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SPIN STRUCTURES ON PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. The notion of Spin-structures on Riemannian manifolds is generalized to manifolds M with an indefinite metric of signature (p,q). The concept of (p,q)-orientability of such manifolds is defined and the group Spin(p,q) is introduced. Then, a Spin(p,q)-structure over M is defined as a principal Spin(p,q)-bundle over M satisfying certain conditions. It is proved that the existence of such a structure is equivalent to the vanishing of the second Stieffel-Whitney classes of two complementary subbundles of the tangent bundle. Examples are provided by manifolds of the form G/T, G compact Lie group, T maximal torus.

INTRODUCTION.

Let M be an n-dimensional oriented Riemannian manifold. A Spin-structure on M is a principal Spin(n)-bundle over M which is also a double covering of the principal SO(n)-bundle of oriented frames. This double covering is such that fibers cover fibers and the corresponding restrictions are equivalent to the universal covering of Spin(n) over SO(n). The existence of a Spin-structure on an orientable manifold is equivalent to the vanishing of the second Stieffel-Whitney class of M ([12]). This structure has been studied and applied in connection with several problems ([1], [2], [3]).

The main objective of this paper is to give a suitable generalization of the above notion for manifolds with an indefinite metric. The special case of dim M = 4 and signature (1,3), the so called gravitational fields, is of interest in Physics and has been previously studied ([4], [5]).

Assume that dim M = n and that the metric has signature (p,q). For technical reasons it is assumed that $p,q > 2$ (see §4). The orientability of M is replaced by the stronger condition of (p,q)-orientability (Definition 1). (p,q)-orientability is somewhat weaker than space-time orientability, as defined in [16], p.341.

For (p,q)-orientable manifolds, the notion of Spin(p,q)-structure is defined in §1. Necessary and sufficient conditions for the existence of such a structure are obtained; the main result being Theorem 2. These conditions are stated in terms of the vanishing of the second Stieffel-Whitney classes of two complementary subbundles of the tangent bundle of M (Corollary 2).

Interesting examples of Spin(p,q)-manifolds are provided by spaces of the form G/T, with G compact connected Lie group and T a maximal torus (§3).

1. (p,q)-ORIENTABLE MANIFOLDS AND SPIN(p,q)-STRUCTURES.

M will denote a connected n-dimensional C^∞ manifold with an indefinite metric g of signature (p,q), $p+q = n \geq 3$. Consider the principal O(p,q)-bundle of orthogonal frames over M, denoted by F'.

DEFINITION 1. M *is* (p,q)-*orientable if the structure group of* F' *admits a reduction to its identity connected component* $SO(p,q)_o$.

For instance, the pseudo-Riemannian sphere S_q^{p+q} is (p,q)-orientable since it is space-time orientable ([16], p.341). This follows easily from the Reduction Theorem ([11], p.83). Another example of (p,q)-orientable manifolds is given by $Q = M \times N$ where M,N are oriented Riemannian manifolds of dimensions p and q respectively and Q has the obvious metric of signature (p,q). Indeed, the structure group of the bundle of linear frames over Q admits a reduction to $GL(p,\mathbf{R}) \times GL(q,\mathbf{R})$ and hence to $SO(p) \times SO(p)$ because of the orientation of M and N. Since $SO(p) \times SO(q) \subset SO(p,q)_o$, it follows that Q is (p,q)-orientable.

The group $SO(p,q)_o$ is homeomorphic to $SO(p) \times SO(q) \times \mathbf{R}^{pq}$. Therefore its fundamental group is, for $p > 2$:

$$\Pi_1(SO(p,q)_o) = \begin{cases} \mathbf{Z}_2 & \text{if } q = 0,1 \\ \mathbf{Z}_2 \times \mathbf{Z} & \text{if } q = 2 \\ \mathbf{Z}_2 \times \mathbf{Z}_2 & \text{if } q > 2. \end{cases}$$

We shall be concerned with the case $p,q > 2$ (see §4 for signature (2,n-2)). First we introduce some notations. The universal covering space of $SO(p,q)_o$ with its natural Lie group structure will be denoted by Spin(p,q). If $K = \mathbf{Z}_2 \times \mathbf{Z}_2$, then Spin(p,q) is a principal K-bundle over $SO(p,q)_o$. It is well known that $H^1(SO(p,q)_o,K)$ classifies the principal K-bundles over $SO(p,q)_o$ ([9]); λ will denote the cohomology class corresponding to the universal covering.

If M is (p,q)-orientable, Π: F $\rightarrow$ M will denote a fixed subbundle of the linear bundle as given by Definition 1.

DEFINITION 2. *A* Spin(p,q)*-structure on the* (p,q)*-orientable manifold* M *is a pair* (P,Θ) *where* P *is a principal* Spin(p,q)*-bundle over* M *and* Θ: P $\rightarrow$ F *a principal* K*-bundle such that the following diagram is commutative:*

$$\begin{array}{ccc} P \times \mathrm{Spin}(p,q) & \xrightarrow{\Theta\times\sigma} & F \times SO(p,q)_0 \\ \downarrow & & \downarrow \\ P & \xrightarrow{\Theta} & F \\ & {\scriptstyle \Pi'}\searrow \quad \swarrow{\scriptstyle \Pi} & \\ & M & \end{array}$$

where σ: Spin(p,q) $\rightarrow$ $SO(p,q)_0$ *is the covering homomorphism and the vertical arrows are the group actions on the total spaces of the respective bundles.*

If follows that if P_m and F_m are the respective fibers over a point $m \in M$, then $\Theta|P_m: P_m \rightarrow F_m$ is equivalent to the universal covering of F_m. This is the key point of the above definition as shown in the following theorem. Put $H = SO(p,q)_0$, $H' = \mathrm{Spin}(p,q)$.

THEOREM 1. *Let* Θ: P $\rightarrow$ F *be a principal* K*-bundle such that for every* $m \in M$, $\Theta: \Theta^{-1}(F_m) \rightarrow F_m$ *is equivalent (as a principal* K*-bundle) to the universal covering of* F_m*. Then* P *can be made into a principal* Spin(p,q)*-bundle over* M*, such that* (P,Θ) *is a* Spin(p,q)*-structure on* M.

Proof. Choose a covering of M by open sets W together with local trivializations $\psi: \Pi^{-1}(W) \rightarrow W \times H$. Then $\psi(\mu) = (\Pi(\mu),(\mu))$, where $\varphi: \Pi^{-1}(W) \rightarrow H$ is a differentiable mapping satisfying $\varphi(\mu.h) = \varphi(\mu)h$, $\mu \in \Pi^{-1}(W)$, $h \in H$. Moreover assume that the sets W are simply connected.

For each $m \in M$ there is a homeomorphism α_m, such that the following diagram is commutative:

$$\begin{array}{ccc} \Theta^{-1}(F_m) & \xrightarrow{\alpha_m} & H' \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\sigma} \\ F_m & \longrightarrow & H \end{array} \qquad (1)$$

Let $\Pi' = \Pi \circ \Theta$. We construct a local trivialization for $\Pi'^{-1}(W)$ as follows. Let e' denote the identity element of H' and define $S: W \longrightarrow \Pi'^{-1}(W)$ by $S(m) = \alpha_m^{-1}(e')$, $m \in W$. We claim that S is a differentiable section. Indeed, let $s : W \longrightarrow \Pi^{-1}(W)$ be the section satisfying $\varphi(s(m)) = e$, the identity element of H, for every $m \in W$. Then the following diagram

$$\begin{array}{ccc} & & \Pi'^{-1}(W) \\ & \overset{S}{\nearrow} & \downarrow \Theta \\ W & \xrightarrow[s]{} & \Pi^{-1}(W) \end{array}$$

is commutative by (1). On the other hand, since W is simply connected, S must be the unique differentiable mapping making the diagram commutative and satisfying $S(m) = \alpha^{-1}m(e')$ for some fixed $m \in W$. Clearly $\Pi' \circ S = id_W$.

For $v \in \Pi'^{-1}(W)$, set $\Phi(v) = \alpha_{\Pi'(v)}(v)$. It follows that $(\Phi \circ S)(m) = e'$ for every $m \in W$ and that the diagram

$$\begin{array}{ccc} \Pi'^{-1}(W) & \xrightarrow{\Phi} & H' \\ \downarrow \Theta & & \downarrow \sigma \\ \Pi^{-1}(W) & \longrightarrow & H \end{array}$$

is commutative. Since Θ and φ are differentiable and σ is a local diffeomorphism, it follows that Φ is differentiable. Define $\psi: \Pi'^{-1}(W) \longrightarrow W \times H'$ by $\psi(v) = (\Pi'(v), \Phi(v))$, $v \in \Pi'^{-1}(W)$.

We have the following commutative diagram

$$\begin{array}{ccc} \Pi'^{-1}(W) & \xrightarrow{\psi} & W \times H' \\ \Theta \downarrow & & \downarrow id \times \sigma \\ \Pi^{-1}(W) & \xrightarrow[\psi]{} & W \times H \end{array} \qquad (2)$$

and can easily check that ψ is a diffeomorphism.

It remains to define a right action of H' on P so that $\Pi': P \longrightarrow M$ is an H'-bundle. For $v \in \Pi'^{-1}(W)$ and $h' \in H'$ let $v.h' = \psi^{-1}(\Pi'(v), \Phi(v)h')$. To check that this is well defined, let W' be another open set with $W \cap W' \neq \emptyset$ and corresponding sections s', S'. Denote by $\times$ the action defined on $\Pi'^{-1}(W')$.

Let $\beta: W \longrightarrow H'$ be the mapping such that $S'(m) = S(m).\beta(m)$, and let $\gamma(m) = \sigma(\beta(m))$.

Then we have

$s'(m) = \Theta(S'(m)) = \Theta(S(m).\beta(m)) = \psi^{-1}(m, \sigma(\beta(m)) = s(m).\gamma(m)$ by (2)

Using this, we obtain:

$$\Theta(S'(m)\times h') = \Theta(S'(m)).\sigma(h') = s'(m).\sigma(h') =$$
$$= s(m).\gamma(m).\sigma(h') = \Theta(S(m)).\gamma(m).\sigma(h') =$$
$$= \Theta(S(m)).\sigma(\beta(m)).\sigma(h') = \Theta(S'(m).h').$$

Since Θ is a local diffeomorphism, this implies that $S'(m).h' = S'(m)\times h'$ if h' is in a suitable neighborhood of e'. But H' is connected, hence the equality holds for every $h'\in H'$. This implies that both definitions agree on $W\cap W'$. Q.E.D.

COROLLARY 1. M *admits a* Spin (p,q)-*structure if and only if there is an element* $\zeta\in H^1(F,K)$ *such that, if* $i_m\colon F_m\to F$ *is the inclusion map,* $i_m^*(\zeta)=\lambda$ *for every* $m\in M$.

Proof. Assume that the conditions of Definition 1 are satisfied and let $\zeta\in H^1(F,K)$ be the cohomology class representing the bundle $\Theta\colon P\to F$. Then for each $m\in M$, $i_m^*(\zeta)$ is the class corresponding to the bundle $\Theta\colon \Theta^{-1}(F_m)\to F_m$, induced by i_m. But this bundle is equivalent, as a K-bundle, to the universal covering $\sigma\colon H'\to H$, with representative $\lambda\in H^1(H,K)$. Hence $i^*(\zeta)=\lambda$ for every $m\in M$. This proves that the condition is necessary. Sufficiency is simply a restatement of Theorem 1. Q.E.D.

2. SPIN(p,q)-STRUCTURES AND CHARACTERISTIC CLASSES.

In this section we obtain a characterization of manifolds with a Spin(p,q)-structure, in terms of the Stieffel-Whitney classes of certain bundles.

Consider the cohomology spectral sequence of the principal H-bundle $\Pi\colon F\to M$ (see [14], p.495). From its second term one can obtain the following exact sequence:

$$(3)\qquad 0\longrightarrow H^1(M,K)\xrightarrow{\Pi^*} H^1(F,K)\xrightarrow{i^*} H^1(H,K)\xrightarrow{\delta} H^2(M,K)$$

where $i\colon H\to F$ is the inclusion of the fiber $F_m = H$ for each $m\in M$, and δ is the transgression (see [8], Th.5.1.2, p.328). Notice that since we are dealing with bundles with pathwise connected structure groups, no orientability questions arise ([13], [9] p.270).

We can now state our main theorem.

THEOREM 2. *A* (p,q)-*orientable manifold* M *admits a* Spin(p,q)-*struc-*

ture if and only if the mapping i_m^* *in the sequence* (3) *is surjective for every* $m \in M$.

Before proving this theorem we draw its main consequences. Let $T = SO(p) \times SO(q)$; this is a maximal compact subgroup of $H = SO(p,q)_o$. Then the structure group H of the bundle $\Pi: F \to M$ has a reduction to T; let $\nu: Q \to M$ be the reduced bundle. Let $ET \xrightarrow{\mu} BT$ be the universal T-bundle and $f: M \to BT$ the classifying map of Q, i.e.: $f^*(ET) \cong Q$. (For details and notations on universal bundles see [6]). Since $BT = BSO(p) \times BSO(q)$, we can write $f(x) = (f_p(x), f_q(x))$ where f_j is the classifying map of the principal SO(j)-bundle $f_j^*(ESO(j))$, $j = p,q$. For each $m \in M$, we have

$$f^*(ET)_m = f_p^*(ESO(p))_m \times f_q^*(ESO(q))_m$$

Since T is a matrix group there is a natural representation of T on $\mathbf{R}^{p+q}$. Let AT be the corresponding bundle associated with $f^*(ET)$. Since this is a subbundle of the bundle of linear frames of M, we have $AT = TM$, the tangent bundle of M.

Similarly let ASO(j) be the bundle associated with $f^*(ESO(j))$ through the natural representation of SO(j) on $\mathbf{R}^j$, $j = p,q$. Hence $TM = AT = ASO(p) \oplus ASO(q)$.

Now consider the cohomology ring of BSO(j) with coefficients in $\mathbf{Z}_2$; it is well known ([6]) that it is a polynomial ring:

$$H^*(BSO(j), \mathbf{Z}_2) = \mathbf{Z}_2 [w_2, \ldots, w_k] \tag{4}$$

with degree $w_i = i$. The universal Stieffel-Whitney class $w_2(j)$ is the nonzero element of $H^2(BSO(j), \mathbf{Z}_2)$. Hence the second Stieffel-Whitney classes of the bundles just introduced are

$$w_2^j = w_2(ASO(j)) = f_j^*(w_2(j))$$

On the other hand $H^1(BSO(j)), \mathbf{Z}_2) = 0$ and an application of the Künneth formula yields:

$$H^2(BT, \mathbf{Z}_2) \cong H^2(BSO(p), \mathbf{Z}_2) \oplus H^2(BSO(q), \mathbf{Z}_2) \tag{5}$$

For the bundles Q and ET one obtains exact sequences analogous to (3). The three sequences can be related in the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(M,K) & \xrightarrow{\Pi^*} & H^1(F,K) & \xrightarrow{i^*} & H^1(H,K) & \xrightarrow{\delta} & H^2(M,K) \\
 & & \downarrow{=} & & \downarrow{g} & & \downarrow{h} & & \downarrow{=} \\
0 & \longrightarrow & H^1(M,K) & \xrightarrow{\nu^*} & H^1(Q,K) & \xrightarrow{i^*} & H^1(T,K) & \xrightarrow{\delta'} & H^2(M,K) \\
 & & \uparrow{f^*} & & \uparrow{f^*} & & \uparrow{=} & & \uparrow{f^*} \\
0 & \longrightarrow & H^1(BT,K) & \xrightarrow{\mu^*} & H^1(ET,K) & \xrightarrow{i^*} & H^1(T,K) & \xrightarrow{\delta''} & H^2(BT,K)
\end{array}
\tag{6}
$$

Since H and T are homotopically equivalent it follows that h is an isomorphism; g is also an isomorphism because of the Five Lemma. Finally, $H^1(ET,K) = 0$ ([15], p.102). Thus δ'' is injective. But

$$H^2(BT,K) \cong H^2(BT,\mathbf{Z}_2) \oplus H^2(BT,\mathbf{Z}_2) \ ; \tag{7}$$

hence by (4) and (5), $H^2(BT,K) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.
An application of the Künneth formula shows that
$H^1(T,K) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.
Therefore δ'' is an isomorphism.
We can now prove the following

COROLLARY 2. *A* (p,q)*-orientable manifold* M *admits a* Spin(p,q)*-structure if and only if* $w_2^j = 0$, $j = p,q$.

Proof. By Theorem 2 and the exactness of (3), M admits a Spin(p,q)-structure if and only if $\delta = 0$. But this is equivalent to the vanishing of $f^*: H^2(BT,K) \longrightarrow H^2(M,K)$, because of the diagram (6). Now by (5) and (7) an element in $H^2(BT,K)$ can be written

$$c_1 w_2(p) + c_2 w_2(q) + c_3 w_2(p) + c_4 w_2(q) = a_1 w_2(p) + a_2 w_2(q)$$

with $c_i \in \mathbf{Z}_2$, $a_i \in K$.
Hence $f^* = 0$ if and only if

$$0 = f^*(a_1 w_2(p) + a_2 w_2(q)) = a_1 f_p^*(w_2(p)) + a_2 f_q^*(w_2(q))$$

for arbitrary $a_1, a_2 \in K$. This is equivalent to

$$w_2^j = f_j^*(w_2(j)) = 0 \qquad \text{for } j = p,q. \qquad\qquad \text{Q.E.D.}$$

COROLLARY 3. *If a* (p,q)*-orientable manifold* M *admits a* Spin(p,q)*-structure, then* $w_2(M) = 0$.

Proof. $\qquad w_2(M) = w_2(TM) = w_2(ASO(p) \oplus ASO(q)) =$

$$= 1 \vee w_2^q + w_1^p \vee w_1^q + w_2^p \vee 1$$

But $\quad w_1^j = w_1(ASO(j)) = f_j^*(w_1(j)) = f_j^*(0) = 0$, by (4) Q.E.D.

The preceding corollaries show that Definition 2 is a natural generalization for pseudo-Riemannian manifolds of the concept of Spin-structure.

We now turn to the proof of the main theorem.

Proof of Theorem 2. Assume that i_m^* is onto for every $m \in M$. Then for each m there is $\zeta_m \in H^1(F,K)$ such that $i_m^*(\zeta_m) = \ell_m(\lambda)$ where $\ell_m : H^1(H,K) \longrightarrow H^1(F_m,K)$ is an isomorphism.

Let $U \subset M$ such that $F|_U = \Pi^{-1}(U) \cong U \times H$. The projection $p_{H,U} : F|_U \longrightarrow H$ induces a mapping $\ell_U = p_{H,U}^* : H^1(H,K) \longrightarrow H^1(F|_U,K)$, which is a monomorphism, by the Künneth formula. If $U = \{m\}$ then $\ell_U = \ell_m$. For each $m \in U$, the inclusion $i_{m,U} : F_m \longrightarrow F|_U$ induces $i_{m,U}^*$ and we have a commutative diagram

(*) $$\begin{array}{ccc} H^1(F_m,K) & \xleftarrow{\ \ell_m\ } & \\ \uparrow{\scriptstyle i_{m,U}^*} & & H^1(H,K) \\ H^1(F|_U,K) & \xleftarrow{\ \ell_U\ } & \end{array}$$

If $W \subset U$ then we have a similar diagram.

Let V be another subset of M with $F|_V$ trivial and $U \cap V \neq \emptyset$.

By (*), $\ell_U(\lambda)$ and $\ell_V(\lambda)$ coincide in $U \cap V$; that is the inclusions of $F|_{U\cap V}$ into $F|_U$ and $F|_V$ satisfy

$$i^*_{U\cap V,U}(\ell_U(\lambda)) = i^*_{U\cap V,V}(\ell_V(\lambda))$$

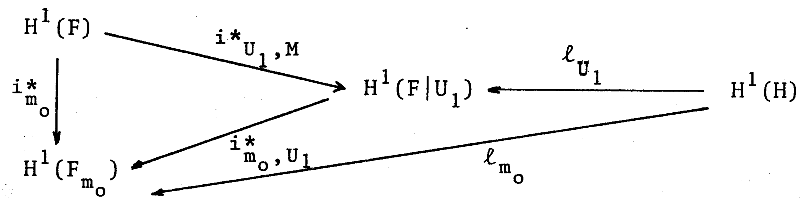
Take $m_o \in M$ and the corresponding $\zeta_o \in H^1(F,K)$ and let m be any other point and α a curve with $\alpha(0) = m_o$, $\alpha(1) = m$. Cover the image of α with open sets $U_1,\dots,U_n$ such that $F|_{U_i}$ is trivial and assume that $m_o \in U_1$ and U_i is homeomorphic to the unit ball in $\mathbf{R}^{p+q}$.

By the Künneth formula we have isomorphisms

$$\ell_{U_i} : H^1(H,K) \longrightarrow H^1(F|_{U_i},K) \ ;$$

moreover all mappings in (*) are isomorphisms.

We also have



where all mappings in the lower right corner are isomorphisms. Thus,

$$i^*_{U_1,M}(\zeta_o) = \ell_{U_1}(\lambda)$$

Therefore, for all $m_1 \in U_1$, $i^*_{m_1}(\zeta_o) = \ell_{m_1}(\lambda)$; in particular for $m_1 \in U_1 \cap U_2$. Repeating the process we obtain $i^*_{m_2}(\zeta_o) = \ell_{m_2}(\lambda)$. Continuing along α, we reach m and obtain $i^*_m(\zeta_o) = \ell_m(\lambda)$. Now we can apply Corollary 1 to obtain the sufficient part of the Theorem. Conversely, assume that M has a Spin(p,q)-structure and let $\gamma\colon H'' \longrightarrow H$ be a principal K-bundle, represented by a class $\omega \in H^1(H,K)$. We will find and element $\tau \in H^1(F,K)$ (i.e.: principal a K-bundle over F) such that $i^*_m(\tau) = \omega$ for every $m \in M$. We proceed in several steps.

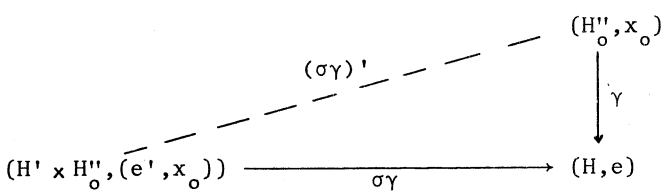
(i) First define a left action of H' on H". Notice that H" can have either two or four connected components. Assuming that $\omega \neq 0$ we can restrict ourselves to the case of two components; they are diffeomorphic by right multiplication by some $k \in K$.

Let H''_o be one of the two components of H" and choose $x_o \in \gamma^{-1}(e) \cap H''_o$, where e is the identity element of H. Then $\gamma\colon (H''_o,x_o) \to (H,e)$ is a covering space. If $\sigma\colon H' \to H$ is the universal covering of H, define

$$\sigma\gamma : H' \times H'' \longrightarrow H$$

by $(\sigma\gamma)((a,b)) = \sigma(a)\gamma(b)$, (product in the group H).

Then, we have the following commutative diagram



The mapping $(\sigma\gamma)'$ is given by the "lifting criterion" since $(\sigma\gamma)_*(\Pi_1(H' \times H''_o,(e',x_o)) = \gamma_*(\Pi_1(H''_o,x_o))$.

Now define $\Phi_o(g,h'') = (\sigma\gamma)'(g,h'')$, $g \in H'$, $h'' \in H_o$. Then,

(a) $\Phi_o(e',h'') = h''$ for every $h'' \in H''_o$.

(b) $\Phi_o(g_1,\Phi_o(g_2,h'')) = \Phi_o(g_1g_2,h'')$.

In fact, $\Phi_o(e',x_o) = e$ and the following diagram

$$\begin{array}{ccc} & \overset{id_{H''_o}}{\nearrow} & (H''_o,x_o) \\ & & \downarrow \\ (e' \times H''_o,(e',x_o)) & \longrightarrow & (H,e) \end{array}$$

is commutative. Hence (a) follows by uniqueness.

On the other hand

$$\Phi_o(g_1,\Phi_o(e',h'') = \Phi_o(g_1e',h'')$$

for every $h'' \in H''_o$. Let $\alpha\colon I \longrightarrow H''$ be a continuous curve such that $\alpha(0) = e'$, $\alpha(1) = g_2$ and

$$F(t,g_1,h'') = \Phi_o(g_1,\Phi_o(\alpha(t),h''))$$
$$G(t,g_1,h'') = \Phi_o(g_1\ \alpha(t),h'')$$

F and G both make the following diagram commute

$$\begin{array}{ccc} & \overset{F}{\nearrow}\ \overset{G}{\nearrow} & (H''_o,\Phi_o(g_1,x_o)) \\ & & \downarrow \gamma \\ (I \times \{g_1\} \times H''_o,(0,g_1,\Phi_o(g,x_o))) & \xrightarrow[\sigma(g,\alpha(t))\gamma]{} & (H,\sigma(g_1)) \end{array}$$

Indeed, $F(0,g,x_o) = \Phi_o(g_1,x_o) = G(0,g_1,x_o)$ and

$$\gamma(F(t,g_1,h'')) = \gamma(\Phi_o(g_1,\Phi_o(\alpha(t),h''))) = \sigma(g_1)\gamma(\Phi_o(\alpha(t),h'')) =$$
$$= \sigma(g_1)\ \sigma(\alpha(t))\gamma(h'') \ ;$$
$$\gamma(G(t,g_1 h'')) = \sigma(g_1\alpha(t))\gamma(h'')$$

Therefore $F = G$ and for $t = 1$ we obtain (b).

Now let H''_1 be the other connected component of H'' and let $k \in K$ be such that $k\ H''_o = H''_1$. Put $x_1 = k\ x_o$ and define

$$\Phi_1(g,h'') = k(\Phi_o(g,kh''))$$

Notice that if x_1 is fixed beforehand then k is uniquely determined. For fixed x_o, x_1 define

$$\Phi\colon H' \times H'' \longrightarrow H''$$

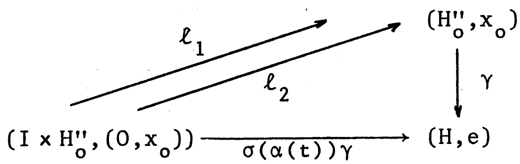
by $\Phi(g,h'') = \Phi_i(g,h'')$, for $h'' \in H''_i$, $g \in H'$, $i = 0,1$.

It is clear that Φ satisfies the group action properties.

(ii) Now we show that the action of H' on H" just defined, commutes with the action of K. Let $K = \{i,k,j_1,j_2\}$ where i = identity, $j_1 = kj_2$, $j_2 = kj_1$. It is clear that $\Phi(g,kx) = k\Phi(g,x)$.
Let j_1 be the element leaving both H''_o and H''_1 invariant. Set

$$\ell_1(g,x) = \Phi_o(g,j_1x) \quad , \quad \ell_2(g,x) = j_1(\Phi_o(g,x))$$

and let α be a continuous curve joining e' with g.
Then $\gamma(\ell_1(\alpha(t)),x)) = \sigma(\alpha(t))\gamma(x) = \gamma(\ell_2(\alpha(t),x))$ and the following diagrams are commutative:



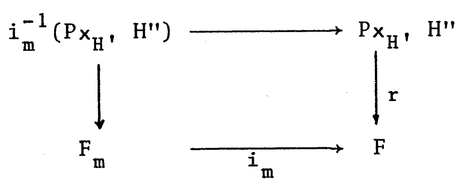
Thus $\ell_1 = \ell_2$. This also holds for Φ_1 and $j_2 = kj_1$, proving our claim.

(iii) H' acts on the right on P and on the left on H", while K acts on the right on H". Then there is a right action of K on $P \times_{H'} H''$ defined by $[x,y]t = [x,yt]$, $x \in P$, $y \in H''$, $t \in K$ (see Bredon's "Introduction to Compact Transformation Groups", p.73). This action is free, as it can be easily verified. Then $P \times_{H'} H''$ is a principal K-bundle over the K-orbit space $P \times_{H'} H''/H$ (again, see Bredon's book p.88).
But $[[x,y]]_K = [x,[y]_K]$, so that by (ii)

$$P \times_{H'} H''/K = P \times_{H'} (H''/K) = P \times_{H'} H$$

Using the homomorphism from H' onto H we obtain $P \times_{H'} H \cong F$.

(iv) Hence we have a principal K-bundle $P \times_{H'} H'' \xrightarrow{r} F$.
Let $\tau \in H^1(F,K)$ be its representative. Considering the diagram



one sees that $i_m^*(\tau) = \omega$. Q.E.D.

3. A CLASS OF EXAMPLES.

In this section we discuss the existence of Spin(p,q)-structures on manifolds of the form G/T, where G is a compact Lie group and T a maximal torus.

Let $\mathcal{G}$ denote the Lie algebra of G. The adjoint representation of T in $\mathcal{G}$ is fully reducible, so that there is a direct sum decomposition

$$\mathcal{G} = L_1 \oplus L_2 \oplus \dots \oplus L_k \oplus L(T)$$

into $Ad_{\mathcal{G}}$ T-invariant subspaces. L(T) is the largest subspace on which T operates trivially and $\dim L_i = 2$, $i = 1,\dots,k$.
The tangent space $(G/T)_o$ of G/T at $o = [T]$ can be identified with the subspace

$$\mathcal{M} = L_1 \oplus L_2 \oplus \dots \oplus L_k$$

In particular $\dim G/T = 2k$.

One can put several invariant indefinite metrics on G/T by choosing an invariant subspace of $(G/T)_o$ and translating it by G. (see [15], p.207).
Thus consider a decomposition

$$(G/T)_o = (L_{j_1} \oplus \dots \oplus L_{j_r}) \oplus (L_{j_{r+1}} \oplus \dots \oplus L_{j_k})$$

and the subbundles of the tangent bundle T(G/T)

$$\xi_r = G\,(L_{j_1} \oplus \dots \oplus L_{j_r})$$
$$\xi_{(k-r)} = G\,(L_{j_{r+1}} \oplus \dots \oplus L_{j_k})$$

obtained by translation by G. The signature of the metric so defined is $(p,q) = (2r, 2(k-r))$ and the Whitney sum of ξ_r and $\xi_{(k-r)}$ is the whole tangent bundle.

Let $\widetilde{T}$ denote the linear isotropy group. Since T is connected we have $\widetilde{T} \subset SO(p,q)_o$. But the structure group of the bundle of linear frames over G/T admits a reduction to $\widetilde{T}$. This shows that G/T is (p,q)-orientable.

According to Corollary 2, G/T admits a Spin(p,q)-structure if and only if $w_2(\xi_r) = 0 = w_2(\xi_{(k-r)})$. By Corollary 3, a necessary condition is satisfied, since $w_2\,(G/T) = 0$ ([7], II)).

The second Stieffel-Whitney classes can be computed as follows. Let $\Theta_1,\dots,\Theta_k$ be the positive roots for a suitable ordering. G/T can be given an almost complex structure having roots $\Theta_1,\dots,\Theta_k$ ([7], I, 12.3).

Then $T(G/T)$, ξ_r and ξ_{k-r} are $U(k)$, $U(r)$ and $U(k-r)$-bundles respectively and $w_2(\xi_r)$, $w_2(\xi_{(k-r)})$ are the mod 2-reductions of the Chern classes $c_1(\xi_r)$, $c_1(\xi_{k-r})$ respectively ([7], I, (13.4)).
Moreover, without loss of generality, we may assume that G is simply connected and semisimple and thus identify $H^2(G/T,\mathbf{Z})$ with the weights of G ([7], I, pp.489-90). Then we have:
"$w_2(\xi_r) = 0 = w_2(\xi_{k-r})$ *if and only if* $\frac{1}{2}c_1(\xi_r)$ *and* $\frac{1}{2}c_1(\xi_{k-r})$ *are weights*".

The Chern classes can be computed by the formulas ([7], II, p.322)

$$c_1(\xi_r) = \sum_{i=1}^{r} \Theta_{j_i} \quad ; \quad c_1(\xi_{k-r}) = \sum_{i=r+j}^{k} \Theta_{j_i}$$

Therefore the problem of whether G/T admits a Spin(p,q)-structure reduces to the problem of whether $\frac{1}{2}\sum_{i=1}^{r}\Theta_{j_i}$ is a weight or not.
For instance, consider the case $G = SU(\ell+1)$. The Lie algebra of G is of type A_ℓ, with simple roots $\alpha_1,\ldots,\alpha_\ell$. The positive roots can be written

$$\Theta_{m,n} = \sum_{i=m}^{n} \alpha_i \qquad n \geqslant m\ ,\ m = 1,\ldots,\ell$$

The space $M \cong (G/T)_o$ can be written

$$M = \sum_{\substack{m,n \\ m\geq n}} L_{(m,n)}$$

Let $\xi_1 = G(L_{(1,1)} \oplus L_{(2,2)} \oplus \ldots \oplus L_{(\ell,\ell)} \oplus L_{(1,\ell)})$ and let ξ_2 be the sum of the remaining $L_{(m,n)}$. This defines a metric of signature $(p,q) = (2(\ell+1),\ell(\ell-1)-2)$. Since

$$\frac{1}{2}\left[\sum_{i=1}^{\ell}\Theta_{i,i} + \Theta_{1,\ell}\right] = \Theta_{1,\ell}$$

which is clearly a weight, we obtain $w_2(\xi_1) = 0$. But $0 = w_2(G/T) = w_2(\xi_1) + w_2(\xi_2)$. Hence $w_2(\xi_2) = 0$ and G/T admits a Spin(p,q)-structure.
Notice that taking $\eta_1 = G(L_{(1,1)} \oplus \ldots \oplus L_{(\ell,\ell)})$ and η_2 its obvious complement one has

$$c_1(\eta_1) = \Theta_{1,\ell} \quad \text{and} \quad \frac{2\langle\Theta_{1,\ell}/2\ ,\ \alpha_1\rangle}{\langle\alpha_1\ ,\ \alpha_1\rangle} = \frac{1}{2}$$

which shows that $w_2(\eta_1) \neq 0$.

4. SIGNATURE (2,n-2).

We conclude with a few observations concerning the case of a metric of signature (2,n-2). The maximal compact subgroup of $SO(2,n-2)_o$ is $SO(2) \times SO(n-2)$ which is not semisimple, unlike the case $p,q > 2$. Because of this we shall not define Spin(2,n-2) = U, the universal covering of $SO(2,n-2)_o$ but rather proceed as follows.

Let $\rho: U \longrightarrow SO(2,n-2)_o$ be the covering homomorphism; then

$\text{Ker}\ \rho = \mathbf{Z} \times \mathbf{Z}_2$ and we define

$$\text{Spin}(2,n-2) = U/\mathbf{Z}$$

Clearly $\sigma: \text{Spin}(2,n-2) \longrightarrow SO(2,n-2)_o$ is a double covering and $\Pi_1(\text{Spin}(2,n-2)) = \mathbf{Z}$. We make this choice taking into consideration that Spin(2,n-2) contains the universal covering of the maximal semisimple connected compact subgroup of $SO(2,n-2)_o$, as in the case of Spin(p,q), $p,q > 2$.

Now the group K is $\mathbf{Z}_2$ and if $H = SO(2,n-2)_o$, there is an exact sequence

$$0 \longrightarrow H^1(M,\mathbf{Z}_2) \xrightarrow{\Pi^*} H^1(F,\mathbf{Z}_2) \xrightarrow{i^*} H^1(H,\mathbf{Z}_2) \xrightarrow{\delta} H^2(M,\mathbf{Z}_2).$$

Thus Theorem 2 is clearly valid in this case. Let T denote the maximal compact subgroup of H.

Then $H^2(BT,\mathbf{Z}_2) = H^2(BSO(2),\mathbf{Z}_2) \oplus H^2(BSO(n-2),\mathbf{Z}_2)$ and Corollary 2 also holds for $p = 2$, $q = n-2$.

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