The objective of this article is to present some results about the classification of one-parameter groups of isometries of $H^p$ and to calculate their Lie algebra using hermitian operators.

It is known that all conformal self-maps of $|z| < 1$ can be written as

$$\varphi(z) = \frac{az+b}{bz+a}$$

(1)

where $a, b$ are complex numbers and

$$|a|^2 - |b|^2 = 1.$$  

(2)

The representation (1) is unique up to sign: if $a, b$ are replaced by $-a, -b$ we get the same $\varphi$, and only in this case. Thus the group $G$ of conformal self-maps of $|z| < 1$ is isomorphic to the group $PU(2) = \mathbb{P}U(2)/\mathbb{Z}_2$ where $PU(2)$ (see [4]) is the group of $2 \times 2$ complex matrices

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with condition (2).

Suppose now that $\{\varphi_t\}$ is a one-parameter subgroup of $G$ represented by

$$\varphi_t(z) = \frac{a(t)z + b(t)}{b(t)z + a(t)}$$

(3)

with (2) holding for each $t$. The composition rule $\varphi_t \circ \varphi_s = \varphi_{t+s}$ implies

$$a(t+s) = a(t)a(s) + b(t)b(s)$$

$$b(t+s) = a(t)b(s) + b(t)a(s)$$

Taking derivatives with respect to $s$ yields:

$$a' = aa + \beta b$$

(4)

$$b' = \beta a + \bar{a} b$$
where \( \alpha = a'(0), \beta = b'(0) \). In order to solve (4) we need \( \exp(tM) \) where

\[
M = \begin{bmatrix}
\alpha & \beta \\
\beta & -\alpha
\end{bmatrix}
\]

First we differentiate \( a(t)a(t) - b(t)b(t) = 1 \) to get \( \alpha + \bar{\alpha} = 0 \) so that \( \bar{\alpha} = -\alpha \) and therefore

\[
M = \begin{bmatrix}
\alpha & \beta \\
\beta & -\alpha
\end{bmatrix}
\]

It follows that if \( \xi^2 = |\beta|^2 + \alpha^2 = |\beta|^2 - |\alpha|^2 \) we have

\[
M^{2n} = \xi^{2n}I_2
\]

\[
M^{2n+1} = \xi^{2n}M
\]

where \( I_2 \) is the 2x2 identity matrix.

Thus

\[
\exp(tM) = I_2 + tM + \frac{t^2}{2!} \xi^2 I_2 + \frac{t^4}{4!} \xi^4 M + \ldots =
\]

\[
= (1 + \frac{(tt)^2}{2!} + \frac{(tt)^4}{4!} + \ldots) I_2 + \]

\[
+ (t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots)M =
\]

\[
= \cosh(t\xi)I_2 + t^{-1} \sinh(t\xi)M
\]

(and then, quite clearly, the choice of square root of \( \xi^2 \) is irrelevant for these formulas).

We conclude that the solution of (4) with initial conditions \( a(0) = 1, \ b(0) = 0 \) is

\[
a(t) = \cosh(t\xi) + \alpha t^{-1} \sinh(t\xi)
\]

\[
b(t) = \beta t^{-1} \sinh(t\xi)
\]

unless \( t = 0 \) in which case \( t^{-1} \sinh(t\xi) \) is to be interpreted as \( t \). The parameters \( \alpha \) and \( \beta \) satisfy \( (\alpha, \beta) \in \mathbb{R} \times \mathbb{C} \) and \( \xi^2 = |\beta|^2 - |\alpha|^2 \).

Using (3) and (4) we get that \( q(z) = \frac{\partial}{\partial t} \varphi_t(z) \big|_{t=0} \) is the polynomial

\[
q(z) = \frac{\partial}{\partial t} \varphi_t(z) \big|_{t=0} = -\beta z^2 + 2\alpha z + \beta
\]

whose roots are
One readily verifies that \( q(1/z) = -q(z)/z^2 \) so that if \(|τ_1| ≠ 1 \) then necessarily \( τ_2 = 1/τ_1 \) (or else \( τ_1, τ_2 \) and \( 1/τ_1 \) would all be roots). It is also clear that \(|τ_1| |τ_2| = 1 \) when \( β ≠ 0 \), since \( τ_1 τ_2 = -β/β \).

These considerations simplify the study of \( (ψ_e) \) and \( q \) done in [1] and [2] (where \( q \) was introduced and called the "invariance polynomial" of \( \{ψ_e\} \)).

Consider now the representation of pairs \( (α, β) ∈ \mathbb{R} × \mathbb{C} \) by \( α = ix_1 \), \( β = x_2 + ix_3 \) with \( x_1, x_2, x_3 \) real. Then the sign of \( τ^2 = |β|^2 - |α|^2 = x_1^2 + x_2^2 + x_3^2 \) changes on a cone \(-x_1^2 + x_2^2 + x_3^2 = 0\) and correspondingly we have:

- \( τ^2 < 0 \); this means that \(|α| > |β| \) and \( α, β \) are both purely imaginary. Thus \(|α + β| ≠ |α - β| \) and therefore \(|τ_1| ≠ |τ_2| \) which can only occur if \(|τ_1| < 1 < |τ_2| \) or \(|τ_2| < 1 < |τ_1| \). In either case \( q \) has only one root in \(|z| < 1 \); \( τ^2 = 0 \); in this case we get \( τ_1 = τ_2 \) from (6) and so also \(|τ_1| = |τ_2| = 1 \). In other words, the roots of \( q \) coincide and belong to \(|z| = 1 \). \( τ^2 > 0 \); in this case \( τ \) is real and \( α \) being purely imaginary we get \( |α + β| = |α - β| \) which implies \(|τ_1| = |τ_2| = 1 \) with \( τ_1 ≠ τ_2 \). Thus \( q \) has two distinct roots, both in \(|z| = 1 \).

These three possibilities are referred to as type (i), type (ii), and type (iii), respectively.

In the following we will exclude the case \( α = β = 0 \) (or \( \{ψ_e\} = \{id\} \)).

Using the proof of (1.7) in [2] and the discussion above we get the following theorem (that gives additional information or generalizes (1.5), (1.7) and (1.10) in [2] and simplifies (1.6) in [1] (cf. also (1.5) in [1]).

**Theorem 1.** There are three mutually exclusive types of one-parameter groups of conformal self-maps of the disc \(|z| < 1\) (labeled types (i), (ii) and (iii)), parametrized by three real parameters \( x_1, x_2, x_3 \), each type corresponding to the set where \( τ^2 = -x_1^2 + x_2^2 + x_3^2 \) satisfies \( τ^2 < 0 \), \( τ^2 = 0 \) or \( τ^2 > 0 \), respectively. For each point \( (x_1, x_2, x_3) \) the corresponding one-parameter group is given by (6) above, where \( α = ix_1 \), and \( β = x_2 + ix_3 \), and the fixed points are given by

\[
τ_1, τ_2 = (ix_1 ± τ)/(x_2 + ix_3)
\]

\((τ_1 = τ_2 = 0\) if \(x_2 = x_3 = 0\)). Further \( τ_1, τ_2 \) are the roots of the invariance polynomial

\[q(z) = -(x_2 + ix_3)z^3 + 2x_1iz + (x_2 - ix_3)\].
An alternative characterization of $\{\varphi_t\}$ can be obtained from this as follows. First a matrix

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

can be represented by $(g_1, g_2, g_3, g_4) \in \mathbb{R}^4$ with $a = g_1 + ig_2$, $b = g_3 + ig_4$. Then the group $\text{SU}(2)$ can be replaced by the hyperboloid

$$V = \{g_1^2 + g_2^2 - g_3^2 - g_4^2 = 1\},$$

which then becomes a two sheeted cover of $G$. The one-parameter subgroups of $G$ appear as curves in $V$, as follows:

**type (i):** write $\xi = i\eta$ with $\eta$ real ($\eta$ agrees with the "angular velocity" of $\{\varphi_t\}$; see [2] paragraph preceding (1.13)). Then $\{\varphi_t\}$ corresponds to

$$g_1(t) = \cos(\eta t)$$

$$(g_2(t), g_3(t), g_4(t)) = \eta^{-1} \sin(\eta t) \ U,$$

where $U \in \mathbb{R}^3$.

**type (ii):** $\{\varphi_t\}$ corresponds to

$$g_1(t) = 1$$

$$(g_2(t), g_3(t), g_4(t)) = tU,$$

**type (iii):** $\{\varphi_t\}$ corresponds to

$$g_1(t) = \cosh (\xi t)$$

$$(g_2(t), g_3(t), g_4(t)) = t^{-1} \sinh (\xi t) \ U.$$ 

Let now $H^p$, $1 < p < \infty$, $p \neq 2$, denote the Hardy space of the disc. It is known (see [1], Th. (2.1)) that each one-parameter group of isometries $(T_t)$ of $H^p$ has the form

$$T_t f = \Phi_t f(\varphi_t), \quad f \in H^p,$$

where $(\varphi_t)$ is a one-parameter group in $G$. Thus the characterization and classification of the theorem above carries over to the $(T_t)$. Further, if $U^p$ denotes the group of all isometries of $H^p$ onto $H^p$, it is proved in [3], Th. 2.1, that $U^p$ is a 4-dimensional Lie group and there is an exact sequence

$$(7) \quad 1 \longrightarrow T \longrightarrow U^p \longrightarrow G \longrightarrow 1$$

where $T$ is the circle group $T = \mathbb{R}/\mathbb{Z}$, $\mu(\theta) = e^{i\theta} I$ and $\tau T = \varphi^{-1}$ if $T f = \Phi f(\varphi)$. It is also known that the extension (7) does not split,
but it suffices to calculate the Lie algebra $L(U^P)$ of $U^P$ as the product

$$L(U^P) = R \ast L(G)$$

where $L(G)$ is the Lie algebra of $G$. Since $G$ is isomorphic to $U(2)/\pm I$, then $L(G)$ is isomorphic to the Lie algebra $\mathfrak{u}(2)$ of matrices

$$M = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

with $(\alpha, \beta) \in i\mathbb{R} \times \mathbb{C}$ the correspondence being given by assigning to the tangent vector $v = \frac{d\varphi_t}{dt}
\bigg|_{t=0}$, where $(\varphi_t)$ is given by (3), the ma-
trix $M(v)$ with $\alpha = \alpha'(0), \beta = \beta'(0)$. (Of course $U(2) \cong SL(2, \mathbb{R})$ im-
plies also $\mathfrak{u}(2) \cong \mathfrak{s}(2, \mathbb{R})$).

Our next goal is to interpret $L(U^P)$ in terms of hermitian operators in $H^P$. Let us recall that an operator $A$ in $H^P$ is hermitian if $A = -i \frac{dA}{dt}
\bigg|_{t=0}$ for some one-parameter group $(T_t)$ in $U^P$ (notice that $A$
will be unbounded, in general). According to (2.4) in [2]:

$$iA\lambda = i\dot{\lambda} f + g f' + (1/p)q' f$$

where primes denote $d/dz$, $q$ is the invariance polynomial of $(\varphi_t)$ and $\lambda$ is a real constant. But then the Lie algebra $L(U^P)$ can be identi-
fi ed to the space $\mathfrak{v}^P$ of operators

$$V(\lambda, q)f = i\lambda f + qf' + (1/p)q' f$$

where $q(z) = -\beta t^2 + 2az + \beta$, with $(\alpha, \beta) \in i\mathbb{R} \times \mathbb{C}$ and $\lambda \in \mathbb{R}$.

**THEOREM 2.** The Lie algebra $L(U^P)$ is isomorphic to the algebra $\mathfrak{v}^P$ of all operators $V(\lambda, q)$ with the bracket defined by operator composi-
tion $[V_1, V_2] = V_1 V_2 - V_2 V_1$.

**Proof.** First we calculate

$$(V(\lambda_1, q_1) - i\lambda_1 I)(V(\lambda_2, q_2) - i\lambda_2 I)f =$$

$$= (V(\lambda_1, q_1) - i\lambda_1 I)(q_2 f' + (1/p)q_2' f) =$$

$$= q_1(q_2 f' + (1/p)q_2' f)' + (1/p)q_1'(q_2 f' + (1/p)q_2' f)' =$$

$$= q_1 q_2'' + (q_1 q_2' + (1/p)(q_1 q_2'))' f' + (1/p^2)q_1 q_2' f,$$

so that, using $[\lambda I, T] = 0$ for all $\lambda, T$ we get
Observe now that if we put $[\lambda_1,\lambda_2] = 0$ and $[q_1,q_2] = q_1q_2' - q_1'q_2$, then (9) reads

$$[V(\lambda_1, q_1), V(\lambda_2, q_2)] = (q_1q_2' - q_1'q_2)f' + \frac{1}{p}(q_1q_2'' - q_1''q_2)f.$$ 

A routine calculation shows that $[q_1,q_2]$ is again an invariance polynomial (and that $[q_1,q_2]$ is a Lie bracket, although this follows from (9) and the fact that $[T,S]$ is a Lie bracket for operators). Then $V^P$ is isomorphic via $V(\lambda, q) \mapsto R \otimes Q$ where $Q = \{q\}$ has the bracket $[q_1, q_2] = q_1q_2' - q_1'q_2$. The proof can be finished by observing that $q(M)$ and $Q$ are also isomorphic under the correspondence $q(M) = -\beta z^2 + \alpha z + \beta$ for $M$ as in (8), i.e., by showing that $q[M_1, M_2] = [q(M_1), q(M_2)]$. This is a routine calculation.

We close with the remark that the identity representation of $V^P$ in $H^P$ is irreducible (see [3]).

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