INVERSION OF ULTRAHYPERBOLIC BESSEL OPERATORS

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ABSTRACT. Let $G_{\alpha} = G_{\alpha}(P \pm io, m, n)$ be the causal (anticausal) distribution defined by

$$G_{\alpha}(P \pm io, m, n) = H_{\alpha}(m,n) (P \pm io)^{\frac{1}{2}(\alpha-n)} \frac{1}{2} K_{\frac{n-\alpha}{2}}[m(P \pm io)^2],$$

where $m$ is a positive real number, $\alpha \in \mathbb{C}$, $K_{\mu}$ designates the modified Bessel function of the third kind and $H_{\alpha}(m,n)$ is the constant defined by

$$H_{\alpha}(m,n) = \frac{\frac{\pi}{2} q^{n} \frac{\pi}{2} \alpha^{\frac{1}{2}} 1 - \frac{\alpha}{2} (m^2)^{\frac{1}{2}} (n-\alpha)}{(2\pi)^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}.$$

The distributions $G_{2k}(P \pm io, m, n)$, where $n$ = integer $\geq 2$ and $k = 1, 2, \ldots$, are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator, iterated $k$-times:

$$K^k\{G_{2k}\} = \delta;$$

$$K = \left\{ \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_n^2} - m^2 \right\}^k.$$

Let $B^\alpha f$ be the ultrahyperbolic Bessel operator defined by the formula

$$B^\alpha f = G_{\alpha} \ast f,$$

$f \in S$. 
Our problem consists in the obtainment of an operator \( T^\alpha = (B^\alpha)^{-1} \) such that if
\[
B^\alpha f = \varphi ,
\]
then
\[
T^\alpha \varphi = f .
\]
In this Note we prove (Theorem III.1, formula (III.7)) that
\[
T^\alpha = G_{-\alpha} ,
\]
for all \( \alpha \in \mathbb{C} \).

We observe that the distribution \( G_\alpha(P \pm i\omega, m, n) \) is a causal (anticausal) analogue of the kernel due to N.Aronszajn - K.T. Smith and A.P.Calderón (cf.[1] and [2], respectively).
The particular radial case of our problem was solved by Nogin, for \( \alpha \neq 1, 2, 3, \ldots \) (cf.[3]).

I. INTRODUCTION

Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Consider a non-degenerate quadratic form in \( n \) variables of the form
\[
P = P(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 ,
\]
where \( n = p+q \). The distribution \( (P \pm i\omega)^{\lambda} \) is defined by
\[
(P \pm i\omega)^{\lambda} = \lim_{\epsilon \to 0} \{P \pm i\epsilon|x|^2\}^{\lambda} ,
\]
where \( \epsilon > 0, \ |x|^2 = x_1^2 + \ldots + x_n^2 , \ \lambda \in \mathbb{C} \).
The distributions \( (P \pm i\omega)^{\lambda} \) are analytic in \( \lambda \) everywhere except at \( \lambda = -\frac{n}{2} - k \), \( k = 0, 1, \ldots \); where they have simple poles (cf.[4], p.275).
The distributions \((m^2 + Q \pm i \alpha)^\lambda\) are defined in an analogue manner as the distributions \((P \pm i \alpha)^\lambda\). Let us put (cf. [4], p.289)
\[
(m^2 + Q \pm i \alpha)^\lambda = \lim_{\varepsilon \to 0} (m^2 + Q \pm i \varepsilon |y|^2)^\lambda, \quad (I,3)
\]
where \(m\) is a positive real number, \(\lambda \in \mathbb{C}\), \(\alpha\) is an arbitrary positive number. In the formula (I,3) we have written
\[
Q = Q(y) = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2, \quad (I,4)
\]
p + q = n,
and
\[
|y|^2 = y_1^2 + \ldots + y_n^2.
\]
It is useful to state an equivalent definition of the distributions \((m^2 + Q \pm i \alpha)^\lambda\).
In this definition appear the distributions
\[
(m^2 + Q)^\lambda = (m^2 + Q)^\lambda_+ \quad \text{if } m^2 + Q > 0, \quad (I,5)
\]
\[
(m^2 + Q)^\lambda_+ = 0 \quad \text{if } m^2 + Q < 0.
\]
\[
(m^2 + Q)^\lambda_+ = (m^2 - Q)^\lambda_- \quad \text{if } m^2 + Q < 0, \quad (I,6)
\]
We can prove, without difficulty, that the following formula is valid (cf. [7], p.566)
\[
(m^2 + Q \pm i \alpha)^\lambda = (m^2 + Q)^\lambda_+ + e^{\pm i \pi \lambda} (m^2 + Q)^\lambda_-. \quad (I,7)
\]
From this formula we conclude immediately that
\[
(m^2 + Q + i \alpha)^\lambda = (m^2 + Q - i \alpha)^\lambda = (m^2 + Q)^\lambda, \quad (I,8)
\]
when \(\lambda = k = \text{positive integer}.
We observe that \((m^2 + Q \pm i \alpha)^\lambda\) are entire distributional functions of \(\lambda\).
Let \(G_\alpha(P \pm i \alpha, m, n)\) be the causal (anticausal) distribution defined by
\[ G_\alpha(P \pm io, m, n) = H_\alpha(m, n)(P \pm io)^{\frac{1}{2}(\alpha - n)} \frac{1}{2} K_{n-\alpha}[m(P \pm io)], \quad (I,9) \]

where \( m \) is a positive real number, \( \alpha \in \mathbb{C} \), \( K_\alpha \) designates the well-known modified Bessel function of the third kind (cf. [5], p. 78, formulae (6) and (7)):

\[
K_\nu(z) = \frac{\pi}{\sin \nu \pi} \left( I_{-\nu}(z) - I_\nu(z) \right), \quad (I,10)
\]

\[
I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{2m+\nu}}{m! \Gamma(m+\nu+1)} \quad (I,11)
\]

and \( H_\alpha(m, n) \) is the constant defined by

\[
H_\alpha(m, n) = \frac{\pi}{(2\pi)^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} e^{\frac{\pi}{4} (\alpha q + i \alpha)} \left( \frac{\pi}{2} \right)^{\frac{n}{2}} \frac{(m^2 + Q)^{-\frac{\alpha}{2}}}{(m^2 + Q + \frac{i}{2})^{\frac{n}{2}}}. \quad (I,12)
\]

The following formula is valid (cf. [6], p. 35, formula (II,1.8)):

\[
[G_\alpha(P \pm io, m, n)]^\Lambda = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{\frac{\pi}{2} (m^2 + Q + \frac{i}{2})^{-\frac{\alpha}{2}}} \cdot (I,13)
\]

Here \( \Lambda \) denotes the Fourier transform of a distribution.

We observe that the right-hand member of (I,13) is an entire distribution of \( \alpha \); therefore \( G_\alpha \) is also an entire distributional function of \( \alpha \).

II. THE PROPERTIES OF \( G_\alpha (P \pm io, m, n) \)

The Bessel potential of order \( \alpha \) (\( \alpha \) being any complex number) of a temperate distribution \( f \), denoted by \( J^\alpha f \), is defined by

\[
(J^\alpha f)^\Lambda = (1 + 4\pi^2 |x|^2)^{-\frac{\alpha}{2}} f^\Lambda, \quad (II,1)\]
was introduced by N.Aronszajn - K.T.Smith and A.P.Calderón (cf. [1] and [2], respectively).

A.P.Calderón proves in [2], Theorem 1, that

\[ J^0 f = G_\alpha \ast f \]  

(II,2)

where

\[ G_\alpha = G_\alpha (x) = \gamma(\alpha) e^{-|x|} \int_0^\infty e^{-|x|t} \left( t + \frac{t^2}{2} \right)^{\frac{n-\alpha-1}{2}} dt \]  

(II,3)

Re \( \alpha < n+1 \), and

\[ \left[ \gamma(\alpha) \right]^{-1} = (2\pi)^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{n-\alpha+1}{2}\right) \]  

(II,4)

We start by observing that the distributional function

\( G_\alpha (P \pm i\omega, m, n) \) (cf. formula (I,9)) is an (causal, anticausal) analogue of the kernel defined by the formula (II,3).

The distributions \( G_\alpha = G_\alpha (P \pm i\omega, m, n) \) share many properties with the Bessel kernel of which they are (causal, anticausal) analogues.

The following theorems hold:

**THEOREM II.1.** Let us put \( \alpha \in C, k = 0, 1, \ldots \), then

\[ \{G_\alpha \ast G_{-2k}\}^\Lambda = (2\pi)^{\frac{n}{2}} \{G_\alpha\}^\Lambda \cdot \{G_{-2k}\}^\Lambda \]  

(II,5)

Here \( \ast \) designates, as usual, the convolution.

**THEOREM II.2.** The following formula is valid

\[ G_\alpha \ast G_{-2k} = G_{\alpha-2k} \]  

(II,6)

when \( \alpha \in C, k = 0, 1, 2, \ldots \).

More generally, the following formulae are valid for all \( \alpha, \beta \in C \),

\[ G_\alpha (P \pm i\omega, m, n) = \delta \]  

(II,7)
Let us define the n-dimensional ultrahyperbolic Klein-Gordon operator, iterated 2-times:

\[ K^2 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2} - m^2 \]

or

\[ K^2 = (\Box - m^2)^2 \]

where \( p+q = n \), \( m \in \mathbb{R}^+ \), \( \ell = 1,2,\ldots \).

From the preceding results we deduce the explicit expression of a family of elementary (causal, anticausal) solution of the ultrahyperbolic Klein-Gordon operator, iterated k-times.

In fact, the following proposition is valid.

**THEOREM II.3.** The distributional functions \( G_{2k}(P \pm io,m,n) \) where \( n = \text{integer} \geq 2 \) and \( k = 1,2,\ldots \), are elementary causal (anticausal) solutions of the ultrahyperbolic Klein-Gordon operator, iterated k-times:

\[ K^k \{ G_{2k}(P \pm io,m,n) \} = \delta . \]

The proofs of the formulae (II,5), (II,6), (II,7), (II,8), (II,9) and (II,11) appear in [6].

It may be observed that the elementary solutions \( G_{2k}(P \pm io,m,n) \) have the same form for all \( n \geq 2 \). This does not happen for other elementary solution, whose form depends essentially on the parity of \( n \) (cf. [7], p.580 and [8], p.403).

We observe that the particular case of Theorem II.3 corresponding to \( n=4, k=1, q=1 \) is especially important.

The corresponding elementary solutions can be written
The formula (II,12) is a useful expression of the famous "magic function" or "causal propagator" of Feynman.

For this reason we have decided to call "causal" ("anticausal") the distributions $G_a(P \pm io,m,n)$.

III. THE INVERSE ULTRAHYPERBOLIC BESSEL KERNEL

Let $B^a f$ be the ultrahyperbolic Bessel operator defined by the formula

$$B^a f = G_a \star f,$$  \hspace{1cm} (III,1)

for all complex $\alpha$.  

Our objective is the attainment of $T^a = (B^a)^{-1}$ such that if $\varphi = B^a f$, then $T^a \varphi = f$.

We note that the inverse ultrahyperbolic Bessel kernel $(B^a)^{-1}$ is, formally, by virtue of (I,13) and (II,10), a fractional power of the differential Klein-Gordon operator:

$$(B^a)^{-1} = (\Box - m^2)^{\frac{\alpha}{2}}.$$  \hspace{1cm} (III,2)

Therefore, here we are seeking an explicit expression for $(B^a)^{-1}$.  The following theorem expresses that if we put, by definition,

$$B^a = G_a,$$  \hspace{1cm} (III,3)

then

$$(B^a)^{-1} = (G_a)^{-1} = G_{-a}$$  \hspace{1cm} (III,4)

for all complex $\alpha$.  

\[
G_2(P + io,m,4) = \frac{mi}{4\pi^2} \frac{K_1[m(P + io)^{1/2}]}{(P + io)^{1/2}}, \hspace{1cm} (II,12)
\]

\[
G_2(P - io,m,4) = \frac{mi}{4\pi^2} \frac{K_1[m(P - io)^{1/2}]}{(P - io)^{1/2}}. \hspace{1cm} (II,13)
\]
Now we shall state our main theorem.

**THEOREM III.1.** If

$$\varphi = B^\alpha f$$

(III,5)

where \(B^\alpha f\) is defined by (III,1), \(f \in S\); then

$$T^\alpha \varphi = f$$

(III,6)

where

$$T^\alpha = (B^\alpha)^{-1} = G_{-\alpha}$$

(III,7)

\(\alpha \in \mathbb{C}\).

Here \(G_{\alpha}\) is defined by (I,9) and \(\alpha\) being any complex number.

**Proof.** From the definitory formula (III,1) we have

$$B^\alpha f = G_\alpha \ast f = \varphi$$

(III,8)

where \(G_\alpha\) is given by (1,9), \(\alpha \in \mathbb{C}\) and \(f \in S\).

Then, in view of (II,9) and (II,7), we obtain

$$G_{-\alpha} \ast (G_\alpha \ast f) = (G_{-\alpha} \ast G_\alpha) \ast f = G_{-\alpha+\alpha} \ast f =$$

$$= G_0 \ast f = \delta \ast f = f$$

(III,9)

Therefore

$$G_{-\alpha} = (B^\alpha)^{-1}$$

(III,10)

Formula (III,10) is the desired result and this finished the proof of Theorem III.1. \(\blacksquare\)
REFERENCES


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