A NOTE ON HOLLOW MODULES

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Let $R$ be a ring with identity and $M$ a right unitary $R$-module. Let $N$ be an $R$-submodule of $M$. $N$ is called a small submodule in $M$ if it satisfies the following condition: the fact that $M = T + N$ for some $R$-submodule $T$ implies $T = M$. If every proper submodule of $M$ is small, we call $M$ a hollow module $[4]$. P. Fleury studied some conditions under which the endomorphism ring of a hollow module is a local ring. We shall call $M$ completely indecomposable when its endomorphism ring is local.

In § 1 we shall show that every finitely generated and uniform hollow module is completely indecomposable, when $R$ is a left or right perfect ring. In § 2 we shall give some relations between injective hollow modules and QF-3 rings. In § 3 when $R$ is a commutative Dedekind domain, we can completely determine all hollow modules and we know that they are completely indecomposable.

1. PERFECT RINGS.

Let $R$ be a ring with identity and $M$ a right unitary $R$-module. By $J(M)$ we shall denote the Jacobson radical of $M$. Since every small submodule in $M$ is contained in $J(M)$, we have

**LEMMA 1.1** ([4]). $M$ is a finitely generated hollow module if and only if $J(M)$ is maximal and small in $M$. In this case $M$ is cyclic.

If $M = mR$, $M \cong R/A$, where $A$ is a right ideal in $R$. Since $J(M)$ is always small in $M$ whenever $M$ is finitely generated, we have

**COROLLARY 1.2** ([4]). $R/A$ is hollow if and only if $A$ is contained in a unique maximal right ideal.

It is clear that every hollow module is indecomposable and so a hollow module of finite length is always completely indecomposable.

**THEOREM 1.3.** We assume that $J(R)$ is nil and $R/J(R)$ is artinian. Then every finitely generated and uniform hollow module is completely indecomposable, where $J(R)$ is the Jacobson radical of $R$.

**Proof.** Let $M$ be an $R$-module with the property above. Then $M \cong R/A$ for
some right ideal \(A\) from Lemma 1.1. If \(A \subseteq J(R)\), then \(R\) is a local ring from Corollary 1.2. It is well known that \(\text{End}_R(R/A) = I(A)/A\); \(I(A) = \{x \in R \mid xA \subseteq A\}\). Let \(\bar{x}_1, \bar{x}_2\) be non-epimorphic elements in \(I(A)/A\). Then \(x_1R \subseteq J(R)\). Hence, \(\bar{x}_1 + \bar{x}_2\) is not epimorphic. Let \(\bar{y}\) be an epimorphism with \(\bar{T} = \bar{x}_1 + \bar{y}\). Since \(x_1 \in I(A) \cap J(R)\), \((1-x_1)^{-1} = 1 + x_1 + \ldots + x_1^{n-1} \in I(A)\), where \(x_1^n = 0\). Hence, \(\bar{y}\) is isomorphic. Let \(\bar{x}_1\) and \(\bar{x}_2\) be epimorphic but not isomorphic, then \(y_1^{-1}(0) \cap y_2^{-1}(0) \neq (0)\) from the assumptions. Hence, \(\bar{x}_1 + \bar{x}_2\) is not isomorphic. Therefore, \(I(A)/A\) is a local ring. Next, we assume \(A \not\subset J(R)\).

Since \(R/(A+J(R))\) is semi-simple and hollow module, \(A+J(R)\) is a maximal right ideal in \(R\). Let \(R/J(R) = (A+J(R))/J(R) \oplus \mathbb{B}\), where \(\mathbb{B}\) is a minimal right ideal in \(R = R/J(R)\). Since \(R\) is semi-simple and artinian and \(J(R)\) is nil, there exist idempotents \(e\) and \(f\) such that \(e \in A\), \(\mathbb{B} = \mathbb{B} \oplus \mathbb{B}\) and \(R = eR \oplus fR\). Hence, \(R/A \cong fR/(fR \cap A) = fR/fC\), where \(C = fR \cap A\). Then \(\text{End}_R(R/A) \cong \text{End}_R(fR/fC) = \text{I'}(fC)/fC\); \(\text{I'}(fC) = \{x \in fR \mid xC \subseteq fC\}\). Now, \(fR/fC\) contains a unique maximal submodule \(fJ(R)/fC\). Hence, we can prove, similarly to the first part, that \(\text{End}_R(fR/fC)\) is local.

**COROLLARY 1.4.** If \(R\) is a left or right perfect ring\(^3\), then every finitely generated and uniform hollow module is completely indecomposable.

2. **QF-3 RINGS.**

Let \(R\) be a commutative ring. If Krull dimension of \(R\) is equal to zero, \(R\) is never small in any ring extension \([9]\). We shall study a similar situation on \(R\)-modules. First we take any ring \(R\), which is not necessarily commutative.

**PROPOSITION 2.1.** Let \(M\) be an \(R\)-module. Then the following conditions are equivalent:

1) \(M\) is not a small submodule in any extension module \(M'\) of \(M\).

2) \(M\) is not small in an injective hull \(E(M)\) of \(M\).

3) There exists an injective module \(E\) containing \(M\) such that \(M\) is not small in \(E\).

**Proof.** 1) \(\rightarrow\) 2) \(\leftrightarrow\) 3) are clear. 2) \(\rightarrow\) 1). We assume \(M' \supseteq M\). Then \(E(M') = E(M) \oplus E_1\). Hence, \(M\) is not small in \(E(M')\). Therefore, \(M\) is not small in \(M'\).

If \(M\) satisfies one of three equivalent conditions in Proposition 2.1, we say \(M\) is non-small in injectives. It is well known that any non-zero submodule is not small in \(M\) if and only if \(J(M) = (0)\).
Hence, we have

**Proposition 2.2.** The following conditions are equivalent:

1) Any non-zero module is non-small in injectives.
2) \( R \) is a right \( V \)-ring.

**Proof.** See [2], p. 356.

We note that if \( M \) is non-small in injectives, then so is any module extension \( M' \) of \( M \).

**Lemma 2.3.** Let \( M \supseteq M_1 \) be \( R \)-modules. If \( M/M_1 \) is non-small in injectives, then so is \( M \).

**Proof.** It is clear from the definitions and the above remark.

**Proposition 2.4.** A ring \( R \) is small in \( E(R) \) if and only if \( E = J(E) \) for any injective module \( E \).

**Proof.** If \( E \not= J(E) \) for some injective module \( E \), there exists a homomorphism \( f \) of \( R \) to \( E \) such that \( f(R) \not\subseteq J(E) \). Hence, \( R \) is not small in \( E(R) \) by Proposition 2.1 and Lemma 2.3. Next, we assume \( R \) is not small in \( E(R) \). Then there exists a submodule \( T \not= E(R) \) such that \( E(R) = R + T \). Hence, \( E(R)/T \) contains a maximal submodule.

**Corollary 2.5.** If \( R \) is a perfect ring, \( R \) is non-small in injectives as an \( R \)-module. If \( R \) is a commutative domain, \( E = J(E) \) for any injective module \( E \).

**Proof.** It is clear from [1], Lemma 2.6 and [8], Theorem 2.

From now on in this section, we assume \( R \) is a right perfect ring. Then there exists a complete set \( \{g_i\} \) of mutually orthogonal primitive idempotents such that \( 1 = \sum g_i \). We shall divide \( \{g_i\} \) into two parts: \( \{g_i\} = \{e_i\}_{i=1}^n \cup \{f_j\}_{j=1}^m \), where the \( e_i R \) is non-small in injectives and the \( f_j R \) is small in \( E(f_j R) \). We know \( n \geq 1 \) by Corollary 2.5. If we denote primitive idempotents by \( e \) and \( f \), respectively, we mean \( e \) belongs to the first class and \( f \) does to the second.

Next, we shall consider two conditions

(*) Every non-small module in injectives contains a non-zero injective module.

and

(**) Every indecomposable injective module is hollow, namely contains a unique maximal submodule.

Let \( K \) be a field and \( R \) a \( K \)-algebra of finite dimension. Then \( \text{Hom}_K(\cdot,K) \) is a dual functor and so the condition (**) is dual to (**)_1 (resp. (**)_2). Every indecomposable, projective left (resp. right) module contains a unique minimal submodule.
LEMMA 2.6. Let $R$ be a right perfect ring. Then (*) holds if and only if every indecomposable, non-small module in injectives is injective. (***) holds if and only if every indecomposable, injective module is of the form $e_iR/e_iA$, where $A$ is a right ideal. (*) implies (**).

Proof. We assume (*) and $M$ is indecomposable, non-small in injectives. Then $M$ is injective and hollow. Hence, (***) holds. Since $M \not\cong J(M)$, $M/J(M) \cong g_iR/g_iJ(R)$ by Lemma 1.1. Therefore, some $e_iR$ is a projective cover of $M$ by Lemma 2.3. Conversely, let $M$ be non-small in injectives and $E = E(M)$. Since $M \not\cong J(E)$, we have $m \in M$ in $M-J(E)$. Then $mR$ is non-small in injectives by Proposition 2.1. Since $mR/mJ(R)$ is of finite length, $mR$ contains an indecomposable and non-small module in injectives. Hence (*) holds.

LEMMA 2.7. Let $R$ be as above. If $M$ is a non-small submodule in $\sum g_iR/g_iA_i$, then there exists a module $E$ such that $E(M) = g_iR/g_iA_i$, where $A_i$ is a right ideal and $E$ is the projection on $g_iR/g_iA_i$.

Proof. Since $M \not\cong \sum g_iJ(R)/g_iA_i$, $E(M) \not\cong g_iJ(R)/g_iA_i$ for some $i$. Hence, $E(M) = g_iR/g_iA_i$, since $g_iR$ is hollow.

PROPOSITION 2.8. Let $R$ be a right artinian ring. Then $R$ is a QF-ring if and only if (*) holds and $e_iR = (0)$ for all $i$ and $j$.

Proof. Let $R$ be a QF-ring and $M$ non-small in injectives. Let $E = E(M)$. Then $E = \sum e_iR$ by [3]. Since $M$ is not small in $E$, $M$ contains a direct summand isomorphic to $e_iR$ by Lemma 2.7. Since $f_j = 0$ for all $j$, $e_iR = (0)$. Conversely, we assume (*). Then the $e_iR$ is injective by Lemma 2.6. If $f_j \neq 0$, $E(f_jR) = \sum e_iR/e_iA_i$. Hence, $0 \neq f_jR = (0)$ for some $k$, which is a contradiction to the assumption.

LEMMA 2.9. Let $R$ be as above. If (***) holds, every $f_iR$ is isomorphically contained in a direct sum $\sum e_iR$ and there exists a right ideal $A_i$ such that $e_iR/e_iA_i$ is non-zero injective for each $i$.

Proof. Let $E = E(fR)$. Then $E = \sum e_iR/e_iA_i$ by Lemma 2.6.

Let $\psi: fR \longrightarrow E$ be the inclusion and $\varphi(f) = \sum e_iR/e_iA_i$. We define $\psi: fR \longrightarrow E$ by setting $\psi(fx) = \sum e_iR/e_iA_i$. It is clear that $\psi$ is monomorphic. Let $F = E(eR)$ and $F = \sum e_iR/e_iA_i$ as above. Since $eR$ is not small in $F$, $eR$ is epimorphic to some $e_iR/e_iA_i$. Hence, $eR = e_iR$. 

\[ e_iR = \sum e_iR/e_iA_i \]
PROPOSITION 2.10. Let $R$ be right artinian. Then $R$ is right QF-3 if either (*) holds or (**) holds and each $e_i R$ contains a unique minimal submodule.

Proof. If (*) holds, each $e_i R$ is injective. Hence, $R$ is right QF-3 by Lemma 2.9 and [10]. In the second case $E(e_i R) = e_i R/e_i A$ and $e_i R = e_i R$ from the proof of Lemma 2.9. Hence, $e_i A = (0)$.

COROLLARY 2.11. Let $R$ be a $K$-algebra of finite dimension over a field $K$. If (**) and (**) hold, $R$ is QF-3.

The examples below show that the converse is not true. Now, we shall study QF-3 rings satisfying (*) or (**) 

THEOREM 2.12. Let $R$ be right artinian. When either $R$ is hereditary or $J(R)^2 = (0)$, the following conditions are equivalent:

1) (*) holds.

2) (**) holds and each $e_i R$ contains a unique minimal submodule.

3) $R$ is a right QF-3 ring.

Proof. 1) $\rightarrow$ 2) $\rightarrow$ 3) are clear. 3) $\rightarrow$ 1). First, we assume that $R$ is hereditary. We may assume $R$ is basic and two-sided indecomposable. Then $R$ is a ring of upper triangular matrices over a division ring by [6], Theorem 2. Hence, only one $e_1 R$ is injective and $f_j R/f_j J(R)$ is isomorphic to submodule of $e_1 R/e_1 A$. Therefore, every injective module is isomorphic to a direct sum of some $e_1 R/e_1 A_i$, where the $A_i$ is a right ideal. Let $M$ be non-small in injectives. Then we have an epimorphism $f: M \rightarrow e_1 R/e_1 A_i$ from Lemma 2.7. Hence, we have $h: e_1 R \rightarrow M$ such that $fh \neq 0$. Since $R$ is hereditary, $M$ contains an injective module.

Next, we assume $J(R)^2 = (0)$ Since $R$ is right QF-3, some $e_i R$ is injective. Let $\{e_i R\}_1$ be the set of such an injective right ideal. We assume $t < n$. Then $e_n R$ is non-small in an injective module $\bigoplus e_R; e_R \in \{e_i R\}_1$ by [10]. Hence, $e_n R$ is isomorphic to some $e_i R$ from Lemma 2.7, which is a contradiction. Since $f_j R \subseteq \bigoplus e_i R$, $f_j R$ is simple. Hence, $f_j R$ is monomorphic to some $e_k R$. We assume $e_1 R/e_1 J(R)$ is not injective.

Then $E = E(e_1 R/e_1 J(R))$ is indecomposable. Take $a \in E - J(E)$. Since $a = \sum a_i$, $a_i \notin J(E)$ for some $i$. Hence, we may assume $a \in \bigoplus e_k - J(E)$ by Lemma 2.3. Then we have either $e_k R = e_k R$ or $e_k R/e_1 J(R)$. Since $a \notin J(E) \not\subseteq e_1 R/e_1 J(R)$, $e_k R$ is injective. Hence $E = e_k R$. Thus we have proved that any indecomposable injective module is isomorphic either to some $e_i R$ or $e_j R/e_j J(R)$. Let $M$ be indecomposable, non-small in injectives and $E$ its injective hull. Let $S(M)$ be the socle of $M$. Then
E = E(S(M)) and S(E) = S(M). Let E = \sum_{k} e_{ik} R / e_{ik} R \cdot e_{ik} R / e_{ik} J(R).

Since \sum_{k} e_{ik} R / e_{ik} J(R) \subseteq S(E) \subseteq M, M = \sum_{k} e_{ik} R / e_{ik} J(R) \oplus M \cap (\sum_{k} e_{ik} R).

If K \neq \emptyset, M \approx e_{k} R / e_{k} J(R). If K = \emptyset, M is not small in \sum_{k} e_{ik} R.

Hence, M \approx e_{k} R by Lemma 2.7. Therefore, (*) holds.

EXAMPLES. 1) Let K be a field, M a K-vector space of finite dimension and M* = Hom_K(M,K). We put

\[
R = \begin{pmatrix}
K & M* & K \\
K & M \\
0 & K
\end{pmatrix}
\]

Then R is a QF-3 ring by the natural multiplication M* \otimes K M \rightarrow K (see [7]). If [M:K] \geq 2, (**) does not hold, since Re_{22} contains two minimal submodules.

2) Put

\[
R = \begin{pmatrix}
K & K & K & K \\
K & 0 & 0 \\
K & 0 \\
0 & K
\end{pmatrix}
\]

Then (**) holds but R is not QF-3.

3) Let S be the ring of upper triangular matrices over K with degree n and R a K-subalgebra of S containing \{e_{ii}\}_{i=1}^{n}. We assume R is a two-sided indecomposable ring.

Then R is QF-3 if and only if (**) holds and e_{ii} R contains a unique minimal submodule. R is QF-3 and hereditary if and only if (*) holds.

Proof. First, we assume e_{ii} R is injective. Then we shall show that e_{ii} R is not injective for all i \geq 2. Let \{e_{ii} R; e_{ii} R = e_{ii} R\}_{i=1}^{t} be the set of such an injective right ideal. We note if e_{kk} R e_{ii} \neq (0), e_{ii} R is monomorphic to e_{kk} R. Hence, since e_{ii} R is indecomposable, e_{ii} R e_{ii} = (0) for t \geq 2. Let e_{ii} R e_{pp} \neq (0) and e_{ii} R e_{qq} \neq (0) for t \geq 2. Then e_{pp} e_{qq} = e_{qq} e_{pp} \neq (0), because if e_{pp} e_{qq} \neq (0), e_{pp} R is monomorphic to e_{ii} R and so e_{ii} R e_{ii} \neq (0), since e_{ii} R is injective. Therefore, R is a direct sum of two ideals A_{i} such that A_{1} = \sum_{k} e_{pp} R; e_{ii} R e_{pp} \neq (0) and A_{2} = \sum_{k} e_{qq} R; e_{ii} R e_{qq} \neq (0) for some t \geq 2. Since R is indecomposable, s = 1. We assume R is QF-3. Then e_{ii} R is only one injective ideal among e_{ii} R. Hence, e_{ii} R e_{ii} = e_{ii} R e_{mn} = K for all i.

We shall show E(e_{ii} R / e_{ii} J(R)) is isomorphic to e_{ii} R / e_{ii} A for some right
ideal A. We assume \( e_j^1 \mathbb{R} e_j^2 \mathbb{R} \ldots \mathbb{R} e_j^t \mathbb{R} = (0) \) and
\[
e_{j+1}^t \mathbb{R} e_j^i = \ldots = e_{j+1}^{t-t+1} \mathbb{R} e_j^i = K,
\]
where
\[
(j < j_2 < \ldots < j_t, 1 = j_{t+1} < \ldots < j_t = i) = \{1, \ldots, i\}.
\]
Put \( e_j^1 A = e_{j_1}^1 + e_{j_2}^1 + \ldots + e_{j_t}^1 + e_{j_{t+1}}^1 R/A \). Then \( e_j^1 A \) is a right ideal
and \( e_j^1 \mathbb{R} e_j^1 A = e_{j+1}^t \mathbb{R} K + e_{j+2}^1 \mathbb{R} \ldots + e_{j+t}^1 \mathbb{R} e_{j_{t+1}}^1 R/A \) and \( e_j^1 \mathbb{R} e_j^1 A \) is injective. Therefore, (*) holds. The converse is clear from the first part and
Proposition 2.10. If \( R \) is QF-3 and hereditary, (*) holds by Theorem 2.12. We assume (*) holds. Then \( R \) is QF-3 by Proposition 2.10. Let
\( E = E(1 \mathbb{R} e_j^1 R) \) and \( E \cong \bigoplus_k e_j^1 \mathbb{R} e_j^1 A_k \). If \( e_j^1 \mathbb{R} e_j^1 R \) is small in \( E \),
\( e_j^1 \mathbb{R} e_j^1 R \subseteq \bigoplus_k e_j^1 \mathbb{R} e_j^1 A_k \). However, \( e_j^1 \mathbb{R} e_j^1 R \) and so \( e_j^1 \mathbb{R} e_j^1 R \) is non-small in injectives. Hence, \( e_j^1 \mathbb{R} e_j^1 R \) is injective by (*)..
Since \( \text{Hom}_K(e_j^1 \mathbb{R} e_j^1 R, K) \) is projective and isomorphic to \( e_j^1 \mathbb{R} e_j^1 K \),
\( e_j^1 \mathbb{R} e_j^1 A \) is injective for all \( i < j \). Therefore, \( R \) is hereditary by [6], Theorem 2.

Concerning with Example 3, we have

PROPOSITION 2.13. Let \( R \) be right artinian and right QF-3. Then \( R \) is hereditary if and only if \( e_j^1 \mathbb{R} e_j^1 A \) is injective for all \( i \) and any right
ideal \( A \).

Proof. Since \( R \) is QF-3, \( \{e_j^1 \mathbb{R} R\}^n \) is a complete set of indecomposable,
injective right ideals (see the first part in the proof of Theorem 2.12). Hence, "only if" part is clear. Conversely, we assume \( e_j^1 \mathbb{R} e_j^1 A \)
is injective for each \( i \) and \( A \). Let \( E \) be an injective module and
\( a \in E/J(E) \). Then \( e_j^1 \mathbb{R} R \) is injective for some \( k \) from the assumption and
Lemma 2.3. Hence, \( R \) satisfies (**) and \( E/M \) is injective for any submodule \( M \). Let \( S(M) \) be the socle of \( M \). We define Loewy se ries \( S^i(M) \) as follows: \( S^0(M)/S^0(-1(M) = S(M/S^0(M)) \). We show the above
fact by induction on \( S^i(M) \). Let \( E = E(M) \cdot E_1 \) and \( E_2 = E(M) =
\bigoplus_i e_j^1 \mathbb{R} e_j^1 A \). Since \( S(M) = S(E_2) \) and \( E_2/S(E_2) \cong M/S(M) \) and \( E_2/S(E_2) \)
is injective from the assumption. Hence, if \( M = S(M) \), \( E/M \) is injective.
We assume \( E'/N' \) is injective for \( E' \) whenever \( E' \) is injective and
\( S^i(N') = N' \). Let \( M = S^{i+1}(M) \). Then \( E/S(M) \) is injective and \( S^i(M/S(M)) =
M/S(M) \). Hence, \( E/M \cong (E/S(M))/(M/S(M)) \) is injective by the induction.

3. MINIMAL NON-SMALL MODULES.

Since any extension of a non-small module in injectives is always non-
small in injectives, we are interested in a minimal one among non-small modules in injectives.

**Proposition 3.1.** If M is minimal one among non-small modules in injectives, then M is a maximal one among hollow modules.

*Proof.* Let E = E(M). Then a proper submodule M_1 of M is small in E from Proposition 2.1. If M = M_1 + M_2 and M_2 ≠ M, M is small in E from the above, which is a contradiction. It is clear that M is maximal one among hollow modules.

We do not know whether the converse of Proposition 3.1 is true. We shall show an affirmative answer when R is a commutative Dedekind domain.

From now on, we assume that R is a commutative domain.

**Proposition 3.2.** Let M be a torsion-free and maximal hollow module. Then M is isomorphic to the quotient field Q of R.

*Proof.* We may assume M ⊆ M ⊗_R Q. Let t ≠ 0 be in R. Then t^{-1}M (⊆ M ⊗_R Q) is also a hollow module containing M. Hence, M = t^{-1}M and M is injective and indecomposable. Therefore, M ≃ Q.

**Theorem 3.3.** Let R be a Dedekind domain. Then a hollow module is isomorphic to one of the following:

1) R/p^n, 2) E(R/p), where p is a prime ideal and 3) R or Q when R is local. In this case every hollow module is completely indecomposable.

*Proof.* Let M be a hollow module. If M is not torsion-free, M contains a direct summand isomorphic to either E(E/p) of R/p^n by [11], Theorem 9. Hence, M is isomorphic to one of them. We assume M is torsion-free. Then E(M) = ∑ u_i Q by [11], Theorem 7. We put M ∩ u_i Q = u_i M_i ≠ (0).

If M = ∑ u_i M_i, I consists of one element and we may assume R ⊆ M ⊆ Q.

Let p and q be prime ideals in R. Since M/pq is a torsion hollow module, p = q by the above argument. Hence, R is local and M ≃ R or M ≃ Q.

Let M ≠ ∑ u_i M_i, ∑ u_i M_i is a small submodule in M. Since M/∑ u_i M_i is torsion and hollow, M/∑ u_i M_i is isomorphic to E(R/p) or R/p^n.

When M/∑ u_i M_i ≠ E(R/p), M = aM + ∑ u_i M_i for any a ≠ 0 in R. Hence, M is injective and so M ≃ Q. When M/∑ u_i M_i ≠ R/p^n, b^M ⊆ ∑ u_i M_i for b ≠ 0 in p. If u_i M_i = u_i Q ⊆ M for some i, M ≃ Q. Hence, we may assume u_i M_i ≠ u_i Q for all i. Now, let π_i be the projection of M to u_i Q, then π_1(M) is a non-zero hollow module in u_1 Q. Hence, R is local from the above. Accordingly, every u_i M_i is projective and so M is projective.

Therefore, M ≃ R.
From Theorem 3.2 and Proposition 3.1 we have

**THEOREM 3.3.** Let $R$ be a Dedekind domain. Then the following conditions are equivalent for an $R$-module $M$.

1) $M$ is a minimal one among non-small modules in injectives.
2) $M$ is a maximal one among hollow modules.
3) $M$ is isomorphic to $E(R/p)$ or to $Q$ if $R$ is local, where $p$ is a prime ideal and $Q$ is the quotient field of $R$.

**REMARK.** Let $R$ be a Dedekind domain which is not local. Then $Q$ is not small in injectives, however $Q$ does not contain a minimal non-small module in injectives.

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Recibido en agosto de 1977.