Let $G$ be a connected complex semisimple Lie group and $G_0$ an inner real form of $G$. In this paper we study the space $\mathfrak{g}$ of all orbits in $G/G_0$ of the totality of unipotent maximal subgroups of $G$.

**INTRODUCTION**

Let $G$, $G_0$, $\mathfrak{g}$ be as above. In this paper we provide a cross section of the action of $G$ in $\mathfrak{g}$ (Theorem 4). We also prove that the orbits of a unipotent maximal subgroup of $G$ in $G/G_0$ are closed (Proposition 6) and an analogous of the Bruhat decomposition for $G$ (Proposition 1).

**STATEMENTS AND PROOFS**

Let $G$ be a complex, connected, semisimple, Lie group. $G_0$ an inner real form of $G$, and $B$ a Borel subgroup of $G$, such that $H_0 = B \cap G_0$ is a compact, Cartan subgroup of $G_0$. Let $H$ be the complexification of $H_0$. Lie groups will always be denoted by capital Roman letters. The corresponding Lie algebra will be denoted by the corresponding lower case german letter. The complexification of a real vector space, will be denote by adding the upperscript $C$.

Let $\Phi(g,h)$ denote the root system of the pair $(g,h)$. Fix $K$ a maximal compact subgroup of $G_0$, such that $H_0 \cap K$ is a maximal torus of $K$. $K$ determines a Cartan decomposition of $g_0 = k \oplus p$. If $\alpha$ is a root of the pair $(g,h)$ its corresponding root space lies in $k^C$ or $p^C$. In the former case $\alpha$ is called compact and in the second case noncompact. Let $\sigma$ or $\overline{\sigma}$ denote the conjugation of $g$ with respect to $g_0$. Then for each $\alpha$ in $\Phi(g,h)$ it is possible to find root vectors $Y_\alpha$, $\overline{Y_\alpha}$ such that:

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Y_{\alpha} lies in k^C, \sigma(Y_{\alpha}) = -Y_{\alpha}, [Y_\alpha, Y_{-\alpha}] = Z_\alpha and \alpha(Z_\alpha) = 2 for \alpha compact.

Y_{\alpha} lies in p^C, \sigma(Y_{\alpha}) = Y_{\alpha}, [Y_\alpha, Y_{-\alpha}] = Z_\alpha and \alpha(Z_\alpha) = 2 for \alpha noncompact.

Two roots are called strongly orthogonal if neither their sum nor their difference is a root. Let \Psi denote the system of positive roots in \Phi(g, h) determined by the Lie algebra n of the unipotent radical N of B. For each noncompact root \alpha in \Psi, let

\[ c_\alpha = \exp(\frac{t}{2} (Y_{-\alpha} - Y_{\alpha})) \]

the inner automorphism associated to \( c_\alpha \) is usually called "The Cayley transform associated \( \alpha \)." In [S] it is proved that if two positive noncompact roots are strongly orthogonal, then the associated Cayley transform commutes. Thus, if \( S \) denotes a subset of \( \Psi \) consisting of noncompact strongly orthogonal roots, then, the product

\[ c_S = \prod_{\alpha \in S} c_\alpha \]

is well defined.

For any two Lie groups \( G \supset H \), let

\[ W(G, H) \] "The Weyl group of H in G"

denote the normalizer of \( H \) in \( G \) divided by \( H \). Keeping in mind the notation written from the beginning we state and prove the first result of this paper.

**PROPOSITION 1.** \( G = \cup G_{\alpha} c_S W B. \)

Here, the union runs over a set of representatives of \( W(G, H) \) and all the subsets \( S \) of \( \Psi \), such that \( S \) consists of strongly orthogonal noncompact roots.

**Proof.** Since every Borel subgroup of \( G \) is equal to its own normalizer, the coset space \( G/B \) can be identified with the set of maximal solvable subgroups of \( G \) via the map \( xB \rightarrow xBx^{-1} \). It follows that this map is equivariant. Now if \( B_1 \) is any Borel subgroup of \( G \), \( B_1 \) contains a \( \sigma \)-invariant Cartan subgroup. Because \( \sigma(B_1) \) is another Borel subgroup of \( G \) and by Bruhat's lemma \( B_1 \cap \sigma(B_1) \) contains a Cartan subgroup. Fix a regular element \( h \) in \( B_1 \cap \sigma(B_1) \) since, in [F] pag.479 is proved that for any regular element \( h \), \( zh + z\sigma(h) \) is regular for suitable \( z \) in \( C \), we have that \( B_1 \) contains a \( \sigma \)-invariant regular element. Thus, \( B_1 \) contains a \( \sigma \)-invariant Cartan subgroup \( T \). In [S] it is proved that if \( T \) is any \( \sigma \)-invariant Cartan
subgroup of $G$, there exists a strongly orthogonal subset $S$ of the set of noncompact roots in $\Phi$, such that $T$ is $G_o$-conjugated to $c_S H c_S^{-1}$. Therefore $B_1$ is $G_o$-conjugated to a Borel subgroup containing $c_S H c_S^{-1}$, for some strongly orthogonal set of noncompact roots in $\Phi$. Since any two Borel subgroups of $G$ containing $H$ are $W(G,H)$ conjugated, we conclude that

$$B_1 = g c_S w B w^{-1} c_S^{-1} g^{-1}$$

($g \in G_o$, $w \in W(G,H)$ and $c_S$ a Cayley transform)

Now, if $x$ is in $G$, $B_1 = x B x^{-1}$ is a Borel subgroup, hence, via the map between $G/B$ and the set of maximal solvable subgroups of $G$ described above, we have that $x = g c_S w b$.

Q.E.D.

**COROLLARY.** If $A$ is any Borel subgroup of $G$, then $A$ contains a $\sigma$-invariant Cartan subgroup.

Towards the uniqueness of the decomposition in proposition 1 we prove

**LEMMA 2.** Let $a$ be any Borel subalgebra of $g$ and $h_1, h_2$ two $\sigma$-invariant Cartan subalgebras of $g$ contained in $a$. Then, there exists $x \in A \cap G_o$ such that $h_2 = \text{Ad}(x) h_1$. Here, $A$ stands for Borel subgroup of $G$, corresponding to $a$.

**Proof.** Let $n$ be the nilpotent radical of $a$. Since $a = h_1 \oplus n$ and that a Cartan subalgebra of $g$, is a Cartan subalgebra of $a$ ([F] 17.7), and any two Cartan subalgebras of $a$ are $A$-conjugated ([F] 17.8) we have that $h_2 = \text{Ad}(n) h_1$, where $n$ is an element of the unipotent radical of $A$. Because $h_1$ and $h_2$ are $\sigma$-invariant we have that

$$h_2 = \sigma(h_2) = \sigma(\text{Ad}(n) h_1) = \text{Ad}(\sigma(n)) \sigma(h_1) = \text{Ad}(n) h_1$$

Hence, $\text{Ad}(n^{-1} \sigma(n)) h_1 = h_1$, so if $H_1$ is the Lie group with Lie algebra $h_1$, we have that $n^{-1} \sigma(n) = w$ is in $W(G,H)$.

If $z \in h_1$, in [F] is proved, for any $n$ in the unipotent radical of $a$ that

$$\text{Ad}(n) Z = Z + n(Z)$$

where $n(Z)$ is an element of $n$, which depends on $Z$ and $n$.

Therefore, for any $Z \in h_1$, since $n = \sigma(n) w$ we have that

$$Z + n_1(Z) = \text{Ad}(w) Z + n_2(Z)$$

where $n_1(Z)$ is in $n$ and $n_2(Z)$ is an element of $n$ plus its opposite, Lie algebra. Because $g$ is the direct sum of $a$ plus the opposite Lie
algebra to \( n \), we have that
\[ \text{Ad}(w)Z = Z \quad \text{for any } Z \text{ in } h_1. \]
This allows us to conclude that
\[ n = \sigma(n)h_0 \quad \text{with } h_0 \text{ in } H_1. \]
The equality \( h_0 = \sigma(n)^{-1}n \) implies \( \sigma(h_0) = h_0^{-1} \), and because \( H_1 \) is abelian connected and \( \sigma \)-invariant, we can find \( h_1 \) in \( H_1 \) such that
\[ h_0 = h_1^2; \quad \sigma(h_1) = h_1^{-1}. \]
Let \( n_1 = nh_1^{-1} \), then \( n_1 \) is in \( B \). On the other hand,
\[ \sigma(n_1) = \sigma(n)\sigma(h_1^{-1}) = \sigma(n)h_1 = \sigma(n)h_0h_1^{-1} = \sigma(n)\sigma(n)^{-1}nh_1^{-1} = nh_1^{-1} = n_1. \]
Thus \( n_1 \) is in \( A \cap G_0 \). Also \( \text{Ad}(n_1)k_1 = \text{Ad}(n)\text{Ad}(h_1^{-1})k_1 = \text{Ad}(n)k_1 = h_2. \)

Q.E.D.

**COROLLARY.** If \( A \) is any Borel subgroup of \( G \) and \( H_1, H_2 \) are \( \sigma \)-invariant Cartan subgroups of \( G \) which are in \( A \), then \( H_1 \) is \( G_0 \cap A \) conjugated to \( H_2 \).

We keep the hypothesis and notation as in proposition 1 and lemma 2.

**LEMMA 3.** We write
\[ G_0 \ c_s \ w \ B = G_0 \ c_s \ w' \ B \quad (*) \]
Here \( c_s, c_s' \) are Cayley transforms and \( w, w' \) are in \( W(G,H) \).
Then, the equality (*) holds if and only if there exists \( w_3 \) in \( W(G_0,H) \) such that \( w_3(S \cup (-S)) = S' \cup (-S') \) and there exists \( w \) in \( W(G,H) \) which is \( c_s \)-conjugated to an element of \( W(G_0,(c_sHc_s^{-1}) \cap G_0)) \),
and if \( w_4 \) is in \( W(G,H) \) satisfying
\[ c_s^{-1} = w_3 c_s w_4 c_s^{-1} \]
them \( w' = w_3 w_4 w_5 w \) in \( W(G,H) \).

**Proof.** If the equality holds, then, there are \( g \) in \( G_0 \) and \( b \) in \( B \), such that
\[ g \ c_s \ w \ b = c_s', \ w' \]
Hence, \( A = g \ c_s \ w \ b \ B \ b^{-1}w^{-1}c_s^{-1}g^{-1} = c_s', \ w' \ B \ w^{-1}c_s^{-1}, \) or
\[ A = g \ c_s \ w \ B^{-1}w^{-1}c_s^{-1}g^{-1} = c_s', \ w' \ Bw'^{-1}c_s^{-1}. \]
Thus \( g \ c_s \ w \ H \ w^{-1}c_s^{-1}g^{-1} = g \ c_s \ H \ c_s^{-1}g^{-1} \) and \( c_s, \ w' \ H \ w'^{-1}c_s^{-1} = c_s, \ c_s^{-1}, \) are \( \sigma \)-invariant Cartan subgroups of \( A \) [S]. Because of
lemma 2, there exists \( b_1 \in A \cap G_0 \) which carries \( g c_s H c_s^{-1} g^{-1} \) onto \( c_s' H c_s' \). Thus, \( c_s H c_s^{-1} \) and \( c_s' H c_s'^{-1} \) are \( G_0 \)-conjugated. [S] implies that there exists \( w_3 \in W(G_0, H) \) such that

\[
w_3(S \cup (-S)) = S' \cup (-S').
\]

Now, if \( \beta \) is any noncompact root \( c_\beta^2 \) is equal to "the reflection about \( \beta \". Thus, \( c_\beta \) is equal to \( c_\beta \) times an element of \( W(G, H) \). Moreover, in \([V]\) is proven that, if \( w \in W(G_0, H) \), then \( c_\beta w \) is equal to \( w c_\beta w^{-1} \) or \( w c_\beta'^{-} w^{-1} \) (\( c_\beta = "reflection about \beta\") depending on whether \( \text{Ad}(w) Y_\beta = Y_w(\beta) \) or \( \text{Ad}(w) Y_\beta = -Y_w(\beta) \).

Therefore, we conclude that the equality \( gc_3 wb = c_3' w' \) implies that there exist \( w_3 \in W(G_0, H) \), \( w_4 \) product of reflections about roots in \( S \), such that

\[
g c_s w b = w_3 c_3 w_4 w_3^{-1} w'
\]

Set \( g_1 = w_3^{-1} g \), and \( w_6 = w_4 w_3^{-1} w' \) then we have that \( g_1 \) is in \( G_0 \), \( w_6 \) is in \( W \) and

\[
g_1 c_s w b = c_s w_6
\]

Thus, \( w b B b^{-1} w^{-1} = w B w^{-1} \) is a Borel subgroup containing \( w H w^{-1} = H \), hence \( (w B w^{-1}) \cap G_0 = H_0 \) \( (G_0 \) is inner!).

Now, \( c_\beta^{-1} g_1 c_\beta^{-1} w H w_6^{-1} c_\beta^{-1} g_1^{-1} c_\beta^{-1} = c_\beta^{-1} g_1 c_\beta^{-1} H c_\beta^{-1} g_1 c_\beta = w b H b^{-1} w^{-1} \) is a \( \sigma \)-invariant Cartan subgroup of \( w B w^{-1} \). By lemma 2, there exists \( h \) in \( (w B w^{-1}) \cap G_0 = H_0 \) such that \( w b H b^{-1} w^{-1} = h H h^{-1} = H \). Therefore, \( b \) lies in the normalizer of \( H \) in \( G \) and in \( B \), which implies \( b \) is in \( H \). Thus \( w \) and \( w b \) represent the same element of \( W(G, H) \). Finally, let \( w_5 = c_\beta^{-1} w_3 g c_\beta \). Because \( w_5 = w_4 w_3^{-1} w' w b \), we have that \( w_5 \in W(G, H) \).

Hence \( w_5 \) is in \( W(G, H) \cap c_\beta^{-1} W(G_0, (c_\beta H c_\beta^{-1}) \cap G_0) c_\beta \). In words, \( w_5 \) is conjugated to an element of the Weyl group of \( c_\beta H c_\beta^{-1} \) in \( G_0 \).

Therefore we have proven

\[
G_0 c_s w B = G_0 c_s w' B \quad \text{implies that there are} \quad w_3 \text{ in } W(G_0, H_0), \ w_5 \text{ in } W(G, H) \text{ such that} \quad w_3(S \cup (-S)) = S' \cup (-S' ) \]

\( w_5 \) is in \( W(G, H) \) and is conjugated by \( c_s \) to an element of \( W(G_0, (c_\beta H c_\beta^{-1}) \cap G_0); \) and if \( w_4 \) is in \( W(G, H_0) \) such that

\[
c_s' = w_3 c_\beta w_4 w_3^{-1}
\]

then

\[
w' = w_3 w_4 w_5 w.
\]
Conversely. Let \( w, w_3, w_4, w_5 \) and \( w' \) as in the hypothesis of the lemma. Then

\[
G_0 c_{w_3} w'B = G_0 w_3 c_{w_4} w_5^{-1} w_3 w_4 w_5 wB = \\
= G_0 c_{w_5} wB = G_0 c_{w} c^{-1}_{w_5} c_{w} B = G_0 c_{w} B.
\]

Q.E.D.

Lemmas 2 and 3 allow us to parametrize in a useful manner the space of orbits of \( G_0 \setminus G \) by the action of the maximal unipotent subgroup of \( G \).

Let \( N_1 \) be any maximal unipotent subgroup of \( G \). The orbit of \( N_1 \) by \( G_0 \times \) in \( G_0 \setminus G \) is the set \( \{ G_0 x n : n \in N_1 \} \).

Let \( \theta \) be the set of all orbits of the totality of maximal unipotent subgroups of \( G \). Since the conjugated of a maximal unipotent subgroup of \( G \) is a maximal unipotent subgroup of \( G \), we have that \( G \) acts on \( \theta \) by the rule

\[
(G_0 \times N_1)g = G_0 x g^{-1}(g N_1 g^{-1}) (x, g \in G).
\]

From now on, we will only consider this action of \( G \) in \( \theta \). Let \( G_0, H, B \) as in the beginning of the paper. Let \( N \) be the unipotent radical of \( B \). If \( N_1 \) is any maximal unipotent subgroup of \( G \), there is \( g \) in \( G \) such that \( N_1 = g N g^{-1} \). Thus, \( G_0 \times N_1 = G_0 x g N g^{-1} = \\
= (G_0 \times g N).g.
\]

Therefore we conclude:

Any element of \( \theta \) is the translate by the action of \( G \) to an orbit of \( N \) (\( N \) being the unipotent radical of \( B \)).

Now, lemma 2 says that any \( N \) orbit is equal to an orbit of the type \( G_0 c_{S} c_{w} h N \) (where, \( h \in H, c_{S} \) is a Cayley transform and \( w \) is in \( W(G,H) \)). Thus, we have proved

**Theorem 4.** A family of representatives of the set \( \theta \) of all the orbits of the totality of maximal unipotent subgroups of \( G \) in \( G_0 \setminus G \) by the action of \( G \) in \( \theta \) is given by

\[
\{ G_0 c_{S} c_{w} h N : c_{S}, ..., w \in W(G,H), h \in H \} \text{ and } G_0 c_{S} c_{w} h N = G_0 c_{S'} c_{w'} h' N
\]

if and only if \( S, S', w, w' \) are related as in lemma 3.

**Lemma 5.** Let \( V \) be a real finite dimensional vector space and \( N \) a unipotent subgroup of \( GL(V) \). Let \( V_C \) be a complexification of \( V \) and \( N_C \) the Zariski closure of \( N \) in \( GL(V_C) \) (we think of \( GL(V) \) included in \( GL(V_C) \) in the usual way). Then

i) For every \( x \) in \( V \), \( N_C x \) is equal to the Zariski closure of \( N.x \).
ii) \((N^C, x) \cap V = N.x\).

**Proof.** Since \(N^C\) is a unipotent subgroup of \(Gl(V_C)\), we have that \(N^C x\) is closed in \(V_C\) [H], therefore \(N^C x\) contains the Zariski closure of \(N.x\). On the other hand, the map \(T + T(x)\) is a polynomial map from \(N^C\) to \(V_C\), hence, it is continuous if we set the Zariski topology in both \(N^C\) and \(V_C\).

Besides in [B] is proved that the Zariski closure of \(N\) is \(N^C\), thus \(N^C x\) is contained in the Zariski closure of \(N.x\), and we have proved i).

In order to prove ii) we need to verify that \((N^C x) \cap V\) is contained in \(N.x\). We do it by induction on dimension of \(V\). If \(\dim V = 1\), the unipotent subgroup of \(Gl(V)\) is \(\{1\}\).

If \(\dim V > 1\). Since, \(N\) is a unipotent subgroup of \(Gl(V)\), Engel's theorem implies that there exists a non zero \(v\) in \(V\) such that \(n(v) = v\) for every \(n\) in \(N\).

Since \(N^C\) is the Zariski closure of \(N\), we have that \(n(v) = v\) for every \(n\) in \(N^C\). By the inductive hypothesis, we conclude that if \(T\) is in \(N^C\), \(a\) in \(V\), \(c\) in \(C\) and \(Tx = a + cv\), then there exists \(S\) in \(N\) such that \(Tx = Sx + dv\), \((d\) in \(C)\).

Now, let \(T\) be in \(N^C\), such that \(Tx\) belongs to \(V\). Owing to the inductive hypothesis, there exist \(S\) in \(N\), \(d\) in \(C\) such that \(Tx = Sx + dv\). Since \(Tx\) and \(Sx\) belong to \(V\), we have that \(d\) is real. If \(d = 0\), we are done.

If \(d \neq 0\), let \(M\) be \(M = \{n \in N^C: n(x) \equiv x (C_v)\}\). It is clear that \(M\) is a Zariski closed subgroup of \(N^C\) and that \(S^{-1}T\) belongs to \(M\) \((S^{-1}T(x) = S^{-1}(Sx + dv) = x + dS^{-1}(v) = x + dv, S^{-1}(v) = v \equiv x)\).

Since \(x\) and \(v\) are in \(V\), it follows that \(M\) is invariant under the conjugation of \(N^C\) with respect to \(N\). Therefore \(M\) has a real form \(M_1\). In other words, \(M_1 = M \cap N\) is a real form of \(M\). Now the map \(n + n(x) - x\) from \(M\) into \(Cv\) is non constant, because \(S^{-1}T\) goes to \(dv\), which is nonzero. Besides it is a polynomial map. Since the unique non trivial Zariski closed subgroup of \(Cv\) is itself, we have that the map \(n + n(x) - x\) is onto. Since, for \(n\) in \(M_1\), \(n(x) - x\) is a real multiple of \(v\) we conclude that there exists \(R\) in \(N\) such that \(R(x) - x = -dv\) \((d\) is real!).

Therefore \(-dv = S^{-1}Tx - x = R(x) - x\), hence \(Tx = SR(x)\). Since \(SR\) belongs to \(N\) we conclude the proof of the lemma.
PROPOSITION 6. Let \( N \) be any maximal unipotent subgroup of \( G \). Then the orbit \( G_0 \times N \) of \( G_0 \times N \) by \( N \) in \( G_0 \backslash G \) is closed in \( G_0 \backslash G \).

Proof. Think of \( G \) as a real Lie group and let \( G^C \) be its complexification. Since \( G \) is a linear Lie group ([W] Wallach) \( G \) is contained in \( G^C \). Let \( G^C_0 \) be the complexification of \( G_0 \) in \( G^C \). Since \( G^C \) and \( G^C_0 \) are semisimple Lie groups, the complex homogeneous manifold \( G^C_0 \backslash G^C \) is a non-singular affine variety. Since \( G^C_0 \cap G = G_0 \), we have that \( G_0 \backslash G \) is a real submanifold of \( G^C_0 \backslash G^C \). Let \( N^C_0 \) be the complexification of \( N_1 \) in \( G^C \). Then ([H], page 125) the orbit \( G^C_0 \times N^C_0 \) is closed in \( G^C_0 \backslash G^C \). Since, for \( x \) in \( G \), the orbit \( G_0 \times N_1 \) is equal to \((G^C_0 \times N^C_0) \cap G \), we have that the orbit \( G_0 \times N_1 \) is closed in \( G_0 \backslash G \).

PROPOSITION. Let \( N_1 \) be any maximal unipotent subgroup of \( G \) and let \( G_0 \times N_1 \) be an orbit of \( N_1 \) in \( G_0 \backslash G \). Let \( \sigma_x \) be the conjugation of \( G \) with respect to the real form \( x^{-1}G_0 x \). Then: 1) The isotropy subgroup of \( N_1 \) at \( G_0 \times N_1 \) is the real form of \( N_1 \cap \sigma_x(N_1) \) determined by \( \sigma_x \). 2) The isotropy subgroup of \( N_1 \) at \( G_0 \times N_1 \) is connected.

Proof. \( \{ n \in N_1 : G_0 \times n = G_0 \times x \} = \{ n \in N_1 : x n x^{-1} \in G_0 \} = \{ n \in N_1 : n \in x^{-1}G_0 x \} = N_1 \cap (x^{-1}G_0 x) \).

Thus, if \( n \in N_1 \) and \( G_0 \times n = G_0 \times x \), then \( \sigma_x(n) = n \).

Hence \( \sigma_x(N_1 \cap (x^{-1}G_0 x)) = N_1 \cap (x^{-1}G_0 x) \). Therefore

\[
N_1 \cap (x^{-1}G_0 x) = (N_1 \cap (x^{-1}G_0 x)) \cap (\sigma_x(N_1 \cap (x^{-1}G_0 x))) = (N_1 \cap (x^{-1}G_0 x) \cap \sigma_x(N_1) \cap (x^{-1}G_0 x)) = (N_1 \cap \sigma_x(N_1)) \cap (x^{-1}G_0 x).
\]

Which proves 1. Let's prove the second affirmation. Since, [H], \( N_1 \cap \sigma_x(N_1) \) is a unipotent algebraic group, it is connected. Moreover, because of a theorem of [B], the group of real points of the algebraic group \( N_1 \cap \sigma_x(N_1) \) has finitely many connected components. Hence, if \( n \) belongs to \( N_1 \cap \sigma_x(N_1) \cap (x^{-1}G_0 x) \), then some power is in the connected component of \( N_1 \cap \sigma_x(N_1) \cap (x^{-1}G_0 x) \).

Say \( x^k \) is in the connected component of \( N_1 \cap \sigma_x(N_1) \cap (x^{-1}G_0 x) \).

Since the exponential map on any real nilpotent connected group is onto, there exists \( y \) in the Lie algebra of \( N_1 \cap \sigma_x(N) \cap (x^{-1}G_0 x) \) such that \( x^k = \exp(y) \). On the other hand, because of Engels theo-
rem and [F] any unipotent algebraic subgroup of $\text{GL}(n, \mathbb{C})$ is simply connected, and hence, [F] the exponential map of $N_1 \cap \sigma_x(N_1)$ is bijective. Thus, the equality $x^k = \exp(y) = (\exp(1/k y))^k$ implies that $x = \exp(1/k y)$. Hence the group $N_1 \cap \sigma_x(N_1) \cap (x^{-1}G_o x)$ is connected.

The following fact is useful.

**Proposition.** Let $K$ be a complex Lie group, such that the exponential map of $K$ is bijective (for example, $K$ unipotent and connected). Let $\sigma$ be an involutive real automorphism of $K$; let $K_0$ be the fixed point set of $\sigma$ and $K_0$ the subset of those elements of $G$ that are taken by $\sigma$ into its inverse. Then $K = K_0 K_0$.

**Proof.** We want to prove that for a given $y$ in $K$, there exist $x$ in $K_0$ and $z$ in $K_0$ such that $y = xz$.

Let $b = b = \sigma(y)^{-1}y$. It is clear that $\sigma(b) = b^{-1}$. Since the exponential map is onto, there exists $Y$ in $k$ such that $b = \exp(Y)$. Since $\sigma(b) = \exp(\sigma(Y)) = b^{-1} = \exp(-Y)$, and the exponential map is injective, we have that $\sigma(Y) = -Y$. Thus $z = \exp(1/2 Y)$ belongs to $K$. Let $x = yz^{-1}$. Then $\sigma(x) = \sigma(y)\sigma(z)^{-1} = \sigma(y)\sigma(z)^{-1} = yb^{-1}\sigma(z)^{-1} = yb^{-1}z = yz^{-1} = x$.

Q.E.D.

**Proposition.** Let $B \subset G$ be any Borel subgroup ($G$ as usual). Let $H$ be a $\sigma$-invariant, Cartan subgroup of $B$. Let $H_o$ be the set of real points of $H$. Then $B \cap G_o = H_o (N \cap G_o) (N$ being the unipotent radical of $B$).

**Proof.** If $hn$ belongs to $B \cap G_o$ then $hn = \sigma(hn) = \sigma(h)\sigma(n)$.

Therefore $\sigma(n) = \sigma(h)^{-1}hn$ belongs to $B$. Since $H$ is $\sigma$-invariant, we have that $\sigma(h)^{-1}h$ is in $H$. Thus (the decomposition $B = H N$) says that $n = \sigma(n)$ and $\sigma(h)^{-1}h = 1$. Hence $\sigma(h) = h$.

Q.E.D.

The next step is to compute the normalizer of an orbit of $N$ in $G/G_o$. Because of the equality $N \times G_o = x(x^{-1}N)x_0$, we have that any orbit in $G/G_o$ is the translate of an orbit through the coset $G_o$. Thus, we conclude.

The normalizer of any $N$-orbit in $G/G_o$ is conjugated (in $G$) to the
normalizer of an orbit of the type $N.0$ ($0 = \text{coset } G_0$).

Now for a fixed unipotent maximal subgroup $N$ of $G$, if $B$ denotes the unique Borel subgroup containing $N$, it is clear that $(B \cap G_0) N$ normalizes the orbit $N.0$. We would like to prove the equality. We have been able to prove this only in particular cases.

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