ON THE ANNIHILATOR IDEAL OF AN INJECTIVE MODULE

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1. INTRODUCTION.

Let $A$ be a noetherian local ring and $N$ a finitely generated $A$-module. In general it doesn’t hold that a $N$-regular element is $A$-regular. However a $N$-regular element may be $A/a$-regular for some ideal $a$ of $A$. So we shall consider the following problem: Characterize a minimal ideal $a$ of $A$ having the property that any $N$-regular element is $A/a$-regular.

First we shall determine the annihilator ideal of an injective module which has a finite set of associated prime ideals. Using this result the above problem will be solved. Finally a slight application to non-zero divisors will be given.

In the following discussion, $(A,m,k)$ is a noetherian local ring and modules are always unitary. The unlabeled Hom means always Hom$_A$. For an $A$-module $N$ $E(N)$ denotes an injective envelope of $N$.

2. THE ANNIHILATOR IDEAL OF AN INJECTIVE MODULE.

Let $a$ be an ideal of $A$ and $N$ an $A$-module. Annih$(a)$, Annih$_N(a)$ denote the ideal of elements $a$ in $A$ with $aa = 0$, the submodule of elements $z$ of $N$ with $za = 0$, respectively.

PROPOSITION 2.1. Let $E$ be an injective $A$-module and $x$ an element of $A$. Then $xE = E$ iff Annih$(x) \subseteq$ Annih$(E)$.

Proof. The "only if" part is obvious. Assume Annih$(x) \subseteq$ Annih$(E)$. From the exact sequence: $0 \to$ Annih$(x) \to A \to xA \to 0$, we obtain the exact sequence: Hom$(xA,E) \to$ Hom$(A,E) \to$ Hom(Annih$(x),E)$. This second map is zero by the hypothesis. So the map: Hom$(xA,E) \to$ Hom$(A,E)$ is surjective. The multiplication map by $x$: Hom$(A,E) \to$ Hom$(A,E)$ factorizes into two epimorphisms: Hom$(A,E) \to$ Hom$(xA,E)$ and Hom$(xA,E) \to$ Hom$(A,E)$, and so $xE = E$.

COROLLARY 2.2. Let $E$ be an injective $A$-module with Annih$(E) = 0$. Then, for $x \in A$, $xE = E$ iff $x$ is non-zero divisor.
COROLLARY 2.3. For \( x \in A \), \( xE(k) = E(k) \) iff \( x \) is a non-zero divisor.

Proof. If is obvious from the following well-known lemma.

LEMMA 2.4. \( \text{Annih}(E(k)) = 0. \)

Proof. \( E(k) \) may be regarded as the injective envelope of the residue field \( \mathbb{A}/mA \) as an \( \mathbb{A} \)-module where \( \mathbb{A} \) is the \( m \)-adic completion of \( A \) [c.f., 4]. So it is sufficient to show this lemma when \( A \) is complete. In this case, by Matlis duality, we have \( \text{Hom}(\text{Hom}(A/a, E(k)), E(k)) \cong A/a \) for any ideal \( a \) of \( A \). If \( a = \text{Annih}(E(k)) \), \( \text{Hom}(A/a, E(k)) \cong E(k) \) and so \( \text{Hom}(\text{Hom}(A/a, E(k)), E(k)) \cong A \) [c.f., 4]. Hence we obtain \( \text{Annih}(E(k)) = 0. \)

PROPOSITION 2.5. Let \( p \) be a prime ideal of \( A \). If \( xE(A/p) = E(A/p) \) for \( x \in A \), then there is an element \( t \) in \( A - p \) with \( t\text{Annih}(x) = 0. \)

Proof. If \( \text{Annih}(x) = 0 \), it is trivial. Assume \( \text{Annih}(x) \neq 0. \) Set \( a = \text{Annih}(x) \). Then we have \( aE(A/p) = 0. \) Since the injective envelope \( E(A/p) \) of the \( A_p \)-module \( A_p/pA_p \) has a zero annihilator ideal, we obtain \( a = 0. \) So there is an element \( t \) in \( A - p \) with \( ta = 0. \)

PROPOSITION 2.6. The annihilator ideal of a non-zero injective module consists of zero divisors.

Proof. Let \( E \) be a non-zero injective module. For any non-zero divisor \( x \), we have \( xE = E \) and so \( xE \neq 0. \)

COROLARIO 2.7. For any injective \( A \)-module \( E \), \( \text{Annih}(E) \) is contained in an associated prime ideal of \( A \).

LEMA 2.8. Let \( p \) be a prime ideal of \( A \). Then \( \text{Annih}(t)E(A/p) = 0 \) for any \( t \in A - p \).

Proof. It follows from \( tE(A/p) = E(A/p) \) for any \( t \in A - p \).

COROLARY 2.9. For any \( t \in A - p \), \( \text{Annih}(t) \subseteq \text{Annih}(E(A/p)). \)

LEMA 2.10. Let \( a = \text{Annih}(E(A/p)) \) with \( p \) a prime ideal of \( A \). Then there exists an element \( t \) in \( A - p \) with \( a = \text{Annih}(t) \) and so \( a = \max\{\text{Annih}(t) : t \in A - p\}. \)

Proof. Since the injective envelope \( E(A/p)_p \) of an \( A_p \)-module \( A_p/pA_p \) has a zero annihilator ideal, there is an element \( t \) in \( A - p \) with \( ta = 0 \) and so \( a \subseteq \text{Annih}(t) \). The statement follows from the above corollary. The following corollary is immediate from the above lemma:
COROLLARY 2.11. Let \( p,q \) be two prime ideals of \( A \) with \( p \subseteq q \). Then \( \text{Annih}(E(A/p)) \) contains always \( \text{Annih}(E(A/q)) \).

COROLLARY 2.12. Let \( 0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \ldots \) be a minimal injective resolution for a finitely generated \( A \)-module \( N \). If \( \text{Annih}(E^1) \neq 0 \), then \( \text{Annih}(E^{i-1}) \supseteq \text{Annih}(E^i) \), where \( E^{-1} = N \).

Proof. It is trivial if \( i=0 \). Assume \( i > 0 \). Put \( a = \text{Annih}(E^i) \). Let \( p \) be any prime ideal of \( A \) with \( \mu^{i-1}(p,N) > 0 \). Then we have \( \text{ht}(p) < n-2 \) where \( n = \dim A \). For, if \( \text{ht}(p) > n-2 \), then \( \mu^i(m,N) > 0 \) [c.f., 2, 3], which is a contradiction to \( a 
eq 0 \). So there is a prime ideal \( q \) of \( A \) such that \( p \subset q \) are distinct with no prime ideal between them. In this case, we have \( \mu^i(q,N) > 0 \) [c.f., 2] and so \( aE(A/p) = 0 \). This completes the proof.

PROPOSITION 2.13. Let \( E \) be an injective \( A \)-module. If there is an associated prime ideal \( p \) of \( A \) which is not contained in the union of the associated prime ideals of \( E \), then \( \text{Annih}(E) \neq 0 \).

Proof. For \( t \in p - \bigcup_{q \in \text{Ass}(E)} q \), we have \( tE = E \), and so \( \text{Annih}(t) \subseteq \text{Annih}(E) \).

THEOREM 2.14. Let \( E \) be an injective \( A \)-module such that \( \text{Ass}(E) = \{p_1, p_2, \ldots, p_n\} \) is a finite set. Then \( \text{Annih}(E) = \max(\text{Annih}(t) : t \in A - \bigcup_{i} p_i) \).

Proof. Put \( a = \max(\text{Annih}(t) : t \in A - \bigcup_{i} p_i) \). Then we have obviously \( a \subseteq \text{Annih}(E(A/p_i)) \) for \( i = 1,2,\ldots,n \) and so \( a \subseteq \text{Annih}(E) \). Let \( x \) be any element of \( \text{Annih}(E) \). Then \( xE(A/p_i) = 0 \) for \( i = 1,2,\ldots,n \). So we may take elements \( t_i \) in \( A - p_i \) with \( t_ix = 0 \) for \( i = 1,2,\ldots,n \) [c.f., (2.4)]. Let us denote \( \{p_1, p_2, \ldots, p_m\} \) the set of associated prime ideals of \( E \) except those contained in another of them.

Choose any element \( u_i \) of \( p_i - \bigcup_{j \neq i} p_j \) for \( i = 1,2,\ldots,m \) and set \( v_i = u_1 \ldots u_{i-1}t_iu_{i+1} \ldots u_m \). Then we have \( v_i \in p_k \) for \( i \neq k \) and \( v_i \notin p_i \), and so \( v_1 + v_2 + \ldots + v_m \notin p_i \) for \( i = 1,2,\ldots,m \). Since \( x(v_1 + v_2 + \ldots + v_m) = 0 \), we obtain \( x \in \text{Annih}(v_1 + v_2 + \ldots + v_m) \subseteq a \) and so \( \text{Annih}(E) \subseteq a \). This completes the proof.

COROLLARY 2.15. Let \( E \) be as above. Then there exists a principal ideal \( I \) of \( A \) such that \( E \) is faithfull over \( I \).

Proof. We have \( \text{Annih}(x) = \text{Annih}(E) \) for some \( x \in A - \bigcup_{i} p_i \).

Set \( I = \mathfrak{a}A \). If \( \mathfrak{a}E = 0 \) for \( \mathfrak{a} \in A \), \( x \) belongs to \( \text{Annih}(E) \) since \( xE = E \), and so \( x = 0 \). This means that \( E \) is faithfull as an \( I \)-module.
3. NON-ZERO DIVISORS.

**Lemma 3.1.** Let $N$ be an $A$-module with $\text{Annih}(E(N)) = 0$. Then a non-zero divisor on $N$ is a non-zero divisor.

**Proof.** For any associated prime ideal $p$ of $A$, there is an associated prime ideal $q$ of $E(N)$ containing $p$. For, otherwise we have $\text{Annih}(E(N)) = \text{Annih}(t) \neq 0$ for $t \in p - \bigcup q \in \text{Ass}(E(N)) q$. Since $\text{Ass}(N) = \text{Ass}(E(N))$, our statement holds.

**Theorem 3.2.** Let $N$ be an $A$-module, $a = \text{Annih}(E(N))$. Then any non-zero divisor on $N$ is a non-zero divisor on $A/a$. If $N$ is finitely generated, then $a$ is the unique minimal ideal of $A$ with respect to this property.

**Proof.** $E(N)$ may be considered as the injective envelope of $N$ as an $A/a$-module. Moreover an $A/a$-module $E(N)$ has a zero annihilator ideal. From the above lemma, $x$ is a non-zero divisor on $A/a$. Let $b$ be an ideal of $A$ such that any non-zero divisors on $N$ are non-zero divisors on $A/b$. Now there is $t \in A$ such that $\text{Annih}(t) = \text{Annih}(E(N))$ and $t$ is a non-zero divisor on $N$. Then $t$ is a non-zero divisor on $A/b$. So we have $a \subseteq b$. For, if $a \not\subseteq b$, $ta = 0$ for $a \in a - b$, which is absurd.

**Corollary 3.3.** Let $N$ be an $A$-module. Put $a = \text{Annih}(E(N))$. Then, if an element $x$ of $A$ is a non-zero divisor on $N$, $\text{inj.dim}_{A/a} N/xN = \text{inj.dim}_{A/a} N - 1$.

**Proof.** It is obvious from the fact that $x$ is a non-zero divisor on $A/a$.

**Corollary 3.4.** Let $N$ be a finitely generated $A$-module and $x_1, x_2, \ldots, x_r$ an $N$-sequence. Then $x_1, x_2, \ldots, x_r$ form an $A$-sequence if $\text{Annih}(\bigoplus_{i=0}^r x_i) = x_i$ for $i = 0, 1, \ldots, r$ where $0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$ is a minimal injective resolution of $N$ and $x_0 = 0$.

**Proof.** By the hypothesis we have $\text{Annih}(E(N)) = 0$ and so $x_i$ is a non-zero divisor. Assume $x_1, x_2, \ldots, x_i$ is an $A$-sequence. Then $\text{Hom}(A/x_i, E^i)$ is the injective envelope of $N/x_iN$ as an $A/x_i$-module [c.f., 1, Theorem 2.2]. By the hypothesis we obtain that the annihilator ideal of $\text{Hom}(A/x_i, E^i)$ as an $A/x_i$-module is zero. So $x_{i+1}$ is a non-zero divisor on $A/x_i$.

**Corollary 3.5.** The following conditions are equivalent:

a) $A$ is Cohen-Macaulay.

b) For a s.o.p. $x_1, x_2, \ldots, x_n$, for $A$ there exists a finitely generated
A-module $N$ such that $x_1, x_2, \ldots, x_n$ is an $N$-sequence and $\text{Annih}(\text{Annih}_i x_i) = x_i$ $(i = 0, 1, \ldots, n)$ where $x_0 = 0, x_1 = (x_1, x_2, \ldots, x_i)$ and $0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots$ is a minimal injective resolution of $N$.

Proof. b) $\Rightarrow$ a). It is obvious from the above corollary.

a) $\Rightarrow$ b). Consider $A$ as $N$. Then the condition b) is satisfied.

COROLLARY 3.6. Let $N$ be a finitely generated $A$-module and $0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots$ a minimal injective resolution of $N$. Then any $N$-sequence forms an $A$-sequence iff $\text{Annih} (E^0) = 0$ and for any $N$-sequence $\text{Annih}(\text{Annih}_i x) = x$ where $i$ is the length of the $N$-sequence and $x$ is the ideal generated by the $N$-sequence.

Proof. The "if part" follows from the above corollary. We shall show the converse. We obtain $\text{Annih}(E^0) = 0$ from the above theorem. Moreover $\text{Hom}(A/x, E^i)$ is the injective envelope of $N/xN$ as an $A/x$-module. By the hypothesis and the above theorem, the annihilator ideal of $\text{Hom}(A/x, E^i)$ as an $A/x$-module is zero and so $\text{Annih}(\text{Annih}_i x) = x$. 


REFERENCES


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