1. INTRODUCTION.

Let $M$ be a (connected) surface in a Euclidean $m$-space $E^m$. For any point $p$ in $M$ and any unit vector $t$ at $p$ tangent to $M$, the vector $t$ and the normal space $T_pM$ of $M$ at $p$ determine an $(m-1)$-dimensional vector subspace $E(p,t)$ of $E^m$ through $p$. The intersection of $M$ and $E(p,t)$ gives rise a curve $\gamma$ in a neighborhood of $p$ which is called the normal section of $M$ at $p$ in the direction $t$. The surface $M$ is said to have planar normal sections if normal sections of $M$ are planar curves. In this case, for any normal section $\gamma$, we have $\gamma' \wedge \gamma'' = 0$ identically. A surface $M$ is said to have pointwise planar normal sections if, for each point $p$ in $M$, normal sections at $p$ satisfy $\gamma' \wedge \gamma'' = 0$ at $p$ (i.e., normal sections at $p$ have "zero torsion" at $p$). It is clear that if a surface $M$ lies in a linear 3-space $E^3$ of $E^m$, then $M$ has planar normal sections and has pointwise planar normal sections.

We shall now define the Veronese surface. Let $(x,y,z)$ be the natural coordinate system in $E^3$ and $(u^1,u^2,u^3,u^4,u^5)$ the natural coordinate system in $E^5$. We consider the mapping defined by

\[
\begin{align*}
  u^1 &= \frac{1}{\sqrt{3}} \ yz, \\
  u^2 &= \frac{1}{\sqrt{3}} \ zx, \\
  u^3 &= \frac{1}{\sqrt{3}} \ xy, \\
  u^4 &= \frac{1}{2\sqrt{3}} (x^2 - y^2), \\
  u^5 &= \frac{1}{6} (x^2 + y^2 - 2z^2).
\end{align*}
\]

This defines an isometric immersion of $S^2(\sqrt{3})$ into the unit hypersphere $S^4(1)$ of $E^5$. Two points $(x,y,z)$ and $(-x,-y,-z)$ of $S^2(\sqrt{3})$ are mapped into the same point of $S^4(1)$, and this mapping defines an imbedding of the real projective plane into $S^4(1)$. This real projective plane imbedded in $E^5$ is called the Veronese surface (see, for instance, [4].)

In [2], we have proved the following.
THEOREM A. Let $M$ be a surface in $E^m$. If $M$ has pointwise planar normal sections, then, locally, $M$ lies in a linear $5$-subspace $E^5$ of $E^m$.

The classification of surfaces in $E^m$ with planar normal sections was obtained in [3].

THEOREM B. Let $M$ be a surface in $E^m$. If $M$ has planar normal sections, then, either, locally, $M$ lies in a linear $3$-subspace $E^3$ or, up to similarity transformations of $E^m$, $M$ is an open portion of the Veronese surface in $E^5$.

In view of Theorems A and B, it is an interesting problem to classify surfaces in $E^5$ with pointwise planar normal sections. As we already mentioned, every surface in $E^3$ has pointwise planar normal sections. A surface $M$ in $E^5$ is said to lie essentially in $E^m$ if, locally, $M$ does not lie in any hyperplane $E^{m-1}$ of $E^m$. According to Theorem A, the classification problem of surfaces in $E^m$ with pointwise planar normal sections remains open only for surfaces which lie essentially either in $E^5$ or in $E^4$.

In this paper, we will solve this problem completely for surfaces which lie essentially in $E^5$. Furthermore, we will obtain three classification theorems for surfaces in $E^4$. As biproducts some new geometric characterizations of the Veronese surface and standard flat tori are then obtained.

2. PRELIMINARIES.

Let $M$ be a surface in $E^m$. We choose a local field of orthonormal frame $(e_1, \ldots, e_m)$ in $E^m$ such that, restricted to $M$, the vectors $e_1, e_2$ are tangent to $M$ and $e_3, \ldots, e_m$ are normal to $M$. We denote by $(\omega^1, \ldots, \omega^m)$ the field of dual frames. The structure equations of $E^5$ are given by

\begin{align*}
(2.1) & \quad \omega^A = -\sum \omega^B A B^B \omega^B, \quad \omega^A + \omega^B = 0, \\
(2.2) & \quad \omega^A B = \sum \omega^C A B C \omega^B, \\
& \quad A, B, C, \ldots = 1, 2, \ldots, m.
\end{align*}

Restricting these forms on $M$, we have $\omega^r = 0$, $r, s, t, \ldots = 3, \ldots, m$. Since

\begin{equation}
(2.3) \quad 0 = \omega^i = -\sum \omega^r i^r \omega^i, \quad i, j, k \ldots = 1, 2,
\end{equation}
Cartan's Lemma implies

\[ \omega_i^2 = \sum\ h_{ij}^r \ \omega_j^j, \quad h_{ij}^r = h_{ji}^r. \]

From these formulas we obtain

\[ d\omega_i^j = -\sum\ \omega_j^k \wedge \omega_j^j, \]

\[ \omega_j^j + \omega_i^i = 0, \]

\[ d\omega_j^i = -\sum\ \omega_k^i \wedge \omega_j^k + \Omega_j^i, \quad \Omega_j^i = \frac{1}{2} \sum\ R_{jkl}^r \ \omega_k^l \wedge \omega_j^j, \]

\[ R_{jkl}^r = \sum (h_{lk}^r h_{jk}^r - h_{lj}^r h_{jk}^r), \]

\[ d\omega_s^r = -\sum\ \omega_t^r \wedge \omega_s^t + \Omega_s^r, \quad \Omega_s^r = \frac{1}{2} \sum\ R_{sij}^r \ \omega_i^j \wedge \omega_s^s, \]

\[ R_{sij}^r = \sum_k (h_{ki}^s h_{kj}^s - h_{kj}^s h_{ki}^s). \]

The Riemannian connection of \( M \) is defined by \( (\omega_i^j) \). The form \( (\omega_i^j) \) defines a connection \( D \) in the normal bundle of \( M \). We call \( h = \sum h_{ij}^r \omega_i^j e_r \) the second fundamental form of the surface \( M \). We call \( H = \frac{1}{2} \text{tr} h \) the mean curvature vector of \( M \). We take exterior differentiation of (2.4) and define \( h_{ijk}^r \) by

\[ \sum h_{ijk}^r \omega_i^k = dh_{ij}^r - \sum h_{ik}^r \omega_j^i - \sum h_{ij}^r \omega_i^k + \sum h_{ij}^s \omega_s^r. \]

Then we have the following equation of Codazzi,

\[ h_{ijk}^r = h_{ikj}^r. \]

If we denote by \( \nabla \) and \( \hat{\nabla} \) the covariant derivatives of \( M \) and \( E^m \), respectively, then, for any two vector fields \( X, Y \) tangent to \( M \) and any vector field \( \xi \) normal to \( M \), we have

\[ \nabla_X Y = \nabla_X h(X,Y), \]

\[ \hat{\nabla}_X \xi = -A_{\xi} X + D_X \xi, \]

where \( A_{\xi} \) denotes the Weingarten map with respect to \( \xi \). If \( \langle , \rangle \) denotes the inner product of \( E^m \), then

\[ \langle A_{\xi} X, Y \rangle = \langle h(X,Y), \xi \rangle. \]

If we define \( \tilde{\nabla} h \) by

\[ (\tilde{\nabla}_X h)(Y,Z) = D_X (h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \]
then equation (2.12) of Codazzi becomes

\[(2.17) \quad (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z).\]

It is well-known that \(\nabla h\) is a normal-bundle-valued tensor of type 
\((0,3)\).

We need the following theorems for the proof of Theorem 1.

**THEOREM C.** (Chen [1]). A surface \(M\) of \(E^m\) has pointwise planar normal sections if and only if 
\((\nabla_t h)(t, t) \wedge h(t, t) = 0\) for any \(t \in TM\).

**THEOREM D.** (Chen [2]). Let \(M\) be a surface in \(E^m\) with pointwise planar normal sections. Then \(\text{Im} h\) is parallel.

### 3. CLASSIFICATION OF SURFACES IN \(E^5\).

In this section we shall prove the following.

**THEOREM 1.** Let \(M\) be a surface which lies essentially in \(E^5\). Then, up to similarities of \(E^5\), \(M\) is an open portion of the Veronese surface in \(E^5\) if and only if \(M\) has pointwise planar normal sections.

**Proof.** Let \(M\) be a surface in \(E^5\) with pointwise planar normal sections. We choose a local field of orthonormal frame \(\{e_1, e_2, e_3, e_4, e_5\}\) such that, restricted to \(M\), \(e_3\) is in the direction of the mean curvature vector \(H\), \(e_1, e_2\) are the principal directions of \(A_3 = A e_3\). Then \(e_3\) is perpendicular to \(h(e_1, e_2)\). We further choose \(e_5\) so that \(e_5\) is in the direction of \(h(e_1, e_2)\). Then, with respect to \(\{e_1, e_2, e_3, e_4, e_5\}\), we have

\[
A_3 = \begin{bmatrix} a & 0 \\ 0 & \beta \end{bmatrix}, \quad A_4 = \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}, \quad A_5 = \begin{bmatrix} \eta & \delta \\ \delta & -\eta \end{bmatrix}
\]

Thus, we have

\[(3.1) \quad h(e_1, e_1) = ae_3 + \gamma e_4 + \eta e_5, \quad h(e_1, e_2) = \delta e_5,
\]

\[h(e_2, e_2) = \beta e_3 + \gamma e_4 + \eta e_5.\]

It is easy to see that \(\dim \text{Im} h = 3\) if and only if 
\(h(e_1, e_1) \wedge h(e_1, e_2) \wedge h(e_2, e_2) \neq 0\). Therefore, \(\dim \text{Im} h = 3\) if and only if \((\alpha + \beta)\gamma \delta \neq 0\). We put

\[(3.2) \quad M_3 = \{p \in M \mid \dim \text{Im} h = 3\}.
\]

Then \(M_3\) is an open subset of \(M\). If \(M_3\) is empty, then Theorem D implies
that $M$ does not lie essentially in $E^5$. From now on, we assume that $M$ lies essentially in $E^5$. Then $M_3$ is not empty. We denote by $N$ a component of $M_3$. On $N$, we have

\[(\alpha + \beta)\gamma \delta \neq 0.\]

From (2.16) and (3.1) we find

\[(\overline{\nabla}_{e_1} h)(e_1,e_1) = [e_1(\alpha) + \gamma \omega_4^3(e_1) + \eta \omega_5^3(e_1)]e_3 + \]
\[+ [\alpha \omega_3^4(e_1) + e_1(\gamma) + \eta \omega_5^4(e_1)]e_4 + \]
\[+ [\alpha \omega_3^5(e_1) + \gamma \omega_5^5(e_1) + e_1(\eta) - 2\delta \omega_1^2(e_1)]e_5,\]

\[(\overline{\nabla}_{e_2} h)(e_2,e_1) = [e_2(\alpha) + \gamma \omega_4^3(e_2) + \eta \omega_5^3(e_2)]e_3 + \]
\[+ [\alpha \omega_3^4(e_2) + e_2(\gamma) + \eta \omega_5^4(e_2)]e_4 + \]
\[+ [\alpha \omega_3^5(e_2) + \gamma \omega_5^5(e_2) + e_2(\eta) - 2\delta \omega_1^2(e_2)]e_5,\]

\[(\overline{\nabla}_{e_1} h)(e_1,e_2) = [\delta \omega_5^3(e_1) + (\alpha - \beta)\omega_1^4(e_1)]e_3 + \]
\[+ [\delta \omega_5^4(e_1) + 2\gamma \omega_1^2(e_1)]e_4 + \]
\[+ [e_1(\delta) + 2\eta \omega_1^2(e_1)]e_5,\]

\[(\overline{\nabla}_{e_2} h)(e_2,e_2) = [e_2(\beta) - \gamma \omega_4^3(e_2) + \eta \omega_5^3(e_2)]e_3 + \]
\[+ [\beta \omega_3^4(e_2) - e_2(\gamma) - \eta \omega_5^4(e_2)]e_4 + \]
\[+ [\beta \omega_3^5(e_2) - \gamma \omega_5^5(e_2) - e_2(\eta) - 2\delta \omega_1^2(e_2)]e_5,\]

Because $M$ has pointwise planar normal sections, Theorem C implies

\[(\overline{\nabla}_{e_1} h)(e_1,e_1) = \lambda_1 h(e_1,e_1), \quad (\overline{\nabla}_{e_2} h)(e_2,e_2) = \lambda_2 h(e_2,e_2),\]

for some local functions $\lambda_1, \lambda_2$. Combining (3.1), (3.4), (3.9) with (3.10) we obtain
(3.11) \[ e_1(\alpha) = \alpha \lambda_1 + \gamma \omega_3^4(e_1) + \eta \omega_5^5(e_1) \]

(3.12) \[ e_1(\gamma) = \gamma \lambda_1 - \alpha \omega_3^4(e_1) + \eta \omega_5^5(e_1) \]

(3.13) \[ e_1(\eta) = \eta \lambda_1 - \alpha \omega_3^5(e_1) - \gamma \omega_4^5(e_1) + 2\delta \omega_1^2(e_1) \]

(3.14) \[ e_2(\beta) = \beta \lambda_2 - \gamma \omega_3^4(e_2) - \eta \omega_5^5(e_2) \]

(3.15) \[ e_2(\gamma) = \gamma \lambda_2 + \beta \omega_3^4(e_2) + \eta \omega_5^5(e_2) \]

(3.16) \[ e_2(\eta) = \eta \lambda_2 + \beta \omega_3^5(e_2) - \gamma \omega_4^5(e_2) + 2\delta \omega_1^2(e_2) \]

Moreover, from (3.5), (3.6), (3.7), (3.8) and equation (2.17) of Coda-

(3.17) \[ e_2(\alpha) = \gamma \omega_3^4(e_2) - \delta \omega_3^5(e_2) + \eta \omega_5^5(e_2) + (\alpha - \beta) \omega_1^2(e_2) \]

(3.18) \[ e_1(\beta) = -\gamma \omega_3^4(e_1) - \delta \omega_3^5(e_2) - \eta \omega_5^5(e_1) + (\alpha - \beta) \omega_1^2(e_2) \]

(3.19) \[ e_1(\delta) = \eta \lambda_2 + (\alpha + \beta) \omega_3^5(e_2) - 2\eta \omega_1^2(e_1) \]

(3.20) \[ e_2(\delta) = -\eta \lambda_1 + (\alpha + \beta) \omega_3^5(e_1) - 2\eta \omega_1^2(e_2) \]

(3.21) \[ \lambda_1 \gamma - (\alpha + \beta) \omega_3^4(e_1) - \delta \omega_4^5(e_2) + 2\gamma \omega_1^2(e_2) = 0 \]

(3.22) \[ \lambda_2 \gamma + (\alpha + \beta) \omega_3^4(e_2) + \delta \omega_4^5(e_1) - 2\gamma \omega_1^2(e_1) = 0 \]

Let \( t = e_1 + ke_2 \). Then, from Theorem C, we have

(3.23) \[ (\overline{h}_{e_1+ke_2})(e_1+ke_2) = 0 \]

for any \( k \). Because \( e_3 \land e_4 \), \( e_3 \land e_5 \), and \( e_4 \land e_5 \) are linearly independent,

(3.1), (3.3), (3.4) - (3.10), and (3.23) imply

(3.24) \[ -\gamma \delta \omega_3^5(e_1) + a \delta \omega_4^5(e_1) - (\alpha + \beta) \gamma \omega_1^2(e_1) = 0 \]

(3.25) \[ (\alpha + \beta) \gamma \lambda_1 + 3 \gamma \delta \omega_3^5(e_2) - 3a \delta \omega_4^5(e_2) + 3(\alpha + \beta) \gamma \omega_1^2(e_2) = 0 \]

(3.26) \[ (\alpha + \beta) \gamma \lambda_2 + 3 \gamma \delta \omega_3^5(e_1) + 3 \delta \omega_4^5(e_1) - 3(\alpha + \beta) \gamma \omega_1^2(e_1) = 0 \]

(3.27) \[ \gamma \delta \omega_3^4(e_2) + (\alpha + \beta) \gamma \omega_1^2(e_2) = 0 \]

(3.28) \[ 2 \gamma \delta \lambda_1 - 3 \gamma \eta \lambda_2 - 3(\alpha + \beta) \gamma \omega_3^5(e_2) + 3 \delta \omega_4^5(e_1) + 6 \gamma \omega_1^2(e_1) = 0 \]

(3.29) \[ -3 \gamma \eta \lambda_1 - 2 \gamma \delta \lambda_2 + 3(\alpha + \beta) \gamma \omega_3^5(e_1) + 3 \delta \omega_4^5(e_2) - 6 \gamma \omega_1^2(e_2) = 0 \]

From (3.25) and (3.27) we find
From (3.24) and (3.26) we find

\[(3.31) \quad \gamma \lambda_2 + 3 \delta \omega_4^5(e_2) - 6 \gamma \omega_1^2(e_2) = 0.\]

Similarly, from (3.21), (3.28) and (3.29), we also have

\[(3.32) \quad 2 \gamma \delta \lambda_1 + 3(\alpha + \beta) \gamma \omega_3^5(e_2) - 3(\alpha + \beta) \gamma \omega_1^5(e_2) = 0.\]

\[(3.33) \quad -2 \gamma \delta \lambda_2 - 3(\alpha + \beta) \gamma \omega_3^5(e_1) + 3(\alpha + \beta) \gamma \omega_1^5(e_1) = 0.\]

From (3.22) and (3.24) we find

\[(3.34) \quad -\alpha \gamma \lambda_2 - \alpha(\alpha + \beta) \omega_3^4(e_2) - \gamma \delta \omega_3^5(e_1) + (\alpha - \beta) \gamma \omega_1^2(e_1) = 0.\]

Similarly, from (3.21) and (3.27) we get

\[(3.35) \quad 2 \gamma \delta \lambda_1 - \beta(\alpha + \beta) \omega_3^4(e_1) + \gamma \delta \omega_3^5(e_2) - (\alpha - \beta) \gamma \omega_1^2(e_2) = 0.\]

From (3.21), (3.30) and (3.31), we obtain, respectively,

\[(3.36) \quad (\alpha + \beta) \omega_3^4(e_1) - 2 \delta \omega_4^5(e_2) + 4 \gamma \omega_1^2(e_2) = 0,\]

\[(3.37) \quad (\alpha + \beta) \omega_3^4(e_2) - 2 \delta \omega_4^5(e_1) + 4 \gamma \omega_1^2(e_1) = 0.\]

From (3.21) and (3.36), we obtain

\[(3.38) \quad 2 \gamma \lambda_2 + 3(\alpha + \beta) \omega_3^4(e_2) = 0.\]

Similarly, from (3.22) and (3.37), we obtain

\[(3.39) \quad 2 \gamma \lambda_1 + 3(\alpha + \beta) \omega_3^4(e_1) = 0.\]

Combining (3.21) and (3.38) we have

\[(3.40) \quad \gamma \lambda_1 - 3 \delta \omega_4^5(e_2) + 6 \gamma \omega_1^2(e_2) = 0.\]

Equations (3.22) and (3.39) imply

\[(3.41) \quad \gamma \lambda_2 + 3 \delta \omega_4^5(e_1) - 6 \gamma \omega_1^2(e_1) = 0.\]

From (3.34) and (3.39) we find

\[(3.42) \quad \alpha \lambda_2 + 3 \delta \omega_4^5(e_1) - 3(\alpha - \beta) \omega_1^2(e_1) = 0.\]

Similarly, we have
From (3.32) and (3.39) we find

\[(3.44) \quad 2\delta \lambda_1 - 2\eta \lambda_2 - 3(\alpha + \beta)\omega_3^5(e_2) = 0.\]

Similarly, we also have

\[(3.45) \quad 2\eta \lambda_1 + 2\delta \lambda_2 - 3(\alpha + \beta)\omega_3^5(e_1) = 0.\]

Now, we want to claim that \( N \) is pseudo-umbilical in \( E^5 \), i.e., \( a = \beta \) on \( N \). Assume that \( \alpha \neq \beta \) at a point \( p \in N \). Then there is an open neighborhood \( U \) of \( p \) in \( N \) such that \( a \neq \beta \) everywhere on \( U \). From (3.38) - (3.45), we obtain the following expression of \( \omega_1^2 \) and \( \omega_3^8 \) on \( U \),

\[(3.46) \quad \omega_1^2 = \left\{ \frac{2\delta \eta \lambda_1 + [\alpha(\alpha + \beta) + 2\delta^2]\lambda_2}{3(\alpha^2 - \beta^2)} \right\} \omega_1 + \left\{ \frac{2\delta \lambda_1 - 2\eta \lambda_2}{3(\alpha + \beta)} \right\} \omega_3^2,

\[(3.47) \quad \omega_3^8 = \left\{ \frac{2\gamma}{3(\alpha + \beta)} \right\} \omega_1 + \left\{ \frac{2\gamma \lambda_2}{3(\alpha + \beta)} \right\} \omega_3^2,

\[(3.48) \quad \omega_5^8 = \left\{ \frac{4\gamma \delta \eta \lambda_1 + \gamma \gamma [(\alpha + \beta)^2 + 4\delta^2]\lambda_2}{3\delta(\alpha^2 - \beta^2)} \right\} \omega_1 + \left\{ \frac{\gamma [(\alpha + \beta)^2 + 4\delta^2]\lambda_2}{3\delta(\alpha^2 - \beta^2)} \right\} \omega_3^2.

Now, we shall make a careful study of the integrability condition to obtain a contradiction. In order to do so, we need to compute the exterior derivatives of \( \omega_3^8 \).

From (3.47) we have

\[(3.50) \quad d\omega_3^8 = d\left( \frac{2\gamma}{3(\alpha + \beta)} \right) (\lambda_1 \omega_1 - \lambda_2 \omega_2) + \left( \frac{2\gamma}{3(\alpha + \beta)} \right) d(\lambda_1 \omega_1 - \lambda_2 \omega_2).

Thus, by applying (3.11) - (3.18), (3.46) and a direct long computation, we may find

\[(3.51) \quad d\omega_3^8 = -\frac{2\gamma}{3(\alpha + \beta)} (e_2(\lambda_1) + e_1(\lambda_2) - \frac{\lambda_1 \lambda_2}{5}).\]
Similarly, we may also obtain

\[
\begin{align*}
(3.52) & \quad \omega_3^5 = \frac{1}{96(\alpha^2 - \beta^2) (\alpha - \beta)} \left\{ 6(\alpha^2 - \beta^2) \delta^2 [e_1(\lambda_1) - e_2(\lambda_2)] - \\
& \quad - 6(\alpha^2 - \beta^2) \delta \eta [e_2(\lambda_1) + e_1(\lambda_2)] - \\
& \quad - 2\{ \delta^2 - \gamma^2 \} [(\alpha + \beta)^2 + 4\delta^2] + 2\delta^2 \eta \} (\lambda_1^2 + \lambda_2^2) + \\
& \quad + 2\delta^2 [\beta(\alpha + \beta) + 2\delta^2] \lambda_1^2 + 2\delta^2 [\alpha(\alpha + \beta) + 2\delta^2] \lambda_2 + \\
& \quad + 2\delta(\alpha^2 - \beta^2) \eta \lambda_1 \lambda_2 \} \omega^1 \wedge \omega^2,
\end{align*}
\]

\[
(3.53) \quad \omega_4^5 = \frac{1}{96(\alpha^2 - \beta^2) (\alpha - \beta)} \left\{ 3(\alpha^2 - \beta^2) \gamma [(\alpha + \beta)^2 + 4\delta^2] [e_1(\lambda_1) - e_2(\lambda_2)] - \\
\right.
\]

\[
\left. \begin{align*}
& \quad - 12\gamma \delta [\alpha^2 - \beta^2] [e_1(\lambda_1) + e_2(\lambda_2)] - \\
& \quad - [(\alpha + \beta)^2 (\alpha^2 - \beta^2) \gamma + \\
& \quad + 2\gamma \delta^2 (5\alpha^2 + 5\beta^2 + 4\eta + 4\delta^2)] (\lambda_1^2 + \lambda_2^2) + \\
& \quad + \gamma [(\alpha + \beta)^2 + 4\delta^2] (\beta^2 \lambda_1^2 + \alpha^2 \lambda_2^2) + \\
& \quad + 4\gamma \delta [\alpha^2 - \beta^2] \lambda_1 \lambda_2 \} \omega^1 \wedge \omega^2.
\end{align*}
\]

On the other hand, by using (2.10) and (3.1), we have

\[
(3.54) \quad R^4_{312} = 0,
\]

\[
(3.55) \quad R^5_{312} = (\beta - \alpha) \delta,
\]

\[
(3.56) \quad R^5_{412} = -2\gamma \delta.
\]

Therefore, by equation (2.9) of Ricci, equations (3.47) - (3.49) and (3.54) - (3.56), we also have

\[
(3.57) \quad \omega_3^4 = -\frac{2\gamma \delta}{96(\alpha^2 - \beta^2) (\alpha + \beta)} (\lambda_1^2 + \lambda_2^2) \omega^1 \wedge \omega^2
\]

\[
(3.58) \quad \omega_4^5 = \frac{1}{96(\alpha^2 - \beta^2) (\alpha + \beta)} \left\{ 2\gamma^2 [(\alpha + \beta)^2 + 4\delta^2] (\lambda_1^2 + \lambda_2^2) - \\
\right.
\]

\[
\left. \begin{align*}
& \quad \left. - 9\delta^2 (\alpha^2 - \beta^2) \right\} \omega^1 \wedge \omega^2,
\end{align*}
\]

\[
(3.59) \quad \omega_5^5 = \frac{-2\gamma \delta}{9(\alpha + \beta)^2} \left\{ 2(\lambda_1^2 + \lambda_2^2) \right\} (\alpha + \beta)^2 \omega^1 \wedge \omega^2.
\]

Comparing (3.51) with (3.57), we find
Comparing (3.52) with (3.58), we find

\[
\delta [e_1(\lambda_1) - e_2(\lambda_2)] - \eta [e_2(\lambda_1) + e_1(\lambda_2)] = \frac{\delta}{3(\alpha^2 - \beta^2)} \left( \lambda_1^2 + \lambda_2^2 \right) - \frac{3}{2} (\alpha^2 - \beta^2) \delta.
\]

Combining (3.53) with (3.59), we get

\[
\frac{1}{3(\alpha^2 - \beta^2)} \left\{ (\alpha + \beta)^2 (\lambda_1^2 + \lambda_2^2) + 2 \gamma^2 \lambda_1 \lambda_2 \right\} = \frac{1}{3(\alpha^2 - \beta^2)} \left\{ (\alpha + \beta)^2 (\lambda_1^2 + \lambda_2^2) - \gamma^2 \lambda_1 \lambda_2 \right\} - \frac{4}{3} \delta \eta \lambda_1 \lambda_2 - 6 \delta^2 (\alpha^2 - \beta^2).
\]

Substituting (3.60) into (3.61), we obtain

\[
e_1(\lambda_1) + e_2(\lambda_2) = \frac{1}{\lambda_1^2 + \lambda_2^2} \left\{ (\alpha + \beta)^2 + 2 \gamma^2 \lambda_1 \lambda_2 \right\} + \frac{1}{\lambda_1^2 + \lambda_2^2} \left\{ (\alpha - \beta)(\alpha + \beta) \right\} - \frac{4}{3} \delta \eta \lambda_1 \lambda_2 - 6 \delta^2 (\alpha^2 - \beta^2).
\]

Substituting (3.60) and (3.63) into (3.62), we may obtain

\[
\alpha^2 - \beta^2 = 0.
\]

This contradicts to (3.3) because we assume that \( \alpha \neq \beta \).

Therefore, we have proved that \( \alpha = \beta \) identically on \( N \), i.e., \( N \) is pseudo-umbilical in \( \mathbb{E}^5 \). Because \( \alpha = \beta \), (3.42), (3.43), (3.44) and (3.45) reduce to

\[
\lambda_2 + 3 \delta \omega_3^5 = 0,
\]

\[
(\alpha + \beta) \lambda_1 = -2 \delta^2 \lambda_1 + 2 \delta \eta \lambda_2,
\]

\[
(\alpha + \beta) \lambda_2 = -2 \delta \eta \lambda_1 - 2 \delta^2 \lambda_2.
\]
From (3.67) and (3.68) we obtain

\begin{equation}
\lambda_1 = \lambda_2 = 0 \, .
\end{equation}

Thus, from (3.30) and (3.31), we have

\begin{equation}
\delta \omega_4^5 = 2\gamma \omega_1^2 \, .
\end{equation}

From (3.38), (3.39), (3.42) and (3.43), we find

\begin{equation}
\omega_3^4 = \omega_3^5 = 0 \, .
\end{equation}

Substituting (3.69) and (3.71) into (3.11), (3.14), (3.17) and (3.18), we find

\begin{equation}
\alpha = \beta = \text{constant on } N \, .
\end{equation}

From (3.12), (3.15), (3.69) and (3.71), we obtain

\begin{equation}
d\gamma = \eta \omega_4^5 \, .
\end{equation}

From (2.9), (2.10), (3.1) and (3.71), we find

\begin{equation}
d\omega_4^5 = -2\gamma \delta \omega_1^1 \wedge \omega_2^2 \, .
\end{equation}

Using (3.13), (3.16), (3.69), (3.70) and (3.71), we have

\begin{equation}
d\eta = (\delta^2 - \gamma^2) \omega_4^5 \, .
\end{equation}

Taking exterior differentiation of (3.73) and applying (2.9), (2.10), and (3.74), we obtain

\begin{equation}
0 = d^2 \gamma = -2 \gamma \delta \eta \omega_1^1 \wedge \omega_2^2 \, .
\end{equation}

From (3.76) we get

\begin{equation}
\eta = 0 \, .
\end{equation}

Since (3.74) shows that \( \omega_4^5 \neq 0 \), (3.75) and (3.77) give \( \delta^2 = \gamma^2 \). Without loss of generality, we may assume that

\begin{equation}
\delta = -\gamma \, .
\end{equation}

From (3.70) and (3.78), we find

\begin{equation}
\omega_4^5 = -2 \omega_1^2 \, .
\end{equation}
From (3.73) and (3.77), we see that $\delta = -\gamma$ is a nonzero constant on $N$. Thus, by the definition of $N$ and continuity, we conclude that $N$ is the whole surface $M$.

From (2.7), (2.9), (3.1), (3.74), (3.78) and (3.79) we find

\begin{equation}
\alpha^2 = 3\gamma^2 .
\end{equation}

Consequently, we may assume that $\alpha = -\sqrt{3} \gamma$. Therefore, by combining (3.71), (3.77), (3.79) and (3.80), we conclude that the connection form $(\omega^a_k)$, restricted to $N$, is given by

\[
\begin{pmatrix}
0 & \omega_1^2 & \sqrt{3} \gamma \omega_1 & -\gamma \omega_1 & \gamma \omega_2 \\
\omega_2 & 0 & \sqrt{3} \gamma \omega_2 & 2\gamma \omega_1 & \gamma \omega_1 \\
-\sqrt{3} \gamma \omega_1 & -\sqrt{3} \gamma \omega_2 & 0 & 0 & 0 \\
\gamma \omega_1 & -\gamma \omega_2 & 0 & 0 & 2\omega_1 \\
-\gamma \omega_2 & -\gamma \omega_1 & 0 & 2\omega_2 & 0
\end{pmatrix}
\]

This shows that, up to similarity transformations of $E^5$, $M$ coincides locally with the Veronese surface [4].

Conversely, if, up to similarity transformations of $E^5$, $M$ is an open portion of the Veronese surface, then $M$ has parallel second fundamental form, i.e., $\bar{V}_h = 0$. Thus, by Theorem C of Chen [1], we conclude that $M$ has pointwise planar normal sections. This completes the proof of Theorem 1.

4. SURFACES IN $E^4$ WITH CONSTANT MEAN CURVATURE.

In this and the next two sections, we will study surfaces in $E^4$. Assume that $M$ is a surface in $E^4$ with pointwise planar normal sections.

We choose a local field of orthonormal frame $(e_1, e_2, e_3, e_4)$ so that, restricted to $M$, $e_3$ is in the direction of $H$, $e_1$, $e_2$ are the principal directions of $A_3$. Then $e_3$ is perpendicular to $h(e_1, e_2)$. With respect to $(e_1, e_2, e_3, e_4)$, we have

\[
A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \eta & \delta \\ \delta & -\eta \end{pmatrix}
\]

Thus we have

\begin{equation}
(4.1) \quad h(e_1, e_1) = \alpha e_3 + \eta e_4, \quad h(e_1, e_2) = \delta e_4, \quad h(e_2, e_2) = \beta e_3 - \eta e_4.
\end{equation}
It is easy to find that the mean curvature, the normal curvature and the Gauss curvature of $M$ in $E^4$ are given respectively by

\[ |H| = \frac{1}{2} |\alpha + \beta| , \quad K^N = 2(\alpha - \beta)^2 \delta^2 \quad \text{and} \quad K = \alpha \beta - \pi^2 - \delta^2. \]

Since $M$ has pointwise planar normal sections, Theorem C implies

\[
(\nabla_{e_1} h)(e_1, e_2) = \lambda_1 h(e_1, e_2), \quad (\nabla_{e_2} h)(e_2, e_2) = \lambda_2 h(e_2, e_2)
\]

for some local functions $\lambda_1, \lambda_2$. Using the same method as before, we have the following

\[
\begin{align*}
(4.3) \quad e_1(a) &= \alpha \lambda_1 + \eta \omega_3^4(e_1), \\
(4.4) \quad e_1(\beta) &= -\eta \omega_3^4(e_1) - \delta \omega_3^4(e_2) + (\alpha - \beta) \omega_1^2(e_2), \\
(4.5) \quad e_1(n) &= \eta \lambda_1 - \alpha \omega_3^4(e_1) + 2 \delta \omega_1^2(e_1), \\
(4.6) \quad e_1(\delta) &= \eta \lambda_2 + (\alpha - \beta) \omega_3^4(e_2) - 2 \eta \omega_1^2(e_1), \\
(4.7) \quad e_2(a) &= -\delta \omega_3^4(e_1) + \eta \omega_3^4(e_2) + (\alpha - \beta) \omega_1^2(e_1), \\
(4.8) \quad e_2(\beta) &= \delta \lambda_2 - \eta \omega_3^4(e_2), \\
(4.9) \quad e_2(n) &= \eta \lambda_2 + \beta \omega_3^4(e_2) + 2 \delta \omega_1^2(e_2), \\
(4.10) \quad e_2(\delta) &= -\eta \lambda_1 + (\alpha - \beta) \omega_3^4(e_1) - 2 \eta \omega_1^2(e_2), \\
(4.11) \quad 2 \delta \lambda_1 - 3 \alpha \lambda_2 - 3 \eta \delta \omega_3^4(e_1) - 3 \alpha (\alpha - \beta) \omega_1^2(e_1) + 3 (\alpha - \beta) \eta \omega_1^2(e_1) &= 0, \\
(4.12) \quad (2 \alpha - \beta) \eta \lambda_1 - 3 \alpha^2 \omega_1^2(e_1) - 3 \pi \omega_1^2(e_2) + 3 (\alpha - \beta) \eta \omega_1^2(e_1) &= 0, \\
(4.13) \quad (\alpha - 2 \beta) \eta \lambda_2 + 3 \eta \delta \omega_3^4(e_1) - 3 \alpha (\alpha + \beta^2 + 2 \delta^2) \omega_3^4(e_2) - 3 (\alpha - \beta) \eta \omega_1^2(e_1) + 6 (\alpha - \beta) \delta \omega_1^2(e_2) &= 0, \\
(4.14) \quad 3 \beta \eta \lambda_1 + 2 \beta \delta \lambda_2 - 3 (\alpha + \beta) \beta \omega_3^4(e_1) + 3 \eta \delta \omega_3^4(e_2) - 3 (\alpha - \beta) \eta \omega_1^2(e_2) &= 0.
\end{align*}
\]

**THEOREM 2.** Let $M$ be a surface which lies essentially in $E^4$. Then $M$ is an open portion of the product surface of two planar circles if and only if $M$ has pointwise planar normal sections and constant mean curvature.
Proof. If $M$ is an open portion of the product surface of two planar circles, then it is easy to check that $M$ has constant mean curvature and pointwise planar normal sections.

Now, let $M$ be a surface which lies essentially in $E^4$. Assume that $M$ has constant mean curvature and pointwise planar normal sections. Then, by using Theorem 4 of [2], we see that $\alpha + \beta \neq 0$. We want to claim that $(\alpha - \beta) \delta = 0$. Assume that $(\alpha - \beta) \delta \neq 0$. If $\eta \neq 0$, then by eliminating $\omega_1^2(e_1)$, $\omega_1^2(e_2)$ from (4.12) and (4.13) with the help of (4.11), (4.14), we have

\begin{align*}
(4.15) & \quad 2[(\alpha + \beta) \eta^2 - 2\alpha \delta^2] \lambda_1 + 2(3\alpha + \beta) \eta \delta \lambda_2 - \\
& \quad - 3(\alpha + \beta)^2 \eta \omega_3^4(e_1) + 6\alpha(\alpha + \beta) \delta \omega_3^4(e_2) = 0, \\
(4.16) & \quad -2(\alpha + 3\beta) \eta \delta \lambda_1 + 2[(\alpha + \beta) \eta^2 - 2\delta^2] \lambda_2 + \\
& \quad + 6(\alpha + \beta) \beta \delta \omega_3^4(e_1) + 3(\alpha + \beta)^2 \eta \omega_3^4(e_2) = 0.
\end{align*}

Combining (4.15) and (4.16), we have

\begin{align*}
(4.17) & \quad [\frac{(\alpha + \beta)^2 \eta^2 - 4\alpha \delta^2}{(\alpha - \beta)^2}] [2\eta \lambda_1 + 2\delta \lambda_2 - 3(\alpha + \beta) \omega_3^4(e_1)] = 0.
\end{align*}

If $(\alpha + \beta)^2 \eta^2 + 4\alpha \delta^2 \neq 0$. We have from (4.11) - (4.17)

\begin{align*}
(4.18) & \quad \omega_1^2 = \frac{2\eta \delta \lambda_1 + (\alpha^2 + \alpha \beta + 2\delta^2) \lambda_2 - 3(\alpha + \beta)^2 \eta \omega_3^4(e_1)}{5(\alpha - \beta)^2}, \\
& \quad + \frac{(\alpha + \beta)^2 \eta \lambda_1 - 2\eta \delta \lambda_2}{3(\alpha - \beta)^2} \omega_2^2.
\end{align*}

If $(\alpha + \beta)^2 \eta^2 + 4\alpha \delta^2 = 0$, differentiating this relation, we have, with the help of (4.3) - (4.10),

\begin{align*}
(4.19) & \quad \omega_3^2 = \frac{2(\eta \lambda_1 + \delta \lambda_2)}{3(\alpha + \beta)} \omega_1^1 + \frac{2(\delta \lambda_1 - \eta \lambda_2)}{3(\alpha + \beta)} \omega_2^1.
\end{align*}

If $(\alpha + \beta)^2 \eta^2 + 4\alpha \delta^2 = 0$, differentiating this relation, we have, with the help of (4.3) - (4.10),

\begin{align*}
(4.20) & \quad [a(\alpha + \beta) \eta^2 - 2\alpha \delta^2] \lambda_1 + 4\alpha \beta \eta \delta \lambda_2 - \\
& \quad - [a(\alpha + \beta)^2 + 2(\alpha - \beta) \delta^2] \eta \omega_3^4(e_1) + \\
& \quad + [4 \alpha \beta (\alpha + \beta) - (\alpha + \beta) \eta^2 - 2 \alpha \delta^2] \delta \omega_3^4(e_2) + 2(\alpha - \beta) \eta \omega_3^2(e_1) + \\
& \quad + (\alpha - \beta)[(\alpha + \beta) \eta^2 + 2\alpha \delta^2] \omega_3^2(e_2) = 0.
\end{align*}

\begin{align*}
(4.21) & \quad -4\alpha \beta \eta \delta \lambda_1 + [8 (\alpha + \beta) \eta^2 - 2 \alpha \delta^2] \lambda_2 + \\
& \quad + [4 \alpha \beta (\alpha + \beta) - (\alpha + \beta) \eta^2 - 2 \delta^2] \delta \omega_3^2(e_1) + \\
& \quad + 2(\alpha - \beta) \eta \omega_3^2(e_1) + 2(\alpha - \beta) \omega_3^2(e_2) = 0.
\end{align*}
\[
\begin{align*}
&+ \left[ \beta(\alpha+\beta)^2 - 2(\alpha-\beta)\delta^2 \right] \eta \omega_3^4(e_2) + \\
&+ (\alpha-\beta)(\alpha+\beta)\eta^2 + 2\beta\delta^2 \omega_1^2(e_1) + \\
&+ 2(\alpha-\beta)^2 \eta \omega_1^4(e_2) = 0.
\end{align*}
\]

From (4.11) - (4.14) and (4.20), (4.21), we still have (4.18), (4.19). Because \(|H|\) is constant, differentiating the relation \(\alpha+\beta = \text{constant},\) we have

\[
(4.22) \quad \alpha \lambda_1 - \delta \omega_3^4(e_2) + (\alpha-\beta)\omega_1^2(e_2) = 0
\]

\[
(4.23) \quad \beta \lambda_2 - \delta \omega_3^4(e_1) + (\alpha-\beta)\omega_1^2(e_1) = 0.
\]

Substituting (4.18), (4.19) into (4.22), (4.23), we get

\[
(4.24) \quad (3\alpha+\beta)\lambda_1 = 0.
\]

\[
(4.25) \quad (3\beta+\alpha)\lambda_2 = 0.
\]

Thus we have (i) \(\lambda_1 = \lambda_2 = 0,\) or (ii) \(3\alpha+\beta = 0, 3\beta+\alpha = 0,\) or (iii) \(3\alpha+\beta = 0, \lambda_2 = 0,\) or (iv) \(3\beta+\alpha = 0, \lambda_1 = 0.\) If case (i) occurs, (4.18) and (4.19) imply \(\omega_1^2 = \omega_3^4 = 0.\) In particular, we have \(H = 0.\) Thus, by applying Theorem 5 of Chen [2], we see that \(M\) is an open portion of the product surface of two planar circles. In particular, we have \(\delta = 0.\) This is a contradiction. If case (ii) occurs, we have \(\alpha+\beta = 0.\) This contradicts to \(\alpha+\beta \neq 0.\) For case (iii), differentiating \(3\alpha+\beta = 0,\) we have

\[
(4.26) \quad 3\omega_2(a) + \omega_2(\beta) = 0.
\]

Since \(\lambda_2 = 0, (4.7), (4.8), (4.18), (4.19),\) and (4.26) imply

\[
(4.27) \quad \eta \delta \lambda_1 = 0.
\]

From this we may again obtain a contradiction. The last case is similar to case (iii). Consequently, we have \(\eta = 0.\)

If \((\alpha-\beta)\delta \neq 0\) and \(\alpha \beta \neq 0,\) then from (4.3) - (4.14) we have \(\alpha \beta + \delta^2 = 0\) and

\[
(4.28) \quad \omega_1^2 = \frac{\alpha \lambda_2}{3(\alpha+\beta)} \omega_1^4 - \frac{\beta \lambda_1}{3(\alpha+\beta)} \omega_2^4,
\]

\[
(4.29) \quad \omega_1^4 = \frac{26 \lambda_2}{3(\alpha+\beta)} \omega_1^4 + \frac{26 \lambda_1}{3(\alpha+\beta)} \omega_2^4.
\]

Differentiating \(\alpha+\beta = \text{constant},\) we have (4.22) and (4.23). By substituting (4.28) and (4.29) into (4.22) and (4.23), we obtain
Thus, (i) \( a_1^2 + a_2^2 + b^2 = 20^2 = 0 \) and
\[ \lambda_1 = \lambda_2 = 0 , \]
(ii) \( 3a^2 + 2a\beta + b^2 - 2\delta^2 = 0 \) and
\[ \lambda_2 = 0 , \]
(iii) \( 3a^2 + 2a\beta + b^2 - 2\delta^2 = 0 \) and
\[ \lambda_1 = 0 , \]
(iv) \( a_1^2 + 2a\beta + b^2 - 2\delta^2 = 0 \).

Case (i) contradicts the assumption. Case (ii) implies \( a^2 = \beta^2 \) which contradicts the assumption too. For case (iii), since \( a\beta + \delta^2 = 0 \), we obtain
\[ 3a^2 + 4a\beta + \beta^2 = 0 . \]
This implies \( 3a + \beta = 0 \). We know that this is impossible. The last case is similar to case (iii).

If \((\alpha - \beta)\delta \neq 0\) and \(a\beta = 0\), then without loss of generality, we may assume \( \beta = 0 \). From (4.3) - (4.14), we have
\[ e_1(\delta) = -\delta w_3^2(e_2) + \alpha w_1^2(e_2) = 0 , \]
(4.34)
\[ e_2(\eta) = 2\delta w_1^2(e_2) = 0 , \]
(4.35)
\[ 2\delta \lambda_1 = 3\alpha w_3^2(e_2) = 0 . \]
These imply \( \lambda_1 = 0 \) and since \( \beta = \eta = 0 \), we have \( h(e_2, e_2) = 0 \). Thus, by (4.2), we may choose \( \lambda_2 = 0 \). From these we obtain a contradiction.
Consequently, we obtain \((\alpha - \beta)\delta = 0\). Thus, \( K^N = 0 \), from which we obtain Theorem 2 by applying Theorem 5 of Chen [2]. (Q.E.D.)

5. SURFACES IN \( \text{E}^4 \) WITH CONSTANT NORMAL CURVATURE.

In this section, we give the following classification result.

THEOREM 3. Let \( M \) be a surface which lies essentially in \( \text{E}^4 \). Then \( M \) is an open portion of the product surface of two planar circles if and only if \( M \) has pointwise planar normal sections and constant normal curvature.

Proof. Let \( M \) be a surface which lies essentially in \( \text{E}^4 \). Assume \( M \) has constant normal curvature and pointwise planar normal sections. As mentioned in the proof of Theorem 2 we may assume that \( a + \beta \neq 0 \). We want to claim that \((\alpha - \beta)\delta = 0\). Assume that \((\alpha - \beta)\delta \neq 0\). Because, \((\alpha - \beta)\delta = \text{constant} \), we have
Assume that \( \eta \neq 0 \). Using (4.3) - (4.10) and (4.18), (4.19), we obtain from (5.1),

\[
\delta \left[ e_i (a) - e_i (b) \right] + (\alpha - \beta) e_i (\gamma) = 0 , \quad i = 1, 2 .
\]

\( (5.1) \)

From these, we know that either \( \lambda_1 = \lambda_2 = 0 \) or \( \lambda_1^2 + \lambda_2^2 = 0 \) and

\[
\delta (5a - 3B) \lambda_1 - \eta (\alpha + \beta) \lambda_2 = 0 ,
\]

\( (5.2) \)

\[
- \eta (\alpha + \beta) \lambda_1 + \delta (3a - 5B) \lambda_2 = 0 .
\]

\( (5.3) \)

The first case implies that \( w_3 = 0 \) which gives \( (\alpha - \beta) \delta = 0 \). In the second case, we differentiate (5.4) to obtain

\[
\delta \lambda_1 = \eta \lambda_2 , \quad \eta \lambda_1 = - \delta \lambda_2 ,
\]

\( (5.5) \)

where we have used (4.3) - (4.10) and (4.18). From (5.5) we find \( n^2 + s^2 = 0 \) which contradicts to the assumption. Consequently, we have \( \eta = 0 \).

If \( \alpha \beta \neq 0 \) and \( (\alpha - \beta) \delta \neq 0 \), then, from (4.3) - (4.14), we have (4.28) and (4.29) and \( \alpha \beta + s^2 = 0 \). Differentiating \( K^N \), we find

\[
\delta (3a - 3B) \delta e_i (a) + (\alpha - 3B) \delta e_i (b) = 0 , \quad i = 1, 2 .
\]

\( (5.6) \)

Using (4.3), (4.4), (4.5), (4.8), (4.28) and (4.29), we have from (5.6),

\[
(5a - 3B) \lambda_1 = (3a - 5B) \lambda_2 = 0 .
\]

\( (5.7) \)

Since \( \alpha \beta + s^2 = 0 \), \( 5a - 3B \) and \( 3a - 5B \) are nonzero. Thus, \( \lambda_1 = \lambda_2 = 0 \). This will give a contradiction. If \( (\alpha - \beta) \delta = 0 \) and \( \alpha \beta = 0 \), then, by the same argument as given in section 4, we also have a contradiction. Thus, we have \( (\alpha - \beta) \delta = 0 \), i.e., \( K^N = 0 \). Therefore, by Theorem 5 of Chen [2], \( M \) is an open portion of the product surface of two planar circles. The converse of this is clear. (Q.E.D.)

6. SURFACES IN \( E^4 \) WITH CONSTANT GAUSS CURVATURE.

THEOREM 4. Let \( M \) be a surface which lies essentially in \( E^4 \). If \( M \) has pointwise planar normal sections and constant Gauss curvature, then \( M \) has vanishing Gauss curvature.

Proof. Let \( M \) be a surface which lies essentially in \( E^4 \). Assume that \( M \)
has constant Gauss curvature $K$ and pointwise planar normal sections. We may assume that $a+b \neq 0$ by Theorem 4 of [2]. If $(a-b)\delta \neq 0$, then, by differentiating $K$, we have

$$(6.1) \quad be_i(a) + ae_i(b) - 2ne_i(\delta) = 0, \quad i = 1, 2.$$ 

Using (4.3) - (4.10), (4.18), (4.19) and (6.1) we find

$$(6.2) \quad (a\delta - \eta^2 - \delta^2)\lambda_1 = (a\delta - \eta^2 - \delta^2)\lambda_2 = 0.$$ 

From this, we may conclude that $K = a\delta - \eta^2 - \delta^2 = 0$.

If $(a-b)\delta \neq 0$, $a\delta \neq 0$, but $\eta = 0$, then we have (4.28), (4.29) and $a\delta + \delta^2 = 0$. Differentiating $K = a\delta - \delta^2$ = constant, we have

$$(6.3) \quad be_i(a) + ae_i(b) - 2ae_i(\delta) = 0, \quad i = 1, 2.$$ 

From (4.3), (4.4), (4.6), (4.7), (4.8), (4.10), (4.28) and (4.29), we have

$$(6.4) \quad (a\delta - \delta^2)\lambda_1 = (a\delta - \delta^2)\lambda_2 = 0.$$ 

Thus, we have $a\delta - \delta^2 = 0$ which contradicts $a\delta + \delta^2 = 0$. If $(a-b)\delta \neq 0$ but $\eta = a\delta = 0$, then by a similar argument as given in section 4, we have a contradiction too.

When $(a-b)\delta = 0$, $K^N = 0$. In this case, Theorem 5 of [2] implies that $M$ is an open portion of a flat torus. Thus, $K = 0$. (Q.E.D.)
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