A CHARACTERIZATION OF PRÜFER RINGS

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ABSTRACT. Let \( R \) be a domain. In this paper we prove the following result: \( R \) is a Prüfer ring if and only if, for every finitely generated \( R \)-module \( M, t(M) \) (the torsion submodule) is a pure submodule of \( M \). This has as corollary a theorem of Kaplansky characterizing Prüfer rings.

Throughout this paper we assume \( R \) is a commutative ring with identity 1. We recall first, for reference, the definition and a characterization of pure submodules ([1], 2, ex. 24. a.) ([2], th. 2.4.).

DEFINITION. An exact sequence of \( R \)-modules:
\[
0 \longrightarrow M' \twoheadrightarrow M \rightarrowtail M'' \longrightarrow 0
\]
is called pure exact (also we say that \( M' \) is a pure submodule of \( M \)) iff for every \( R \)-module \( N \), the following sequence is exact:
\[
0 \longrightarrow N \otimes_R M' \longrightarrow N \otimes_R M \longrightarrow N \otimes_R M'' \longrightarrow 0
\]

LEMMA. Every direct summand is a pure submodule.

THEOREM 1. Let \( M \) be an \( R \)-module and \( M' \) a submodule of \( M \). \( M' \) is a pure submodule of \( M \) if and only if, for every finite family \( (a'_j)_{j=1}^n \subseteq M' \) such that \( a'_j = \sum_{i=1}^{m} r_{ij} a_i \) (\( a_i \in M, r_{ij} \in R, j = 1, \ldots, n \)), there exist a family \( (b'_i)_{i=1}^m \subseteq M' \) such that:
\[
a'_j = \sum_{i=1}^{m} r_{ij} b'_i \quad j = 1, \ldots, n
\]

In this paper we give the following characterization of Prüfer rings.

THEOREM 2. Let \( R \) be a domain. \( R \) is Prüfer if and only if for every \( R \)-module \( M \) of finite type, the torsion submodule \( t(M) \) is a pure submodule of \( M \).

Before proving theorem 2, let us show that it has as corollary the following theorem ([3]):

THEOREM 3. Let \( R \) be a domain. \( R \) is a Prüfer ring if and only if, for
every R-module \( M \) of finite type, \( t(M) \) is a direct summand of \( M \).

**Proof.** The non trivial part (the implication for the left) follows from the lemma and theorem 2.

**Proof of theorem 2.** If \( R \) is Prüfer, \( \frac{M}{t(M)} \) is projective, then the exact sequence:

\[
0 \longrightarrow t(M) \longrightarrow M \longrightarrow \frac{M}{t(M)} \longrightarrow 0
\]

splits, so is pure exact.

For the converse we use the fact that \( R \) Prüfer is equivalent to say that every finitely generated ideal \( I \neq 0 \) of \( R \) is invertible. Let \( I \neq 0 \) an ideal of \( R \) generated by \( a_1, \ldots, a_n \) with, say, \( a_1 \neq 0 \). Put:

\[
M = R^n / (a_1(a_2, \ldots, a_n))
\]

We have \( (a_1, \ldots, a_n) \in t(M) \), and \( (a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i e_i \) where \( (e_i) \) is the canonical base of \( R^n \). Being, by hypothesis, \( t(M) \) a pure submodule of \( M \), by theorem 1 there exist \( p_{ij} \in R \) with \( (p_{i1}, \ldots, p_{in}) \in t(M) \), such that:

\[
(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i (p_{i1}, \ldots, p_{in})
\]

In particular:

(1) \[
a_1 = \sum_{i=1}^{n} a_i p_{i1} + r a_1 a_1 \quad (r \in R)
\]

We have \( (p_{i1}, \ldots, p_{in}) \in t(M) \), \( i = 1, \ldots, n \), then there must exist \( r_i, s_i \in R \), \( r_i \neq 0 \), \( s_i \neq 0 \), \( i = 1, \ldots, n \), such that:

(2) \[
r_i p_{ij} = s_i a_1 a_j \quad i, j = 1, \ldots, n
\]

Replacing \( p_{i1} = \frac{s_i a_1 a_1}{r_i} \) in (1) and using the fact that \( a_1 \neq 0 \), we obtain:

\[
1 = (r + \frac{s_i a_1}{r_i}) a_1 a_1 + \sum_{i=2}^{n} \frac{s_i a_1}{r_i} a_i
\]

To conclude the proof we only have to verify that:

\[
\frac{s_i a_1}{r_i} a_j \in R \quad i, j = 1, \ldots, n
\]

but this is clear from (2).
REFERENCES


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