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THE PRIME RADICAL OF A SKEW GROUP RING

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Let K be a ring and let G be a totally ordered group whose elements act as automorphisms on K. We denote by K_*G the skew group ring over K. The prime radical $P(K_*G)$ is a homogeneous ideal of K_*G ([5], Theorem 1.2). From this, $P(K_*G) = S(K)_*G$, where S(K) is a G-ideal of K. In this paper we shall apply the same method as in [2] and [3] to study the prime radical of K_*G. By this way, we shall obtain similar results to the one above for a sequence of ideals of K_*G, beginning with the Noether radical of K_*G. Furthermore, we shall give a characterization of S(K). Finally, we shall compare S(K) with the prime radical P(K).

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1. INTRODUCTION.

Throughout this paper we assume that K is a ring and G is a totally ordered group whose elements act as automorphisms on K. The skew group ring $R = K_*G$ is the ring whose elements are the finite sums $\sum_{\sigma \in G} a_\sigma u_\sigma$, $a_\sigma \in K$, with the multiplication defined by $u_\sigma a = \sigma(a) u_\sigma$, for every $a \in K$ and $\sigma \in G$. An ideal I of K is said to be a G-ideal if $\sigma(I) = I$, for every $\sigma \in G$. If I is a G-ideal of K, then I_*G is an ideal of K_*G. If H is an ideal of K_*G, then $H \cap K$ is a G-ideal of K. The prime radical of a ring T will be denoted by P(T).

Following [3] and ([4], p.194), for every ordinal α, we define an ideal $N_K(\alpha)$ and a G-ideal $S(\alpha)$ of K as follows

(i) $N_K(o) = 0$ and $S(o) = 0$.

(ii) Suppose that $N_K(\alpha)$ (respectively $S(\alpha)$) has been defined for every ordinal α less than the ordinal β. Then $N_K(\beta)$ (respectively

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$S(\beta)$ is defined as follows:

If $\beta = \gamma+1$ is not a limit ordinal, then $N_K(\beta)$ is the sum (that is, the union) of all ideals A of K (respectively G-ideals B of K) such that $A^t \subseteq N_K(\gamma)$ (respectively $B^t \subseteq S(\gamma)$) for some integer t).

If β is a limit ordinal, then $N_K(\beta) = \sum_{\gamma<\beta} N_K(\gamma)$ (respectively $S(\beta) = \sum_{\gamma<\beta} S(\gamma)$).

There exists an ordinal τ (respectively ρ) such that $N_K(\tau) = N_K(\tau+1)$ (respectively $S(\rho) = S(\rho+1)$). We write $S(K)$ for $S(\rho) = S(\rho+1)$. As it is known, $N_K(\tau) = N_K(\tau+1)$ coincides with the intersection $P(K)$ of all prime ideals of K.

2. THE MAIN RESULTS.

THEOREM 2.1. *For any ordinal* α, $N_R(\alpha) \cap K = S(\alpha)$ *and* $N_R(\alpha) = S(\alpha)_*G$.

Proof. For any $a \in S(1)$, there exists a G-ideal I of K which is nilpotent and such that $a \in I$. Since I_*G is nilpotent we have $a \in N_R(1)$. On the other hand, since $N_R(1)$ is the union of all nilpotent ideals of R, we obtain that $N_R(1) \cap K = S(1)$.

Suppose now that $N_R(\alpha) \cap K = S(\alpha)$ for every ordinal α less than the ordinal β.

Case I, in which $\beta = \gamma+1$ is not a limit ordinal. Let $a \in S(\beta)$. Then there exists a G-ideal I of K such that $a \in I$ and $I^t \subseteq S(\gamma)$, for some integer t. This shows that $(I_*G)^t \subseteq S(\gamma)_*G = (N_R(\gamma) \cap K)_*G \subseteq N_R(\gamma)$. Hence, $a \in I_*G \subseteq N_R(\beta)$. On the other hand, since $N_R(\beta) = \cup\{A$: A is an ideal of R and $A^s \subseteq N_R(\gamma)$, for some integer s$\}$, it follows that $N_R(\beta) \cap K = \cup\{A \cap K$: $A \cap K$ is a G-ideal of K and $(A \cap K)^s \subseteq S(\gamma)$, for some integer s$\} \subseteq S(\beta)$.

Case II, in which β is a limit ordinal. Since $N_R(\beta) = \sum_{\gamma<\beta} N_R(\gamma)$ we have $N_R(\beta) \cap K = \sum_{\gamma<\beta} (N_R(\gamma) \cap K) = S(\beta)$.

Next we shall show that $N_R(\alpha) = S(\alpha)_*G$ by transfinite induction.

For any $f = \sum_{\sigma\in G} a_\sigma u_\sigma$ in $N_R(1)$ there exists a nilpotent ideal I of R such that $f \in I$. As G is a totally ordered group, among the $\sigma \in G$ with $a_\sigma \neq 0$ there is a maximum τ. Assume that $I^s = 0$ and put

$A = \{a \in K: au_\tau + h \in I$ for some $h = \sum_{\rho \in G} b_\rho u_\rho \in R$ such that $\rho < \tau$ if $b_\rho \neq 0\}$. It is easy to see that A is a nilpotent G-ideal of K and then $a_\tau \in A \subseteq S(1) \subseteq N_R(1)$. Thus we have $f - a_\tau u_\tau \in N_R(1)$. Repeating this procedure we have $f \in S(1)_*G$. Hence $N_R(1) = S(1)_*G$.

We now assume that $N_R(\alpha) = S(\alpha)_*G$ for every ordinal α less than β. Then we can easily complete the proof by transfinite induction, using a similar argument.

The following corollary is a direct consequence of Theorem 2.1. (see [5], Corollary 1.3.).

COROLLARY 2.2. $P(K_*G) = S(K)_*G$ *where* $S(K) = P(K_*G) \cap K$.

Now we are going to give a characterization of S(K) (see 2, Theorem 1.1.). A G-ideal Q of K is said to be G-prime if $A.B \subseteq Q$ for any G-ideals A and B implies that either $A \subseteq Q$ or $B \subseteq Q$. It is easy to see that if P is a prime ideal of R, then $P \cap K$ is a G-prime ideal of K. Moreover, if Q is a G-prime ideal of K, then Q_*G is a prime ideal of R.

THEOREM 2.3. *The* G-*ideal* S(K) *is equal to the following*

(i) *The intersection of all the* G-*prime ideals of* K.

(ii) *The intersection of all the* G-*ideals* I *of* K *such that* K/I *has no nilpotent* G-*ideals*.

Proof. We denote by I_1 and I_2 the ideals of K defined by (i) and (ii) respectively. If Q is a G-prime ideal of K, then K/Q has no nilpotent G-ideals. Hence $I_2 \subseteq I_1$.

Let Q be a G-ideal of K such that K/Q has no nilpotent G-ideals. If I is a G-ideal of K and $I^n = 0$, I+Q/Q is a nilpotent G-ideal of K/Q. Thus $I \subseteq Q$ and then $S(1) \subseteq Q$. Using transfinite induction we have $S(K) \subseteq I_2$.

Finally, using Corollary 2.2. we have $S(K) = P(K_*G) \cap K =$
$= \cap \{I \cap K$: I is a prime ideal of R$\} \supseteq I_1$.

The ideal S(K) is called the G-prime radical of K. It is not equal in general to P(K) as we will see in the following example.

EXAMPLE 2.4. Let F be a field and $X = (X_i)_{i \in Z}$ a set of indeterminates. Put $A = F[X]$ the polynomial ring over X and σ the F-automorphism of A defined by $\sigma(X_i) = X_{i+1}$, for all $i \in Z$. On the ring

$K = A/P$, where P is the ideal generated by $\{X_i^n: i \in Z\}$, we define the automorphism induced by σ, which is denoted by σ again. Then K is a local ring with the maximal ideal M generated by $\{X_i+P: i \in Z\}$. Hence $P(K) = M$ and $S(k) = 0$. The validity of the example follows from the next lemma. We denote by x_i the coset X_i+P.

LEMMA 2.5. *The ring* $K = A/P$ *is a* G-*prime ring*.

Proof. Let A and B be G-ideals of K such that $A.B = 0$. If A is a non-zero ideal of K, it must contain an element of the type $x_1^{n-1}\, x_2^{n-1} \ldots\ldots x_t^{n-1}$ as is easy to see. Similarly, if $B \neq 0$ there is an integer u such that $x_1^{n-1}\, x_2^{n-1} \ldots\ldots x_u^{n-1} \in B$.
Since A and B are G-ideals,
$x_1^{n-1}\, x_2^{n-1} \ldots x_t^{n-1}\, x_{t+1}^{n-1} \ldots x_{u+t}^{n-1} = x_1^{n-1}\, x_2^{n-1} \ldots x_t^{n-1}\, \sigma^t(x_1^{n-1} \ldots x_u^{n-1}) \in$
$\in A.B = 0$, which is a contradiction.

3. ADDITIONAL REMARKS.

It is natural to look for conditions under which $P(K_*G) = P(K)_*G$, that is, $S(K) = P(K)$. We shall prove here that this is true when we assume some finiteness condition of G on $P(K)$.

Let V be a subset of K. We say that G satisfies the condition (F) on V if the following holds

(F) For every $v \in V$ there exists a finite set $H = \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subseteq$ $\subseteq G$ such that $\tau(v) \in T_H(v)$, for every $\tau \in G$, where $T_H(v)$ is the ideal of K generated by $\{\sigma_1(v), \sigma_2(v), \ldots, \sigma_n(v)\}$.

EXAMPLE 3.1. (1) If K is a right Noetherian ring and G is any group, G satisfies (F) on K.

(2) If G is represented by a finite set of automorphisms of K, then G satisfies (F) on K.

If P is an ideal of K, $\Gamma(P) = \{a \in P: \sigma(a) \in P$ for every $\sigma \in G\}$ is the maximum G-ideal of K which is contained in P. If P is a prime ideal, then $\Gamma(P)$ is a G-prime as it can be easily verified. On the other hand, $S(\alpha) \subseteq N_K(\alpha)$ for any ordinal α and we have $S(K) \subseteq P(K)$.

THEOREM 3.2. *If* (F) *is satisfied on* $N_K(\alpha)$, *then* $N_K(\alpha) = S(\alpha)$ *for every ordinal* α.

Proof. Let $a \in N_K(1)$ be and I a nilpotent ideal of K such that $a \in I$. Consider the automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_n$ of G such that $\tau(a) \in T_H(a)$ for all $\tau \in G$, where $T_H(a)$ is the ideal generated by $\{\sigma_1(a), \sigma_2(a), \ldots, \sigma_n(a)\}$. The ideal $I_1 = \sum_{i=1}^{n} \sigma_i(I)$ is a nilpotent ideal of K and $a \in \Gamma(I_1) \subseteq S(1)$. Using transfinite induction we have $N_K(\alpha) = S(\alpha)$ for every ordinal α.

The following corollary is a direct consequence:

COROLLARY 3.3. If (F) is satisfied on P(K), then S(K) = P(K) and $P(K_* G) = P(K)_* G$.

REMARK 3.4. Let us assume that (F) is satisfied on K and let Q be a G-prime ideal of K. By the Zorn lemma, the set of all the ideals I of K wich $\Gamma(I) = Q$ has a maximal member, say P. Then $\Gamma(P) = Q$ and it is easy to see that P is a prime. Then, in this case $S(K) = \cap\{Q: Q \text{ is a G-prime}\} = \cap\{\Gamma(P): P \text{ is prime}\} = \Gamma(P(K)) = P(K)$. We have nearly Corollary 3.3.. This remark makes it clear that the condition S(K) = P(K) holds when every G-prime ideal Q of K is of the type $\Gamma(P)$ for a prime P.

An example of this was recently obtained by M.Ferrero [1]. He proves that if T is a liberal extension of a ring A, G is a group which is represented by A-automorphisms of T and K is an intermediate extension of A such that $\sigma(K) = K$ for every $\sigma \in G$, then Q is a G-prime ideal of K if and only if $Q = \Gamma(P)$, for a prime P of K.

Finally we have,

EXAMPLE 3.5. Let A be a ring, $K = A[X_1, X_2, \ldots, X_n]$ a polynomial ring over A and G a group whose elements act as A-automorphisms of K. Then S(K) = P(K). In fact, if Q is a G-prime ideal of K, then $Q \cap A$ is a prime ideal of A, as is easy to see. Hence $P(A) \subseteq Q \cap A \subseteq$ $\subseteq Q$ and so $P(K) = P(A)[X_1, X_2, \ldots, X_n] \subseteq Q$. The result follows from Theorem 2.3..

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