Revista de la
Unión Matemática Argentina
Volumen 32, 1985.

THREE-VALUED LUKASIEWICZ ALGEBRAS WITH AN ADDITIONAL OPERATION

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SUMMARY. In the present note, we introduce cyclic three-valued Lukasiewicz algebras as a generalization of ordinary Lukasiewicz algebras. We were motivated by certain results from A.Monteiro [14] and S.Lee and Y.Keren-Zvi [7], who introduced a new operation, called rotation, on Boolean algebras. They show a new possible approach to the design of digital circuits using the obtained structure.

Cyclic three-valued Lukasiewicz algebras are a natural generalization of the symmetric Boolean algebras introduced by G.Moisil in [10] (see also [3]) and the cyclic Boolean algebras introduced also by G.Moisil in [8] and studied by A.Monteiro in [13] and [14].

In this paper we recall definitions and basic properties needed for the understanding of the work. We study the homomorphisms and the congruence relations, and we characterize the simple algebras. We prove that these algebras are semi-simple and then we describe the algebra with a finite set of free generators.

1. INTRODUCTION.

It is well known ([2], [11], [12], [15]) that a three-valued Lukasiewicz algebra is a system $(A,1,\sim,\nabla,\wedge,\vee)$ such that A is a nonempty set, 1 is an element of A, $\sim$ and ∇ are unary operations defined on A, and $\wedge$, $\vee$ are binary operations defined on A fulfilling the following axioms:

1. $x \vee 1 = 1$
2. $x \wedge (x \vee y) = x$
3. $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$
4. $\sim\sim x = x$
5. $\sim(x \wedge y) = \sim x \vee \sim y$
6. $\sim x \vee \nabla x = 1$
7. $x \wedge \sim x = \sim x \wedge \nabla x$
8. $\nabla(x \wedge y) = \nabla x \wedge \nabla y$

That is, a three-valued Lukasiewicz algebra can be defined as a De Morgan algebra [4] with an operation ∇ which verifies the conditions 6), 7) and 8).

This notion was introduced by Moisil in 1940 [8] with the purpose of giving an algebraic interpretation of three-valued Lukasiewicz propositional calculus.

We follow the terminology introduced by A.Monteiro in his lectures [11].

1.1. DEFINITION. A k-cyclic three-valued Lukasiewicz algebra is a system (L,T), where L is a three-valued Lukasiewicz algeb. : and ı is a unary operation on L satisfying the properties of an automorphism, and $T^k x = x$, for all $x \in L$.

L is k-periodic if k is the smallest positive integer such that $T^k x = x$ for all $x \in L$.

We will denote by $\mathcal{L}_k$ the class of k-cyclic three-valued Lukasiewicz algebras. Since these algebras are defined by means of equations, $\mathcal{L}_k$ is a variety.

We will refer to a k-cyclic (k-periodic) Lukasiewicz algebra as L, for short, and the terms homomorphisms, subalgebras, etc., will have the usual meaning in universal algebra.

T_k denotes the k-periodic Lukasiewicz algebra formed by the sequences $x = (x_1, x_2, \ldots, x_k)$, with $x_i \in \{0, 1/2, 1\}$, $1 \leqslant i \leqslant k$, with the natural Lukasiewicz operations and the operation T defined by $T(x_1, x_2, \ldots, x_k) = (x_k, x_1, \ldots, x_{k-1})$.
Similarly, B_k will denote the k-periodic Lukasiewicz algebra formed by the sequences $x = (x_1, x_2, \ldots, x_k)$, with $x_i \in \{0,1\}$, $1 \leqslant i \leqslant k$, and the operations defined as above. B_k is then a k-periodic Boolean algebra with k atoms. The subalgebras of the algebras T_k and B_k will play an important role in what follows.

2. CONGRUENCES.

Recall that in any Lukasiewicz algebra we have the necessity operator Δ defined by means of $\Delta x = \sim \nabla \sim x$.

2.1. DEFINITION. A kernel of an algebra $L \in \mathcal{L}_k$ is a subset N of L such that:

N1. N is a filter

N2. If $x \in N$, then $\Delta x \in N$

N3. If $x \in N$, then $Tx \in N$.

A kernel is said to be proper if $N \neq L$.

If N is a kernel of L, we introduce a relation in the following way: for arbitrary $x,y \in L$, $x \equiv y \pmod{N}$ if and only if there exists an element a of N such that $x \wedge a = y \wedge a$.

It is easy to check that $\equiv$ is a congruence relation on A. Since $\mathcal{L}_k$ is a variety, the quotient set L/N becomes a k-cyclic Lukasiewicz algebra under the induced operations. The application $h: L \longrightarrow L/N$ defined by $h(x) = |x|$, where $|x|$ is the equivalence class which contains the element x, is an onto homomorphism, and $N = h^{-1}(\{1\})$. If L_1 and L_2 are in $\mathcal{L}_k$, h_1 and h_2 homomorphisms from L onto L_1 and L_2 respectively such that $h_1^{-1}(\{1\}) = h_2^{-1}(\{1\})$, then L_1 and L_2 are isomorphic, that is, the only homomorphic images of L are the algebras L/N, where N is a kernel of L.

The family Φ of proper kernels of a k-cyclic Lukasiewicz algebra ordered by set-inclusion, is inductive, in the sense that every chain in Φ has an upper bound in Φ ; then Φ contains a maximal element. The maximal elements of Φ are called maximal kernels of L.

We are going to characterize the maximal kernels of a k-cyclic Lukasiewicz algebra L. We begin by recalling that a subset F of a three-valued Lukasiewicz algebra L is said to be a monadic filter if it verifies the conditions N1 and N2. Moreover, the notions of maximal monadic filter and minimal prime filter are equivalent. This result will be used in the following theorem.

2.2. THEOREM. *For a kernel* N *to be maximal it is necessary and sufficient that there exists a minimal prime filter* P *such that* $N = P \cap T(P) \cap \ldots \cap T^{k-1}(P)$.

Proof. First we prove the sufficient condition. Suppose P is a minimal prime filter and $N = P \cap T(P) \cap \ldots \cap T^{k-1}(P)$. It is easy to see that $T(P),\ldots,T^{k-1}(P)$ are also minimal prime filters, and therefore P, $T(P),\ldots,T^{k-1}(P)$ are monadic filters. Then, if $x \in N$ we have $\Delta x \in N$. On the other hand, it is clear that $x \in N$ implies that $Tx \in N$. Now let $N \subseteq M$, M a maximal kernel. From N1 and N2, M is a monadic filter, then M is contained in a maximal monadic filter Q. But Q is a minimal prime filter and $N = P \cap T(P) \cap \ldots \cap T^{k-1}(P) \subseteq Q$. Then there exists i, $0 \leqslant i \leqslant k-1$, such that $T^i(P) = Q$. Then $M \subseteq T^i(P)$ and therefore $M \subseteq P \cap T(P) \cap \ldots \cap T^{k-1}(P) = N$, that is $N = M$.

For the converse, if N is a maximal kernel there exists a minimal prime filter P such that $N \subseteq P$. Then $N \subseteq T(P)$ and therefore $N \subseteq P \cap T(P) \cap \ldots \cap T^{k-1}(P)$. Since $P \cap T(P) \cap \ldots \cap T^{k-1}(P)$ is a kernel we have $N = P \cap T(P) \cap \ldots \cap T^{k-1}(P)$.

As a consequence, the semisimplicity of k-cyclic Lukasiewicz algebras is obtained:

2.3. COROLLARY. *Every* k-*cyclic Lukasiewicz algebra* L *is isomorphic to a subdirect product of simple algebras of the same nature.*

Proof. If $\{N_i\}_{i \in I}$ is the family of all maximal kernels of L, then $\bigcap_{i \in I} N_i = \{1\}$.

In the finite case we have the following theorem which will be used later:

2.4. THEOREM. *If* L *is a finite* k-*cyclic Lukasiewicz algebra, then* L *is isomorphic to the direct product of the algebras* L/M_i, $1 \leqslant i \leqslant p$, *where* $\{M_i\}$ *is the family of all maximal kernels of* L.

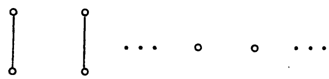
The determination of simple members of the variety $\mathcal{L}_k$ will be our next objective. First we have to introduce the following definition:

2.5. DEFINITION. We say that a minimal prime filter P of L has period d if d is the smallest natural number such that $T^d(P) = P$. In this case, we say that the maximal kernel $N = P \cap T(P) \cap \ldots \cap T^{d-1}(P)$ has period d.

It is easy to see that d must be a divisor of k, and if $N = P \cap T(P) \cap \ldots \cap T^{d-1}(P)$ is a maximal kernel of period d, then the only minimal prime filters which contain N are $P, T(P), \ldots, T^{d-1}(P)$.

Since Lukasiewicz algebras are De Morgan algebras, we can define for any prime filter P of L the Bialynicki-Birula and Rasiowa transformation [4] by means of the formula $\varphi(P) = \complement \sim P$, where $\complement$ denotes the set-theoretical complement and $\sim P = \{\sim x : x \in P\}$. It is clear that in a k-cyclic Lukasiewicz algebra, $\varphi(T(P)) = T(\varphi(P))$.

2.6. REMARK. We know that in a Lukasiewicz algebra L the set $\pi(L)$ of all prime elements of L, with the induced order, consists of pairwise disjoint chains of two elements at most [18] :

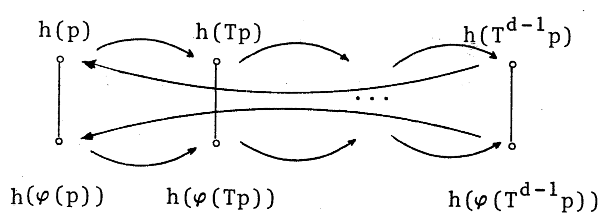


If L is finite, L can be characterized up to isomorphism by means of $\pi(L)$. If T is an automorphism of L, T induces an order isomorphism on $\pi(L)$ of period k. Conversely, it is possible to prove that a k-cyclic Lukasiewicz algebra can be characterized up to isomorphism by the ordered set of its prime elements and an order isomorphism T on $\pi(L)$, with $T^k p = p$, for all $p \in \pi(L)$.

Let M be a maximal kernel of period d, that is
$M = P \cap T(P) \cap \ldots \cap T^{d-1}(P)$, P minimal prime filter, $d|k$.

FIRST CASE.

Suppose $P \neq \varphi(P)$. Since T^i is an automorphism, $0 \leqslant i \leqslant d-1$, we have $T^i(P) \neq T^j(\varphi(P)) = \varphi(T^j(P))$. Thus, the 2d prime filters $P, T(P), \ldots, T^{d-1}(P), \varphi(P), T(\varphi(P)), \ldots T^{d-1}(\varphi(P))$ are pairwise distinct. On the other hand, if Q is a prime filter which contains M, then $\varphi(Q)$ also contains M and either Q or $\varphi(Q)$ is a minimal prime filter. If Q is minimal, there exists i such that $T^i(P) \subseteq Q$, and thus $T^i(P) = Q$. Analogously, if $\varphi(Q)$ is a minimal prime filter.
If $h: L \longrightarrow L/M$ is the canonical homomorphism, the quotient algebra L/M has exactly the following prime filters:
$h(P), h(T(P)), \ldots h(T^{d-1}(P)), h(\varphi(P)), h(T(\varphi(P))), \ldots, h(T^{d-1}(\varphi(P)))$.
Since L is finite, P is a principal filter F(p), generated by a prime element p of L. In addition, $h(F(p)) = F(h(p))$, $h(T(F(p))) = F(h(T(p)))$ and if we suppose that φ is defined on the set of prime elements of L, we can write $h(\varphi(F(p))) = F(h(\varphi(p)))$.
Then from remark 2.4 it follows that L/M is uniquely determined by the following diagram of the ordered set of its prime elements, where the action of bijection is represented by an arrow $\longrightarrow$:



Therefore, L/M is isomorphic to the k-periodic Lukasiewicz algebra T_d, and then it has 3^d elements.

SECOND CASE

Suppose $P = \varphi(P)$. Thus the only prime filters which contain M are $P, T(P), \ldots, T^{d-1}(P)$ and they are all distinct.
Hence the determinant diagram of the set of prime elements of L/M is

$$\begin{tikzcd} \overset{h(p)}{\circ} \arrow[r] & \overset{h(Tp)}{\circ} \arrow[r] & \cdots \arrow[r] & \overset{h(T^{d-1}p)}{\circ} \arrow[lll, bend left] \end{tikzcd}$$

and so, L/M is isomorphic to the k-periodic Boolean algebra with d atoms and then it has 2^d elements [14].

Summing up, the simple k-cyclic Lukasiewicz algebras can be characterized as the algebras T_d and B_d, with d a natural divisor of k.

3. FREE ALGEBRAS.

The aim of this section is to describe the structure of the k-cyclic three-valued Lukasiewicz algebras with a set G of n free generators, n being a finite positive cardinal number. We begin by recalling the following definition:

3.1. DEFINITION. If n is a finite cardinal number, then by a free k-cyclic three-valued Lukasiewicz algebra with n free generators we mean any k-cyclic three-valued Lukasiewicz algebra L(k,n) such that

1. L(k,n) has a set of generators G of power n.
2. Any map f from G into any k-cyclic three-valued Lukasiewicz algebra A can be extended to a homomorphism h_f from L(k,n) into A.

The homomorphism h_f is uniquely determined by the images of the elements g of G.

Since $\mathcal{L}_k$ is a variety it follows from a well known theorem of G. Birkhoff [5] that for any cardinal $n > 0$ there exists L(k,n) and it is unique up to isomorphisms.

On the other hand, if d is a divisor of k, the set of all maximal kernels M of L(k,n) such that L(k,n)/M is isomorphic to T_d, is finite. Similarly for B_d. Then the set of all maximal kernels of

$L(k,n)$ is finite, and from corollary 2.3, $L(k,n)$ is finite. Then it follows that the variety $\mathcal{L}_k$ is locally finite.

Consequently, from theorem 2.4 we conclude that

3.2. THEOREM. *The free* k-*cyclic three-valued Lukasiewicz algebra* $L(k,n)$, n *a finite positive cardinal number, is isomorphic to the direct product of the family* $\{L(k,n)/N_i\}$ $i \in I$, *where* $\{N_i\}$ *is the finite set of all maximal kernels in* $L(k,n)$.

This theorem allows us to apply a technique used by L.Monteiro [16] to describe the free three-valued Heyting algebras. For this we need to know some details about the subalgebras of the simple algebras T_d and B_d.

First, observe that a subalgebra of a simple algebra is a simple algebra. Besides, if d' is a divisor of d and

$S = \{x \in T_d : T^{d'}x = x\}$, S is a subalgebra of T_d , and an easy calculation shows that the number of elements of S is $3^{d'}$ and S is isomorphic to the simple algebra $T_{d'}$.

Analogously, the set $R = \{x = (x_1,\dots,x_d) \in T_d : T^{d'}x = x$, $x_i \in \{0,1\}\}$ is a subalgebra of T_d with $2^{d'}$ elements isomorphic to $B_{d'}$, and such that $R \subset S$. They are the only subalgebras of T_d.

From now on the subalgebras of T_d will be denoted by $T_{d'}$ and $B_{d'}$, d' divisor of d.

In the same way, the subalgebras of B_d are the algebras $B_{d'}$, d' divisor of d.

It is worth noticing that $T_{d'} \cap T_{d''} = T_{d' \wedge d''}$, $B_{d'} \cap B_{d''} = B_{d' \wedge d''}$ and $B_{d'} \cap T_{d''} = B_{d' \wedge d''}$, where $d' \wedge d''$ is the greatest common divisor of d' and d''.

Now we are in the position to study k-cyclic three-valued Lukasiewicz algebras with a finite set of free generators.

Since
$$L(k,n) = \prod_{d|k} B_d^{\alpha_d} \times \prod_{d|k} T_d^{\beta_d}$$

we must to compute α_d and β_d.

For each divisor d of k, the family N of all maximal kernels of $L(k,n) = L$ can be split into the following sets:

$$N_B^d = \{N \in N : L/N \cong B_d\} \quad ; \quad N_T^d = \{N \in N : L/N \cong T_d\}$$

We can write $L \cong \prod_{N \in \mathcal{N}} L/N = \prod_{\substack{N \in \mathcal{N}_B^d \\ d|k}} L/N \times \prod_{\substack{N \in \mathcal{N}_T^d \\ d|k}} L/N \cong$

$$\cong \prod_{d|k} B_d^{N(\mathcal{N}_B^d)} \times \prod_{d|k} T_d^{N(\mathcal{N}_T^d)}$$

$N(X)$ denoting the number of elements of a finite set X.

So the numbers we must to compute are $\alpha_d = N(\mathcal{N}_B^d)$ and $\beta_d = N(\mathcal{N}_T^d)$.

Let $Epi(L,T_d)$ be the set of all epimorphisms from L onto T_d, $Aut(T_d)$ the set of all automorphisms of T_d, $F^*(G,T_d)$ the set of all functions f from G into T_d with the property that T_d is the subalgebra generated by $f(G)$.

It is easy to see that $N[Aut(T_d)] = d$.

Consider the mapping $s: Epi(L,T_d) \to \mathcal{N}_T^d$ defined by $s(h) = Ker\, h = h^{-1}(\{1\})$, $h \in Epi(L,T_d)$. It is not difficult to prove that s is an onto mapping. In addition, if $s(h) = N$, $s^{-1}(N) = \{\alpha \circ h: \alpha \in Aut(T_d)\}$.

Thus $$\beta_d = N(\mathcal{N}_T^d) = \frac{N(Epi(L,T_d))}{N(Aut(T_d))} = \frac{N(Epi(L,T_d))}{d}$$

On the other hand, the mapping r which maps each epimorphism h of $Epi(L,T_d)$ into its restriction to G: $f = h_G$, defines a one to one correspondence between $Epi(L,T_d)$ and $F^*(G,T_d)$.

Therefore $$\beta_d = \frac{N(F^*(G,T_d))}{d}.$$

Now observe that $F^*(G,T_d)$ is the set of all functions f from G into T_d such that there exists no maximal subalgebra S of T_d with $f(G) \subset S$, that is to say, $f(G) \not\subset S$ for all maximal subalgebra S of T_d.

But, if $M(d)$ is the set of maximal divisors of d, the maximal subalgebras of T_d are the algebras T_x, with $x \in M(d)$, and the algebra B_d.

Let $F_T^d = F(G,T_d)$ be the set of all functions from G into T_d; similarly, let F_B^d be the set of all functions from G into B_d.

Then $N(F^*(G,T_d)) = N(F_T^d - \bigcup_{x \in M(d)} F_T^x \cup F_B^d) =$

$$= N(F_T^d) - N(\bigcup_{x\in M(d)} F_T^x \cup F_B^d) =$$

$$= (3^d)^n - [N(F_B^d) + N(\bigcup_{x\in M(d)} F_T^x) - N((\bigcup_{x\in M(d)} F_T^x) \cap F_B^d)] =$$

$$= (3^d)^n - [N(F_B^d) + N(\bigcup_{x\in M(d)} F_T^x) - N(\bigcup_{x\in M(d)} (F_T^x \cap F_B^d))]$$

Observe that the algebras $T_x \cap B_d = B_x$, $x \in M(d)$, are the maximal subalgebras of B_d, and $F_T^x \cap F_B^d$ is the set of all functions from G into $T_x \cap B_d = B_x$.

A.Monteiro [14] proved that

$$N(\bigcup_{x\in M(d)} (F_T^x \cap F_B^d)) = \sum_{X\subseteq M(d)} (-1)^{N(X)}\left\{2^{\left[\bigwedge_{x\in X} x\right]}\right\}^n$$

where the greatest common divisor of the elements of X is denoted $\bigwedge_{x\in X} x$. (If $X = \emptyset$ we put $\bigwedge_{x\in\emptyset} x = d$).

In addition, $N(F_B^d) = 2^{dn}$ and

$$N(\bigcup_{x\in M(d)} F_T^x) = \sum_{X\subseteq M(d)} (-1)^{N(X)}\left\{3^{\left[\bigwedge_{x\in X} x\right]}\right\}^n$$

Therefore

$$N(F^*(G,T_d)) = \sum_{X\subseteq M(d)} (-1)^{N(X)}\left\{3^{\left[\bigwedge_{x\in X} x\right]}\right\}^n + \sum_{X\subseteq M(d)} (-1)^{N(X)}\left\{2^{\left[\bigwedge_{x\in X} x\right]}\right\}^n$$

and so

$$\beta_d = \frac{\sum_{X\subseteq M(d)} (-1)^{N(X)}\left\{3^{\left[\bigwedge_{x\in X} x\right]}\right\}^n - \sum_{X\subseteq M(d)} (-1)^{N(X)}\left\{2^{\left[\bigwedge_{x\in X} x\right]}\right\}^n}{d}$$

$$\alpha_d = \frac{\sum_{X\subseteq M(d)} (-1)^{N(X)}\left\{2^{\left[\bigwedge_{x\in X} x\right]}\right\}^n}{d}$$

If $k = 1$, $\alpha_1 = 2^n$, $\beta_1 = 3^n - 2^n$, then $L(1,n) = B_1^{2^n} \times T_1^{3^n-2^n}$ is the three-valued Lukasiewicz algebra with n free generators. This formula has been obtained by A.Monteiro in [11].

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Recibido en febrero de 1985.
Versión final agosto de 1985.