
\end{array} \right.
\]

In particular if \( \lambda_n \) is a simple pole then from (38) we get

\[
\text{res} \ G(x,y,\lambda) = \Phi_n(x) \Phi_n(y) \text{ res} \ (-M(\lambda)) = \psi_n(x) \psi_n(y) \text{ res} \ (-M(\lambda)).
\]

and

\[
\text{res} \ (-M(\lambda)) = \frac{1}{(2/\pi)(\text{PQ} - P/Q)(\lambda-n)}.
\]

This implies the thesis except for \( r_n \to 0 - 1 \). Assume that \( n \) is so great that \( \lambda_n \) is a simple pole of \( M \) and not a pole of \( m \). Then

\[
\text{res} \ -M(\lambda) = - \frac{\pi + \rho}{(d/d\lambda)(\text{Q}-P)} (\lambda-n) = - \frac{\pi + \rho P/Q}{\text{Q}(d/d\lambda)(\text{Q}-P)/Q} (\lambda-n) = Q^2(\lambda-n) \cdot \left[ m'(\lambda-n) - (P/Q)'(\lambda-n) \right].
\]

From (36) and (32) it follows

\[
\|\Phi_n\|^2 = \left. Q^2(s_n) \cdot \frac{d}{ds} m(s^2) \right|_{s=s_n} = Q^2(\lambda-n) \cdot \frac{d}{d\lambda} m(\lambda) \bigg|_{\lambda=\lambda_n}
\]

and

\[
\text{res} \ -M(\lambda) = \frac{1}{\|\Phi_n\|^2 - [P'Q - PQ'](\lambda-n)}.
\]

also implies that \( \|\Phi_n\|^2 \sim \lambda^{2qP(2p-1)} \). On the other hand,

\[
(P'Q - PQ')(\lambda-n) = \begin{cases} 
O(\lambda^{P+q-1}) & \text{if } p\neq q \\
O(\lambda^{P+q-2}) & \text{if } p=q 
\end{cases} = O(\lambda^{2pq-2}).
\]

Therefore, from (40) and (41) we finally get:

\[
r_n = \frac{\|\Phi_n\|^2}{\|\Phi_n\|^2 - [P'Q - PQ'](\lambda-n)} \to 1 \quad \text{Q.E.D.}
\]

5. MULTIPLE POLES OF THE GREEN'S KERNEL.

The calculation of the residue of \( G_\lambda(f) \) at a pole \( \xi \) is reduced to the computation of the residue at \( \xi \) of \( -M(\lambda) \Phi(x,\lambda) \Phi(y,\lambda) \), as follows from (38) and (39). On the other hand, by means of Picard's approximation method it is not difficult to prove the following proposition:

The functions \( \Phi^{(i)}(x,\lambda), i=0,1,2, \) are entire in \( \lambda \) for fixed \( x \in (0,1) \),
(the exponent means derivation with respect to x); besides, they are continuous in \((x, \lambda) \in [0,1] \times C\), and it holds

\[
\frac{\partial^j \phi}{\partial \lambda^j} (x, \lambda) = \frac{\partial^j}{\partial x^j} \left[ \frac{\partial \phi}{\partial \lambda} \right] (x, \lambda), \quad i=1,2, \quad j=1,2,3, \ldots ,
\]

Furthermore, the functions in (42) are also continuous in \((x, \lambda)\).

(A detailed proof is given in appendix I of [GJ]). Let us define the functions \(U_j(x, \lambda)\), \(j = 0, 1, \ldots\), by

\[
U_j(x, \lambda) = \frac{1}{j!} \frac{\partial^j \phi}{\partial \lambda^j} (x, \lambda).
\]

Therefore

\[
\Phi(x, \lambda) = U_0(x, \xi) + (\lambda - \xi) U_1(x, \xi) + \ldots .
\]

If \(M(\lambda)\) has a pole of order \(r\) at \(\xi\), let us write

\[-(\lambda - \xi)^r M(\lambda) = c_{-r} + c_{-r+1} (\lambda - \xi) + \ldots + c_{-1} (\lambda - \xi)^{r-1} + \ldots .\]

In consequence,

\[-(\lambda - \xi)^r M(\lambda) \Phi(x, \lambda) \Phi(y, \lambda) =
\]

\[-(\lambda - \xi)^r \sum_{j=0}^{r} \sum_{k=0}^{j-1} \frac{c_{j-1}}{j!} U_k(x, \xi) U_{j-k-1}(y, \xi) + \ldots .\]

But \((\lambda - \xi) \Phi(x, \lambda) = -\Phi'(x, \lambda) + (q - \xi) \Phi(x, \lambda)\), \(q = x^{-2} (\nu^2 - 1/4)\).

Because of (42) and (44) we get

\[
\sum_{k=0}^{r} \left[ - U_k'(x, \xi) + (q - \xi) U_k(x, \xi) \right] (\lambda - \xi)^k = \sum_{k=0}^{r} U_k(x, \xi)(\lambda - \xi)^{k+1},
\]

which implies that the functions \(U_k\) satisfy the following equations

\[
\begin{align*}
U_0'' + (\xi - q) U_0 &= 0, \\
U_1'' + (\xi - q) U_1 &= -U_0, \\
& \quad \vdots \\
U_n'' + (\xi - q) U_n &= -U_{n-1},
\end{align*}
\]

For each \(n\) the set of functions \(U_0, \ldots, U_n\), is linearly independent on any interval \(I\) contained in \((0,1)\). In fact, \(U_0(x, \xi) = \Phi(x, \xi) \neq 0\) on \(I\). Let us assume that the set under consideration be linearly dependent on \(I\). Then there exists a first \(k\) such that

\[
U_k = \sum_{m=0}^{k-1} \alpha_m U_m \text{ on } I.
\]

Applying the operator \(d^2/dx^2 + (\xi - q)\) to this identity, it follows
from (46) that $U_k = \sum_{m=1}^{k-1} c_m U_m$, in contradiction with the definition on $k$.

If $r$ is, as before, the order of the pole $\xi$ of $M(z)$, 
$(U_0(x,\xi), \ldots, U_{r-1}(x,\xi))$ is called the principal chain of functions associated to $\Phi(x,\xi)$, and from (45) we see that

$$\text{res } G(x, y, \lambda) = \left( \prod_{j=1}^{r-1} c_{-j} \right) \sum_{k=0}^{r-1} U_k(x, \xi) U_{j-1-k}(x, \xi),$$

and from (39), for $f \in L^2(0,1)$, we get

$$\text{res } G_\lambda(f) = \int_0^1 \left( \prod_{j=1}^{r-1} c_{-j} \right) \sum_{k=0}^{r-1} U_k(x, \xi) U_{j-1-k}(y, \xi) \, f(y) \, dy.$$

The brackets inside (47) equal to

$$(\ldots) = \sum_{k=0}^{r-1} U_k(x, \xi) \mathring{U}_k(y, \xi), \text{ where } (\mathring{U}_k(y, \xi) = \prod_{j=k+1}^{r-1} c_{-j} U_{j-1-k}(y, \xi).$$

That is

$$\begin{vmatrix}
\mathring{U}_0 \\
\vdots \\
\mathring{U}_{r-1}
\end{vmatrix}
= \begin{vmatrix}
c_{-r} & c_{-r+1} & \cdots & c_{-1} & U_{r-1} \\
c_0 & c_{-r} & \cdots & c_{-2} & U_{r-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & c_{-r} & U_0
\end{vmatrix}, \quad c_{-r} \neq 0.$$

Therefore, the set $\mathring{U}_0, \ldots, \mathring{U}_{r-1}$ is also linearly independent on any $I \subset (0,1)$.

Since res $G_\lambda(f)$ is a finite number for any $x \in (0,1)$ and any $f \in L^2(0,1)$, we see that the bracket in (47) defines a function in $L^2(0 < y < 1)$, $\forall x \in (0,1)$. The linear independence of $\{U_k ; k=0,1,\ldots\}$ implies that each $\mathring{U}_j \in L^2$. In fact, there exist $r$ points in $I: x_1, \ldots, x_r$, such that the matrix $(U_k(x_m)), k=0,\ldots,r-1, m=1,\ldots,r$, has a non-null determinant.

Since $\sum_{k=0}^{r-1} U_k(x, \xi) \mathring{U}_k(y, \xi) = F_m(y) \in L^2$, then $\mathring{U}_k \in \sum A_m F_m$ also belongs to $L^2$.

Summing up, we have

**PROPOSITION 5.** For $k=0,\ldots,r-1$, $U_k(x, \xi)$ and $\mathring{U}_k(x, \xi)$ belong to $L^2$, when $\xi$ is a pole of order $r$ of $M(z)$. If $\gamma_k(\xi) = \gamma_k(\xi, \xi)$ denotes the continuous linear functional on $L^2$ defined by

$$\int_0^1 U_k(y, \xi) f(y) \, dy, \text{ then } \text{res } G_\lambda(f) = \sum_{k=0}^{r-1} U_k(x, \xi) \cdot \gamma_k(f).$$
6. EIGENFUNCTION EXPANSIONS FOR CERTAIN REGULAR FUNCTIONS.

Let \( \{c_n\} \) be a family of circumferences with center 0 and radii \( r_n \) to be defined later but such that \( r_n \to \infty \) as \( n \to \infty \).

We shall prove the following result.

**Theorem 4.** Assume that \( f \in C^2(0,1) \) is null on neighborhoods of 0 and 1. Then, it holds uniformly on \( 0 < x < 1 \) that

\[
\lim_{n \to \infty} \frac{1}{2\pi} \int_{c_n} G_\lambda(f) \, d\lambda = f(x).
\]

To prove this theorem we need some auxiliary results. The first of them is the well-known fact that Green's operator is the inverse of the differential operator at least on a set of functions dense in \( L^2 \).

**Lemma 3.** Let \( f \) be as in Theorem 4, and \( \lambda \) not a pole of \( G_\lambda \). Then, on \( 0 < x < 1 \),

\[
f(x) = \lambda G_\lambda(f) + G_\lambda(\tilde{f}) \quad \text{if} \quad \tilde{f} = f'' + x^{-2}[1/4 - \nu^2] f.
\]

(This follows after integrating by parts twice the integrals involved in \( G_\lambda(f'') \), (cf. [T], ch. II)).

**Lemma 4.** If \( g \in L^1(0,1) \) is null on \( (0,\varepsilon) \) for a certain \( \varepsilon > 0 \), then it holds

\[
\lim_{n \to \infty} \int_{c_n} G_\lambda(g) \frac{d\lambda}{\lambda} = 0,
\]

uniformly in \( \varepsilon \in (0,1) \).

**Proof.** The integral that must be estimated is equal to

\[
\int_{c_n} G_\lambda(g) \frac{d\lambda}{\lambda} = \int_{\varepsilon}^{1} dy \, g(y) \int_{c_n} G(x,y,\lambda) \frac{d\lambda}{\lambda}.
\]

Then, it will be sufficient to show that, uniformly on \( y > \varepsilon \),

\[
0 < x < 1,
\]

\[
\int_{c_n} G(x,y,\lambda) \frac{d\lambda}{\lambda} = 2 \int_{D_n} G(x,y,s^2) \frac{ds}{s} = o(1),
\]

holds. Here, \( D_n = \{s; \, |s| = r_n^{1/2}, \, |\arg s| < \pi/2\} \).

If \( \Delta \) and \( D \) are defined as in paragraph 2, from the asymptotic formulae for the Hankel functions, we get:
\[
\begin{align*}
\Delta(x,s^2) &= \frac{2}{\pi s} \cos sl + e^{(1-x)}|\text{Im } s|. 0(1/s^2), \\
D(x,s^2) &= \frac{2}{\pi s} \sin sl + e^{(1-x)}|\text{Im } s|. 0(1/s^2).
\end{align*}
\]

In (52) the O's hold uniformly on \(0 < x < 1\). From (16) and (52) the following estimation for \(\Phi\) is obtained,

\[
\Phi(x,s^2) = s(2q+1)\sqrt{2p} e^{1-x} |\text{Im } s|. 0(1/s).
\]

Let \(a\) and \(b\) be functions of \(p\) and \(q\) defined as follows:

\[
a = 0, \quad b = \sqrt{2/\pi} \text{ if } q \leq p, \quad a = \sqrt{2/\pi}, \quad b = 0 \text{ if } q < p.
\]

Then, from (33) we obtain on \(0 < x < 1\):

\[
\Psi(x,s^2) = -\frac{\sqrt{x}}{s(2q+1)\sqrt{2p}} J_\nu(xs) + \frac{\sqrt{x} J_\nu'(xs)}{sQ(s^2) J_\nu''(s)} + \frac{\sqrt{x} J_\nu'(xs)}{sQ(s^2) J_\nu''(s)}.
\]

Taking as \(\sqrt{x}\) the points in \((0, \infty)\) where the function defined by the square brackets in the second denominator in (54) takes a maximum, it is possible to complement (54) with the following estimation valid for \(s \in D_n\) and \(O(1)\) independent of \(x \in (0,1)\):

\[
\Psi(x,s^2) = s(2q+1)\sqrt{2p} [a \cos(s-\nu\pi/2 - \pi/4) - b \sin(s-\nu\pi/2-\pi/4)] + e^{|\text{Im } s|} O(s^{-1}).
\]

From (53) and (55) it follows that

\[
G(x,y,s^2) = \begin{cases} 
-\Phi(x) \psi(y) = e^{(y-x)} |\text{Im } s|. 0(1/s), & \text{if } x > y, \\
-\Phi(y) \psi(x) = e^{(x-y)} |\text{Im } s|. 0(1/s), & \text{if } y > x,
\end{cases}
\]

where the O's hold uniformly if \(c < y < 1, \quad 0 < x < 1\). Then

\[
\int_{D_n} G(x,y,s^2) \frac{ds}{s} = \int_{D_n} 0(1/s^2) ds = 0(x_n^{-1/2}) = o(1). \quad Q.E.D.
\]

\textbf{Proof of theorem 4.} (49) says that \(G_\lambda(f)(x) = f(x)/\lambda - G_\lambda(f)(x)/\lambda\) and therefore we get:

\[
\frac{1}{2\pi i} \int_{C_n} G_\lambda(f)(x) d\lambda = f(x) - \frac{1}{2\pi i} \int_{C_n} G_\lambda(f)(x) d\lambda/\lambda.
\]

The thesis now follows from Lemma 4.

\textbf{COROLLARIES TO THEOREM 4.} i) \textit{Let } f \textit{ be as in theorem 4. If } M \textit{ denotes}
the finite set of poles of $M(z)$ that are multiple, null or nonreal, and $r(\xi)$ the multiplicity of the pole $\xi$, then it holds uniformly on $(0,1)$ that

$$f(x) = \sum_{\xi \in \mathcal{M}} r(\xi) \gamma_k(f,\xi) + \sum_{m=1}^{\infty} b_m(f) \; \hat{\psi}_m(x),$$

where the last sum extends over the set of normalized eigenfunctions corresponding to non-zero, real and simple eigenvalues.

ii) The system of functions

$$(V_k : k=1,2,3,\ldots) = \{u_j(x,\xi)/\|u_j\|_2^2, \; \hat{\psi}_m(x) ; \xi \in \mathcal{M}, j < r(\xi), m=1,2,3,\ldots \}$$

is complete in $L^2(0,1)$.

These results are easy outcomes of Propositions 4 and 5, and Theorem 4. Thus, iii) of Theorem 1 is verified.

7. EXPANSION OF A SQUARE-INTEGRABLE FUNCTION.

In this section we show that the complete system of functions $\{V_k\}$ verifies the hypotheses of theorems 1 and 2. This proves in particular the possibility of expanding an $L^2$-function into a series of eigenfunctions and associated functions.

LEMMA 5. The gramian of the system $\{V_k\}$ satisfies i), Th.1.

More precisely, for a certain $\alpha > 1/2$ it holds:

i) $$\int_0^1 \hat{\psi}_n(x) \; \hat{\psi}_m(x) \; dx = \frac{O(1)}{s_n s_m}, \quad \text{if} \; n \neq m,$$

ii) $$\int_0^1 u_k(x,\xi) \; \hat{\psi}_n(x) \; dx = \frac{O(1)}{s_n}, \quad \text{if} \; \nu \text{ is not an integer},$$

iii) $$\int_0^1 u_k(x,\xi) \; \hat{\psi}_n(x) \; dx = \frac{O(1) \log s_n}{s_n}, \quad \text{if} \; \nu \text{ is an integer}.$$

Proof. If $z_n^2 \neq z_e^2$, from (37) and Lommel's formula we get that

$$(\hat{\psi}_n,\hat{\psi}_e) = \int_0^1 \hat{\psi}_n(x) \; \hat{\psi}_e(x) \; dx$$

is equal to:

$$\frac{2}{s_n^2 - z_e^2} \left[ s_e \cdot \frac{J'_v(s_e)}{J_v(s_e)} - s_n \cdot \frac{J'_v(s_n)}{J_v(s_n)} \right] D_n D_e$$

if $J_v(s_n) \neq 0 \neq J_v(s_e)$,

$$\frac{-2}{s_n^2 - z_e^2} \left[ s_n \cdot \frac{J'_v(s_n)}{J_v(s_n)} - s_e \cdot \frac{J'_v(s_e)}{J_v(s_e)} \right] D_n D_e$$

if $J'_v(s_n) \neq 0 \neq J'_v(s_e)$. 

We define now the symmetric polynomial \( V(\lambda, \mu) \) as follows,

\[
V(\lambda, \mu) = (P(\lambda)Q(\mu) - P(\mu)Q(\lambda))/(\lambda - \mu).
\]

Then, taking into account the eigenvalue equation (34), it follows easily that \((\hat{\psi}_n, \hat{\psi}_e)\) is equal to

\[
\int_0^1 \hat{\psi}_n(x) \hat{\psi}_e(x) \, dx = 2s_n s_e \cdot V(s_n^2, s_e^2) \cdot \frac{D_n}{s_n Q(s_n^2)} \frac{D_e}{s_e Q(s_e^2)}
\]

if \( J_v(s_n) \neq 0 \neq J_v(s_e) \),

\[
= 2 s_n s_e \cdot V(s_n^2, s_e^2) \cdot \frac{D_n}{s_n Q(s_n^2)} \frac{D_e}{s_e Q(s_e^2)}
\]

if \( J_v'(s_n) \neq 0 \neq J_v'(s_e) \),

\[
= 2 s_n s_e \cdot V(s_n^2, s_e^2) \cdot \frac{D_n}{s_n Q(s_n^2)} \frac{D_e}{s_e Q(s_e^2)}
\]

if \( J_v(s_e) = 0 \), \( J_v'(s_e) = 0 \).

Now, if \( q > p \) then the degree of \( V(\lambda, \mu) \) in \( \lambda \) or \( \mu \) is not greater than \( q - 1 \), and \( |D_n| \rightarrow 1 \). In consequence \( D_n/(s_n Q(s_n^2)) = o(s_n^{-2q-1}) \).

Then, from (59) we obtain

\[
(\hat{\psi}_n, \hat{\psi}_e) = O(s_n^{-2} s_e^{-2}).
\]

On the other hand, if \( p > q \) then \( \text{deg} \, V(\lambda, \mu) = \text{deg} \, V(\lambda, \mu) \leq p - 1 \) and \( |D_n| \rightarrow 1 \). This implies that

\[
(\hat{\psi}_n, \hat{\psi}_e) = O(s_n^{-1} s_e^{-1}).
\]

i) is thus proved. To prove ii) and iii) of lemma 5 we need a more explicit expression for \( U_k(x, \xi) \) which is exhibited next. Assume \( v \neq \text{integer} \). Then \( Y_v(z) \cdot \sin \nu \pi = \cos \nu \pi \cdot J_v(z) - J_{-v}(z) \), and this, together with (16) yields

\[
-2 \frac{\sin \nu \pi}{\nu} \Phi(x, s^2) = (P+Q/2)(s^2) [J_v(xs) J_{-v}(s) - J_{-v}(xs) J_v(s)]
\]

\[+ Q(s^2) s [J_v(xs) J_{-v}^\prime(s) - J_{-v}(xs) J_v^\prime(s)].\]

Thus for \( v \neq \text{integer} \) we have,

\[
\Phi(x, s^2) = \sqrt{x} [s^{-v} J_v(xs) f(s^2) + s^v J_{-v}(xs) g(s^2)],
\]

where \( f \) and \( g \) are certain entire functions.

Assume now \( v = \text{integer} \). In this case: \( Y_v(z) = (2/\pi) \left[ \log(z/2) \cdot J_v(z) \right] \)

\[+ H_{-v}(z) \] where \( H_{-v}(z) \) is equal to \( z^{-v} \) times an entire function of \( z^2 \).
Therefore, again from (16) we get
\[(\pi/2) \Delta(x,s^2) = -\log x J_\nu(s) J_\nu(xs) + J_\nu(xs) H_{-\nu}(s) - J_\nu(s) H_{-\nu}(xs), \]
\[s. (\pi/2) \Delta(x,s^2) = J_\nu(xs)(J_\nu(s) - s \log x J_\nu(s) + s H'_{-\nu}(s)) - H_{-\nu}(xs). s J'_\nu(s). \]
Thus, for \( \nu = \text{integer} (>0) \) we get with \( f, g \) and \( h \) entire functions
\[(64) \Phi(x,s^2) = \sqrt{x} \left[ s^\nu \log x J_\nu(xs) f(s^2) + s^{-\nu} J_\nu(xs) g(s^2) + \right. \]
\[+ \left. s^\nu H_{-\nu}(xs) h(s^2) \right].\]

In consequence, the associated functions defined in (43), after taking into account (63) and (64), take the form
\[(65) \ k! U_k(x,s^2) = \left( \frac{\partial}{\partial s^2} \right)^k \Phi(x,s^2) = (\text{if } \nu \neq \text{integer}) = \]
\[= \sqrt{x} \sum_{j=0}^{k} \left( \frac{\partial}{\partial s^2} \right)^j \left( s^{-\nu} J_\nu(xs) \right) + g_j(s^2) \left( \frac{\partial}{\partial s^2} \right)^j \left( s^\nu J_\nu(xs) \right) \]
\[(66) \ k! U_k(x,s^2) = (\text{for } \nu = \text{integer}) = \sqrt{x} \sum_{j=0}^{k} \left( \frac{\partial}{\partial s^2} \right)^j \left( s^{-\nu} J_\nu(xs) \right) + h_j(s^2) \left( \frac{\partial}{\partial s^2} \right)^j \left( s^\nu H_{-\nu}(xs) \right) \]
where with \( f_j, h_j \) and \( g_j \) we denote entire functions. Observe that \( h_j(\xi) = 0 \) if \( -\nu + 2j \ll -1 \) since \( U_k(x,\xi) \in L^2(0,1) \).

Next we recall some formulae from the theory of Bessel functions:
\[(67) \left\{ \begin{array}{l}
\left( \frac{d}{dz} \right)^m z^\nu J_\nu(z) = z^{\nu+m} J_{\nu+m}(z), \\
\left( \frac{d}{dz} \right)^m z^{-\nu} J_\nu(z) = (-1)^m z^{-\nu-m} J_{\nu+m}(z). 
\end{array} \right. \]
Let \( \lambda = 0 \) be a pole of \( M(a) \) of order \( r \) and \( \nu \neq \text{integer} \). In this situation (cf. (65)):
\[(68) \ U_k(x,0) = \sqrt{x} \sum_{j=0}^{k} \left( A_j x^{\nu+2j} + B_j x^{-\nu+2j} \right), \quad k = 0,1,\ldots,r-1. \]

\( U_k \in L^2 \) implies that \( B_j = 0 \) for \( 2j-\nu \ll -1 \). Let us see ii) when \( \xi = 0 \).
It will be sufficient to prove (cf. Proposition 3) that
\[(69) \int_{0}^{1} x^{2j+\nu+1} J_\nu(xs) \ dx = 0(s^{-3/2}), \]
\[(70) \int_{0}^{1} x^{2j-\nu+1} J_\nu(xs) \ dx = 0(s^{-\alpha-1/2}), \]
since \( (70) \) equals \( 2j_0 - \nu + 3/2 \) if there is a \( j_0 \) verifying
\[1 > 2j_0 - \nu + 3/2 > 1/2; \alpha = 1 \text{ if there is no } j_0 \text{ with such a property}. \]
In (69) and (70) the \( 0^\alpha \text{'s are independent of } j \text{ for fixed } \nu \).
In fact,
\begin{equation}
(71) \quad t^{2j + v + 2} \int_0^1 x^{2j + v + 1} J_v(x^t) \, dx = \int_0^t y^{2j + v + 1} J_v(y) \, dy = \quad (\text{from (67), } m=1) = \pm \int_0^t y^{2j + v + 1} J_{v+1}(y) \, dy = \quad \pm y^{2j + v + 1} \cdot J_{v+1}(y) \quad \left[ t \mp 2 \int_0^t y^{2j + v} J_{v+1}(y) \, dy \right] \quad = \quad t^{2j + v + 1} \cdot 0(t^{-1/2}) + O(1) + 2j \int_1^t y^{2j + v - 1/2} \cdot 0(1) \, dy = \quad = (t^{2j + v 1/2} + 0(1)) \cdot 0(1)
\end{equation}
and this proves (69), and also (70), because in that case we must consider only j's such that \(2j - v > -1\) \((\text{cf. (68)})\).

Let us assume next that \(s^2 = \xi \neq 0\) is a pole of order \(r\) of \(M(\lambda)\) and still \(\nu \neq \text{integer}\).

From (65) is obtained.
\begin{equation}
(72) \quad u_k(x, s^2) = \sqrt{x} \sum_{j=0}^k \left[ A_j \left( \frac{\partial}{\partial s^2} \right)^j (s^{-\nu} J_{v} (xs)) + B_j \left( \frac{\partial}{\partial s^2} \right)^j (s^\nu J_{-v} (xs)) \right]
\end{equation}
with \(B_j = 0\) if \(2j - \nu \leq -1\), and \(k=0, 1, \ldots, r-1\). To prove ii) in this case it is sufficient to show that the following estimations hold:
\begin{align}
(73) & \quad \int_0^1 \left( \frac{\partial}{\partial s^2} \right)^j (s^{-\nu} J_{v} (xs)) J_v(s_n x) \, dx = 0(s_n^{-1/2}), \\
(74) & \quad \int_0^1 \left( \frac{\partial}{\partial s^2} \right)^j (s^\nu J_{-v} (xs)) J_v(s_n x) \, dx = 0(s_n^{-2-1/2}).
\end{align}
The last one must hold only for \(2j - \nu + 1 > 0\). The left-hand sides of (73) and (74) are equal to
\begin{equation}
(75) \quad \left( \frac{\partial}{\partial s^2} \right)^j \int_0^1 (s^{2\nu} J_{2\nu} (xs)) J_v(s_n x) \, dx = \quad \left( \frac{\partial}{\partial s^2} \right)^j \left[ J_{2\nu} (xs) J_v(s_n x) \cdot s_n x - J_v(s_n x) J'_{2\nu} (xs) \cdot s x \right] \right|_{x=1}^{x=0}
\end{equation}
corresponding the upper sign to (73) and the lower one to (74). Then, left-hand side of (73) =
\begin{equation}
= \left( \frac{\partial}{\partial s^2} \right)^j \left[ J_{-\nu} (s) J'_v(s_n) \cdot s_n - J_v(s_n) J'_v(s_n) \cdot s \right] \frac{s^{\nu}}{s^2 - s_n^2} = 0(1/s_n^{3/2});
\end{equation}
Left-hand side of (74) =
\begin{equation}
= \left( \frac{\partial}{\partial s^2} \right)^j \left[ J_{-\nu} (s) s^\nu J'_v(s_n) s_n - J_v(s_n) J'_v(s) s^{1+\nu} \right] \frac{s^{1+\nu}}{s^2 - s_n^2}.
\end{equation}
where \( a_0 = 2^{-v/r(v+1)} \), \( b_0 = 2^v/r(1-v) \). The last equality follows from the fact that here \( v < 2j+1 \); as before, \( \alpha \) is defined by
\[
1 \wedge (2j - v + 3/2 ; 2j - v + 3/2 > 1/2) .
\]
The proof of ii) is thus accomplished.

Let us see iii), i.e., assume \( v = \) positive integer. If zero is a pole of order \( r \) of \( M(\lambda) \), from (66) and for \( k=0,1,...,r-1 \), we obtain
\[
\begin{align*}
(76) \quad U_k(x,0) &= \sqrt{x} \left\{ \sum_{j=0}^{k} x^{v+2j} (A_j + B_j \log x) + \sum_{j=0}^{k} x^{-v+2j} C_j \right\},
\end{align*}
\]
and \( C_j = 0 \) if \(-v+2j < -1\). The \( \alpha \) defined above is now equal to one and (71) can be written as
\[
\int_0^1 x^{v+2j+1} J_v(x s_n) \, dx = O(s_n^{-3/2}) \text{ if } \pm v+2j+1 > 0.
\]

Then if \( \xi = 0 \), iii) will be established if we show that
\[
\begin{align*}
(77) \quad \int_0^1 x^{v+2j+1} \log x \cdot J_v(x s_n) \, dx &= O\left(\frac{\log s_n}{s_n^{3/2}}\right).
\end{align*}
\]

But, as in (71) after integrating by parts we see that
\[
\int_0^1 x^{v+2j+1} \log x \cdot J_v(x t) \, dx =
\]

\[
= t^{-v-2j-2} \left\{ \int_0^t y^{v+1} J_{v+1}(y) y^{2j-1} (2j \log y + 1) \, dy \right\}
\]

\[
= t^{-v-2j-2} O(1) \int_0^t y^{v+2j-1/2} \left( 2j (\log t - \log \xi) + 1 \right) \, dy = O\left(\frac{\log t}{t^{3/2}}\right)
\]

and this implies (77). Here again the 0 does not depend of \( j \). Let us suppose, for the last step, that \( \xi = s^2 \neq 0 \) is a pole of order \( r \) of \( M(\lambda) \). From (66) it follows
\[
(78) \quad U_k(x,s^2) = \sqrt{x} \left\{ \sum_{j=0}^{m} A_j x^{v+2j} + \sum_{j=0}^{m} B_j x^{v+2j} \log x + \sum_{2j-v-1}^{m} C_j x^{-v+2j} \right\},
\]
where the series of coefficients \( A_j, B_j, \) and \( C_j \) are absolutely summable. Then, from the estimations (69), (70) and (77) (recall, that they hold uniformly on \( j \)), (79) is obtained:
\[
(79) \quad \int_0^1 U_k(x,\xi) J_v(x s_n) \sqrt{x} \, dx = (\xi | A_j |) O\left(\frac{1}{s_n^{3/2}}\right) +
\]

\[
+ (\xi | B_j |) O\left(\frac{\log s_n}{s_n^{3/2}}\right) + (\xi | C_j |) O\left(\frac{1}{s_n^{a+1/2}}\right),
\]

and since in this situation \( \alpha = 1 \), iii) is proved. Q.E.D.
THEOREM 5. The system \( \{V_n\} \) defined at the end of section 6 verifies the hypothesis of Theorem 1.

Proof. Lemma 5 proves i) of Th. 1; Propositions 4 and 5 and the Corollaries to theorem 4 prove iii) and iv). It remains only to check ii) and this will follow from the following proposition:

\[ f \in L^2, \quad b_n(f) = (f, \hat{\psi}_n) = 3K \left( \sum |b_n| \right)^2 \leq K \|f\|_2^2. \]

In fact, let \( \sigma_n \) be positive zeroes of \( J_v(t) \) if \( p > q \), and of \( J_v'(t) \) if \( p < q \). It is not difficult to see that if \( s_n \) is a pole of \( N(s^2) \) then eventually after a renumeration of \( \{\sigma_n\} \) it follows that:

\[ |\sigma_n - s_n| \longrightarrow 0 \text{ for } n \longrightarrow \infty. \]

One way of checking this is to prove, using the asymptotic formulae for \( J_v \) and \( J_v' \), that \( \sigma_n \) satisfies formula (26) if \( p > q \), and (25) if \( p < q \), but with different values for the constants \( C \).

Define now \( \varphi_n(x) = \delta \sqrt{x} J_v(\sigma_n x). \sqrt{x} J_v(s_n x) \|^{-1} \) with \( \delta = +1 \) or -1 and such that \( \hat{\psi}_n(x) = \delta \sqrt{x} J_v(s_n x). \sqrt{x} J_v(s_n x) \|^{-1} \).

\( \{\varphi_n(x)\} \) is an orthogonal family of \( L^2 \)-functions, and \( \|\varphi_n\|_2 \longrightarrow 1 \),

(this is a consequence of one of Lommel's formulae that asserts that
\[ \int_0^1 x J_v(ax) \, dx \sim k/a \quad \text{if } a \longrightarrow \infty. \]

Then, calling \( \tilde{b}_n(f) = (f, \varphi_n) \) we have

\[ \frac{\sum |\tilde{b}_n(f)|^2}{\|f\|_2^2} \leq K \frac{\sum |b_n|^2}{\|\varphi_n\|_2^4} \leq K. \|f\|_2^2. \]

ii) will follow from

\[ b_n(f) = \tilde{b}_n(f) + \|f\|_2 0(1/n), \quad n > n_0. \]

But, \( \hat{\psi}_n(x) - \varphi_n(x) = \sqrt{x} J_v(\sigma_n x) - J_v(s_n x) \sqrt{x} J_v(s_n x) \|^{-1} = \sqrt{x} s_n 0(1) \). (\( \sigma_n - s_n \) \( J_v'(s_n x) = \sqrt{s_n \beta_n} \) \( \sigma_n - s_n \) \( \sqrt{s_n \beta_n} J_v'(s_n x) \)).

where \( \beta_n \) is a number between \( s_n \) and \( \sigma_n \). Since \( s_n/\sigma_n \longrightarrow 1 \), it also holds that \( \beta_n/\sigma_n \longrightarrow 1 \), and from the last formula we obtain

\[ |\hat{\psi}_n(x) - \varphi_n(x)| = |\sigma_n - s_n| 0(1). \]

(82), together with next estimation, proves (81).

\[ |\sigma_n - s_n| = O(1/s_n^3). \]

To establish this, first observe that from (34) it follows \( J_v(s_n) = O(s_n^{-3/2}) \) when \( p > q \), and then

\[ O(1/s_n^3) = J_v(s_n) - J_v(\sigma_n) = J_v(s_n) - J_v(s_n) - J_v'(s_n) J_v' = J_v'(s_n). \]
where \( t_n \) is a number between \( \sigma_n \) and \( s_n \). On the other hand \( |J_\nu(s_n)| > |J_\nu(t_n)| \) because of the monotony of \( J_\nu \) between two zeroes of \( J'_\nu \).

Therefore, \( J_\nu(t_n) = O(s_n^{-3/2}) \) and \( |J'_\nu(t_n)| > C_0/\sqrt{s_n} \). After replacing in (84), (83) follows.

Assume next \( p \leq q \). (34) implies \((t_n as before) that

\[
(85) \quad M/s_n^{3/2} > |J'_\nu(s_n)| = |J_\nu'(s_n) - J_\nu'(\sigma_n)| = |\sigma_n - s_n|.|J''_\nu(t_n)|.
\]

From Bessel's equation: \( J''_\nu(t_n) = -J_\nu(t_n) + O(1/t_n^{3/2}) \), and therefore

\[
|J''_\nu(t_n)| > C_1/\sqrt{s_n}
\]

holds. As in the preceding case (85) implies (83).

Q.E.D.

**Theorem 6.** Hypothesis b) of theorem 2 holds for the system \( \{V_n\} \) under consideration. Therefore \( g > 0 \).

**Proof.** From (59) we see that

\[
\int_0^1 \psi_n \hat{\psi}_m \, dx = 0 \Rightarrow V(s_n^2, s_m^2) = 0.
\]

Since \( P \) and \( Q \) have no root in common, for any value of \( s_n \), \( V(s_n^2, y) \) has only a finite number of roots. Then

\[
\int_0^1 \psi_n \hat{\psi}_m \, dx \neq 0 \quad \text{for each } s_n \text{ and infinitely many } s_m.
\]

Q.E.D.
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