AN APPROXIMATION THEOREM FOR CERTAIN SUBSETS
OF SOBOLEV SPACES

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SUMMARY. We show that a class of differentiable functions vanishing together with their derivatives of order less than \( r \) on the boundary of a smooth domain \( \Omega \) is dense in the subset of \( W^{m+r,p}(\Omega) \) defined by the functions already in \( W^{r,p}_0(\Omega) \). We give a direct proof by introducing a particular extension operator and a related reflection operator. These subsets are Banach spaces that we call \( W^p_{r,m+r}(\Omega) \).

1. PRELIMINARIES AND NOTATION. Let \( \Omega \) be a domain in \( \mathbb{R}^n \). By \((.,.)\) and \( \| \cdot \| \) we shall always denote the scalar product and norm in \( L^2(\Omega) \). For \( r \) a nonnegative integer we denote by \( H^r(\Omega) \) the Sobolev space \( H^r(\Omega) := \{ u \in D'(\Omega) ; D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| < r \} \) with the norm \( \| u; H^r(\Omega) \| = (\sum_{|\alpha| < r} \| D^\alpha u \|^2)^{1/2} \) and by \( \tilde{H}^r(\Omega) \) the closure of \( C_0^\infty(\Omega) \) in \( H^r(\Omega) \) (cfr. [A] where \( H^r(\Omega) = W^{r,2}(\Omega) \) and \( \tilde{H}^r(\Omega) = W^{r,2}_0(\Omega) \)). We state some well known facts about these spaces that we shall need in what follows.

LEMMA 1. If \( u \in H^r(\Omega) \), \( v \in \tilde{H}^r(\Omega) \) and \( |\alpha| < r \), then

\[
(D^\alpha u, v) = (u, D^\alpha v).
\]

Proof. If \( v_h \in C_0^\infty(\Omega) \) is a sequence such that \( \| v_h - v; H^r(\Omega) \| \to 0 \),
then

\[ (\mathbf{D}^\alpha u, v) = \lim_{h \to \infty} (\mathbf{D}^\alpha u, v_h) = \lim_{h \to \infty} (u, \mathbf{D}^\alpha v_h) = (u, \mathbf{D}^\alpha v), \quad \text{Q.E.D.} \]

Let \( \Omega \) be a bounded domain with \( C^\infty \) boundary (i.e. there exists a finite open covering of \( \Omega \), \( \{ U_j \mid j = 1, \ldots, N \} \), such that for each \( j \) there is a map \( \phi_j \) from \( U_j \) onto \( B = \{ \mathbf{y} \in \mathbb{R}^n ; |\mathbf{y}| < 1 \} \) with the properties: i) \( \phi_j \) is one to one, ii) \( \phi_j \in C^\infty(U_j) \), \( \phi_j^{-1} \in C^\infty(B) \), iii) \( \phi_j(U_j \cap \Omega) = B^+ = \{ \mathbf{y} \in B; \mathbf{y}_n > 0 \} = B \cap R^+ \).

**Lemma 2.** If \( u \in C^r(\Omega) \) and \( \mathbf{D}^\alpha u = 0 \) on \( \partial \Omega \) for \( |\alpha| < r \), then \( u \in H^r(\Omega) \).

**Proof.** Let \( U_0 \) be an open subset of \( \Omega \) such that \( \bigcup_{j=0}^N U_j \supseteq \Omega \).

Using a \( C^\infty \) partition of unity subordinate to this covering one sees that it is enough to prove that: if \( u \in C^r(R^+_n) \), \( D^\alpha u(x_1, \ldots, x_{n-1}, 0) = 0 \) for \( |\alpha| < r \) and \( \text{supp } u \) is bounded, then \( u \in H^r(R^+_n) \), (cf. [A], T.3.35, particularly formula (15)). Now in that case let \( \tilde{u}(x) := u(x) \) for \( x \in R^+_n \) and 0 otherwise. Then Gauss' theorem yields for \( \phi \in C^\infty_0(R_n) \) and \( |\alpha| < r \)

\[
\int_{R^+_n} (-1)^{|\alpha|} \tilde{u} \mathbf{D}^\alpha \phi \, dx = (-1)^{|\alpha|} \int_{R^+_n} u \mathbf{D}^\alpha \phi \, dx = \int_{R^+_n} \mathbf{D}^\alpha u \, \phi \, dx = \int_{R^+_n} \mathbf{D}^\alpha u \, \phi \, dx
\]

That is, \( \mathbf{D}^\alpha \tilde{u} \) is the function \( \mathbf{D}^\alpha u \) for \( |\alpha| < r \) and so \( \tilde{u} \in H^r(R^+_n) \).

But then \( u = \lim_{\varepsilon \to 0} v_\varepsilon \) in \( H^r(R^+_n) \) where \( v_\varepsilon(x) = \tilde{u}(x_1, \ldots, x_{n-1}, x_n - \varepsilon) \).

Since \( \text{supp } v_\varepsilon \) is compact in \( R^+_n \), \( v_\varepsilon \in H^r(R^+_n) \) and the proof is complete, Q.E.D.

2. **Introduction.** For \( r, R \) positive integers, \( r < R \), let us call \( H_{r,R}^r(\Omega) \) the Hilbert space \( H_{r,R}^r(\Omega) := \mathbf{H}^r(\Omega) \cap H^R(\Omega) \) with the norm of \( H^R(\Omega) \) and call \( D^r(\Omega) := \{ \phi \in C^\infty(\Omega); \mathbf{D}^\alpha \phi = 0 \text{ on } \partial \Omega \text{ for } |\alpha| < r \}. \)
Now let \( \Omega \) be a bounded domain with \( \mathcal{C}^m \) boundary. By Lemma 2, \( D_\tau(\Omega) \subset H_{\tau, R}(\Omega) \). (It also follows that this space contains properly the space \( H^R(\Omega) \), cf. Th. 5.) In this paper we prove that \( D_\tau(\Omega) \) is a dense subset of \( H_{\tau, R}(\Omega) \). That is

**Theorem 1.** If \( G_{\tau, R}(\Omega) := \text{closure of } D_\tau(\Omega) \text{ in } H^R(\Omega) \), then

\[
G_{\tau, R}(\Omega) = H_{\tau, R}(\Omega).
\]

This theorem can be proved in the particular case \( R = 2\tau \) using results of P.D.E. as follows. For \( \lambda > 0 \) the operator \((-\Delta)^\tau + \lambda \) maps \( H_{\tau, 2\tau}(\Omega) \) continuously into \( L^2(\Omega) \). This map is also 1:1 since for \( u \in H_{\tau, 2\tau}(\Omega) \) using Lemma 1 we obtain

\[
((-\Delta)^\tau u + \lambda u, u) = \sum_{|\alpha| = \tau} (r!/\alpha!)(D^{2\alpha}u, u) + \lambda \| u \|^2 =
\]

\[
= \sum_{|\alpha| = \tau} (r!/\alpha!) \| D^\alpha u \|^2 + \lambda \| u \|^2.
\]

On the other hand for \( \lambda \) great enough \((-\Delta)^\tau + \lambda \) maps \( H_{r, 2r}(\Omega) \) continuously into \( L^2(\Omega) \) (cfr. [S], Th. 9-27, pg. 219). In consequence \( G_{r, 2r}(\Omega) = H_{r, 2r}(\Omega) \). We shall give a direct proof of this fact and moreover of Theorem 1. By using a partition of unity as in Lemma 2 it is enough to prove

**Theorem 2.** Let \( K \) be a compact set in \( B \) and \( u \in H_{r, R}(\Omega) \) with \( \text{supp } u \subset K \cap \Omega^+ \). Then there exists a sequence \( u_h \in D_\tau(\Omega) \) such that \( \text{supp } u_h \subset B^+ \text{ and } \| u_h - u \| H^R(\Omega) \rightarrow 0 \text{ for } h \rightarrow \infty \).

Our proof relies on the following result.

3. **Auxiliary Lemma.** Given \( R \) integers \( K_1, K_2, \ldots, K_R \) there exists a polynomial \( p(x) \) of degree \( R-1 \) such that

1) \( p(2^j) \) is an integer for \( j = 0,1, \ldots \)

2) \( p(2^{m-1}) = K_m \text{ (mod 2) for } 1 \leq m \leq R \)
iii) \( p(2^{m-1}) = K_R \pmod{2} \) for \( R < m \).

**Proof.** If \( x_i = 2^{i-1}, i = 1, 2, \ldots, R \), define \( p(x) \) by

\[
p(x) := \prod_{j=1}^{R} h_j \prod_{k=1}^{R} \frac{(x - x_k)/x_k}{(x - x_k)/x_j}
\]

where \( h_j = 0 \) if \( K_j \) is even and \( h_j = 1 \) if \( K_j \) is odd. Observe that \( p(x) \) satisfies i) and ii) since \( (x_j - x_k)/x_s \) is odd for \( s = \min(j,k) \) and is even for \( s < \min(j,k) \). By the same reasoning for \( x = x_m, m > R \), the first product in the definition of \( p \) is odd and the last is even when not empty. So \( p(x_m) - p(x_R) \) is even, Q.E.D.

**COROLLARY.** Given \( R \) integers \( K_1, \ldots, K_R \), there exists an entire function \( f(z) \) without zeroes such that

i) \( f(2^{j-1}) = (-1)^{K_j} \) for \( j = 1, \ldots, R \)

ii) \( f(2^{j-1}) = 1/f(2^{j-1}) \) for \( j \in \mathbb{N} \).

**Proof.** Define

(1) \[
f(z) := \exp(i\pi p(z))
\]

where \( p(z) \) is the polynomial in the preceding lemma. Then both \( f(z) \) and \( g(z) := 1/f(z) \) have the required properties, Q.E.D.

4. **AN EXTENSION OPERATOR.** Let \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) be an entire function. We associate to \( f \) the operator

(2) \[
(T_f u)(x', t) := \sum_{k=0}^{\infty} c_k u(x', -2^k t)
\]

where \( x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, t \in \mathbb{R}_1 \).

\( T_f \) is well defined if \( u \) vanishes outside a sphere.
THEOREM 3. For \( u \in H^R(R_n^+) \), supp \( u \subseteq B^+ \), we have

i) \( \text{supp } T_f u \subseteq B^- \), \( T_f u \in H^R(R_n^-) \)

ii) \( \| T_f u; H^R(R_n^-) \| \leq M(f) \| u; H^R(R_n^+) \| \) and

\[
D^\alpha T_f u = T_{f_h} (D^\alpha u) \quad \text{for } h = \alpha_n, \quad |\alpha| < R
\]

where \( f_h \) is the entire function

\[
f_h(z) := (-1)^h \sum_{k=0}^{\infty} c_k z^{h+k} = (-1)^h f(2^h z).
\]

Proof. The first assertion of i) is immediate. The second follows from ii). Observe that if \( x = (x', t) \in K \), a compact set in \( R_n^- \), then the sum defining \( T_f u(x', t) \) is finite. Therefore (3) is correct in \( D'(R_n^-) \). To prove ii) it is therefore enough to prove

\[
\| T_{f_h} u; L^2(R_n^-) \| \leq M(f_h) \| u; L^2(R_n^+) \|.
\]

But \( \| u(x', -2^k t) \| = 2^{-k/2} \| u \|. \) Summing up, one gets

\[
\| T_{f_h} u \| \leq \left( \sum_{k=0}^{\infty} |c_k| \cdot 2^{k(2h-1)/2} \right) \| u \| \quad \text{Q.E.D.}
\]

Observe that the lemma remains true if the roles of \( R_n^+ \) and \( R_n^- \) are interchanged. Now we define the extension operator \( E_f \) associated to \( f(z) = \sum c_k z^k \) by

\[
E_f u(x', t) := \begin{cases} 
  u(x', t) & \text{for } t > 0 \\
  0 & \text{for } t = 0 \\
  T_f u(x', t) & \text{for } t < 0.
\end{cases}
\]

THEOREM 4. Let \( u \in H_{r,R_n}(R_n^+) \), closure in \( R_n \) of \( \text{supp } u \subseteq B \).
If the entire function \( f(z) \) verifies

\[
f(2^s) = (-1)^s \quad \text{for } s = r, r+1, \ldots, R-1
\]

then \( E_f u \in H^R(R_n^-) \), supp \( E_f u \subseteq B \) and

\[
\| E_f u; H^R(R_n^-) \| \leq C(f) \| u; H^R(R_n^+) \|.
\]
Proof. We shall show that if $|\alpha| \leq R$, $h = \alpha_n$ and $f_h$ is defined by (4), then

$$(9) \quad D^\alpha(E_f u) = E_{f_h}(D^\alpha u).$$

Therefore, the theorem will follow from Th. 3. To prove (9) we consider two cases.

CASE 1: $\alpha = (0, \ldots, 0, h)$. Let $\phi \in C^\infty_0(\mathbb{R}_n)$. Then if we set $x = (x', t)$,

$$(10) \quad <D^\alpha E_{f_h} u, \phi> = (-1)^h <E_{f_h} D^h \phi, u> =$$

$$= (-1)^h \int_{\mathbb{R}_n^+} (u D^h_t \phi + (-1)^h \sum_{k=0}^\infty c_k u(x', 2^k t) D^h_t \phi(x', -t)) \, dx =$$

$$= (-1)^h \int_{\mathbb{R}_n^+} u D^h_t \phi \, dx + \sum_{k=0}^\infty 2^{(h-1)k} c_k \int_{\mathbb{R}_n^+} u(x) D^h_t \phi(x', -2^k t) \, dx =$$

$$= (-1)^h \int_{\mathbb{R}_n^+} u(x) D^h_t \psi_h(x', t) \, dx \, dt$$

with

$$(11) \quad \psi_h(x', t) = \phi(x', t) - \sum_{k=0}^\infty (-2^k)^{h-1} c_k \phi(x', -2^k t).$$

Since $\sum_{k=0}^\infty |c_k| M^k < \infty$ for any $M > 0$, it is possible to interchange $\sum$ and $\int$ in (10). Also $\psi_h \in C^\infty_0(\mathbb{R}_n^+) \cap H^s(\mathbb{R}_n^+)$ for any $s$.

Now we shall show that

$$(12) \quad (-1)^h \int_{\mathbb{R}_n^+} u(x) D^h_t \psi_h(x', t) \, dx = \int_{\mathbb{R}_n^+} D^h_t u \psi_h \, dx.$$

In fact, since $u \in \hat{H}^r(\mathbb{R}_n^+)$, by Lemma 1,

$$(13) \quad (-1)^h \int_{\mathbb{R}_n^+} u D^h_t \psi_h \, dx = (-1)^{h-j} \int_{\mathbb{R}_n^+} D^j_t u \psi_h \, dx \quad \text{for} \quad j = \min(h, r).$$
This proves (12) for h < r. If h > r, then in view of (7), $\psi_h(x', 0) = 0$ and also $D^\gamma \psi_h(x', 0) = 0$ for $|\gamma| < h - r$. Then by Lemma 2, $\psi_h \in H^{h-r}(R_n^+)$, and we can apply again Lemma 1 to the right hand side of (13) ($j = r$ now!) thus obtaining (12).

The combination of (10) with (12) yields

$$<D^\alpha E_{\epsilon} u, \phi> = \int D^\alpha u \psi_h \, dx = <E_{\epsilon h} (D^\alpha u), \phi>.$$ 

CASE 2: $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 0)$. Then (9) is true regardless condition (7) for $u \in H^q(R_n^+)$, $q > |\alpha|$. In fact, let $\eta(t) \in C^\infty(R_1)$, $\eta = 0$ for $|t| < 1/2$, $\eta = 1$ for $|t| > 1$, and call $\eta_\epsilon(t) := \eta(t/\epsilon)$.

Then for $\phi \in C^\infty_c(R_1)$ we have $\eta_\epsilon \phi \in C^\infty_c(R^-_n \cup R^+_n)$ and so

$$<D^\alpha E_{\epsilon} u, \phi> = (-1)^{|\alpha|} <E_{\epsilon} u, D^\alpha \phi> = (-1)^{|\alpha|} \lim_{\epsilon \to 0} <E_{\epsilon} u, \eta_\epsilon D^\alpha \phi> = \lim_{\epsilon \to 0} (-1)^{|\alpha|} <E_{\epsilon} u, D^\alpha (\eta_\epsilon \phi) > = \lim_{\epsilon \to 0} <E_{\epsilon} (D^\alpha u), \eta_\epsilon \phi > = <E_{\epsilon} (D^\alpha u), \phi >.$$

To combine this two cases we write $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 0) + + (0, \ldots, 0, h) = \alpha' + \alpha''$ and obtain $D^\alpha (E_{\epsilon} u) = D^\alpha' E_{\epsilon} (D^\alpha'' u) = E_{\epsilon h} (D^\alpha u)$, Q.E.D.

5. A REFLECTION OPERATOR. Next we define an operator $E$ which is a generalization of $\phi(x', t) \rightarrow -\phi(x', -t)$. Let $f(z) = \sum_{k=0}^\infty c_k z^k$ be the entire function constructed in the Corollary of section 3 for $K_i = i$ if $i < r$ and $K_i = i-1$ for $r < i < R$.

That is

$$f(z) = \sum_{k=0}^\infty c_k z^k \text{ for } 1 \leq i \leq r; \quad f(z) = (-1)^{i-1} \text{ for } r < i < R.$$ 

Further, let $g(z) := 1/f(z) = \sum_{k=0}^\infty d_k z^k$. For $v$ a function with bounded support let us define
LEMMA 3. If $\phi \in C^\infty_o (\mathbb{R}^n)$, then

i) $E\phi \in C^\infty_o (\mathbb{R}^n)$

ii) $E\phi = \phi$ implies $\phi \in D_+ (\mathbb{R}^n)$

iii) $E^2 \phi = \phi$

iv) $\|E\phi; H^s (\mathbb{R}^n)\| \leq M_s \|\phi; H^s (\mathbb{R}^n)\|$ $\forall s \in \mathbb{N}$.

v) Let $\psi \in H^s (\mathbb{R}^n)$, support of $\psi \subset \mathbf{B}$. If the sequence

\begin{align*}
\{\phi_m \} \subset C^\infty_o (\mathbf{B}) & \text{ verifies } \lim_{m \to \infty} \|\phi_m \cdot \psi; H^s (\mathbb{R}^n)\| = 0,
\end{align*}

then

\begin{align*}
\lim_{m \to \infty} \|E\phi_m - E\psi; H^s (\mathbb{R}^n)\| = 0.
\end{align*}

Proof. i) It is clear from the definition that supp $E\phi$ is bounded and that $E\phi \in C^\infty (\mathbb{R}^n_+ \cup \mathbb{R}^n_-)$. Also

\begin{align}
D^\alpha E\phi (x', 0) & \approx \left( \sum_{k=0}^\infty d_k \alpha_n \right) D^\alpha \phi (x', 0) = (-1)^\alpha d_n (2^\alpha) D^\alpha \phi (x', 0) \\
D^\alpha E\phi (x', -0) & \approx \left( \sum_{k=0}^\infty c_k \alpha_n \right) D^\alpha \phi (x', 0) = (-1)^\alpha c_n (2^\alpha) D^\alpha \phi (x', 0).
\end{align}

i) then follows from

\begin{align}
f(z^h) = g(z^h) = \pm 1.
\end{align}

ii) Let $\alpha_n < r$. Using (15) and (14) it follows that

\begin{align}
D^\alpha E\phi (x', 0) = (-1)^\alpha d_n (2^\alpha) D^\alpha \phi (x', 0) = -D^\alpha \phi (x', 0).
\end{align}

But if $E\phi = \phi$ then

\begin{align}
D^\alpha E\phi (x', 0) = D^\alpha \phi (x', 0)
\end{align}

Comparing (17) and (18) we get $D^\alpha \phi (x', 0) = 0$ for $|\alpha| < r$, that
is $\phi \in D_{r}(R^{n}_{+})$.

iii) Observe that $T_{g}T_{f}\phi(x',t) = \sum_{k=0}^{\infty} d_{k}(\sum_{h=0}^{\infty} c_{h}\phi(x',2^{k+h}t)) = \sum_{j=0}^{\infty} \phi(x',2^{j}t)(\sum_{k=0}^{j} d_{k}c_{j-k})$.

Since $f(z)g(z) = 1$ we have $\sum_{k=0}^{j} d_{k}c_{j-k} = 1$ if $j=0$ and 0 otherwise. Therefore, it holds pointwise that

$$T_{g}T_{f}\phi(x',t) = \phi(x',t) = T_{f}T_{g}\phi(x',t).$$

iv) By i), $\|E\phi; H^{s}(R^{n}_{+})\| \leq \|T_{f}\phi; H^{s}(R^{n}_{+})\| + \|T_{g}\phi; H^{s}(R^{n}_{+})\|$. Now Theorem 3 yields iv).

v) By iv), $E_{h}\phi$ is a Cauchy sequence in $H^{s}(R^{n})$. Therefore, there exists $U \in H^{s}(R^{n})$ such that $\|E_{h}\phi - U; H^{s}(R^{n})\|$ tends to zero.

But in virtue of Theorem 3, ii) both norms $\|E_{h}\phi - T_{g}\phi; H^{s}(R^{n}_{+})\|$ and $\|E_{h}\phi - T_{f}\phi; H^{s}(R^{n}_{+})\|$ tend to zero. So $U$ restricted to $R^{n}_{+}$ is equal to $T_{g}\phi$ and $U$ restricted to $R^{n}_{-}$ is $T_{f}\phi$. Since the distribution $U$ is a function of $L^{2}(R^{n})$ it follows that $U = Ev$, Q.E.D.

Note that conditions (14) for $r < i < R$ are not really used in the proof of Lemma 3.

6. PROOF OF THEOREM 2. Let $u \in H_{r,R}(R^{n}_{+})$, supp $u \subseteq K$ and call $u' := E_{x}u$ (cfr. (6)). Observe that by (14) the hypotheses of Theorem 4 are fulfilled. Thereby $u' \in H^{R}(R^{n})$, supp $u' = K' = $ compact in $B$ and $Eu' \in H^{R}(R^{n})$. In consequence, from the definition of $u'$ we have $Eu' = u'$ a.e. (cfr.(19)). Now let $\phi_{h} \in C_{0}^{\infty}(B)$ be a sequence converging to $u'$ in $H^{R}(R^{n})$. By Lemma 3, v), $E\phi_{h}$ converges to $Eu' = u'$ in $H^{R}(R^{n})$ and then
Using Lemma 3, iii), we see that \( E\Phi = \phi \). Then by ii) of the same Lemma we obtain that \( u_h := \phi \) restricted to \( R_n^+ \) belongs to \( D_r(R_n^+) \). Since \( \|u' - u_h; H^R(R_n^+)\| \leq \|u' - \phi; H^R(R_n^+)\| \leq \|u' - \phi; H^R(R_n)\| \), we see by (20) that the sequence \( u_h \) satisifies all the requirements, Q.E.D.

7. THE SPACES \( W^p_{r,R}(\Omega) \). Our method can be applied to prove that \( D_r^+(\Omega) \) is dense in other Banach spaces. For \( 1 \leq p < \infty \), \( 0 < r < R \), \( r, R \) integers, let us define

\[
W^p_{r,R}(\Omega) := W^{r,p}_o(\Omega) \cap W^{r,p}(\Omega) \text{ with the norm } \| \cdot \|_{W^{r,p}(\Omega)}.
\]

**THEOREM 1'.** If \( \Omega \) is a bounded domain with \( C^\infty \) boundary, then \( D_r(\Omega) \) is dense in \( W^p_{r,R}(\Omega) \).

This theorem reduces to prove

**THEOREM 2'.** \( \{ u \in D_r^+(R_n^+) : \text{supp } u \text{ bounded} \} \) is dense in \( W^p_{r,R}(R_n^+) \).

The proof follows the same lines as that of Theorem 2 noticing that the operator \( E_f \) defined by (6) is continuous from \( W^p_{r,R}(R_n^+) \) into \( W^{r,p}(R_n) \), and the operator \( E \) of Lemma 3 is continuous in \( W^{r,p}(R_n) \). Lemma 2 should be replaced by

**LEMMA 2'.** If \( u \in C^r(\bar{\Omega}) \) and \( D^q u = 0 \) on \( \partial \Omega \) for \( |\alpha| < r \), then \( u \in W^{r,p}_o(\Omega) \).

**THEOREM 5.** Let \( r \) be a positive integer and \( R \) a nonnegative one. The completion of \( D_r(\bar{\Omega}) \) in the norm \( \| \cdot \|_{W^{r,p}(\Omega)} \) is isomorphic
to the space \( W^p_0(\Omega) \) if \( R \leq r \) and isomorphic to \( W^p_{r,R}(\Omega) \) if \( R > r \).

**Proof.** In fact, for \( R \leq r \), because of Lemma 2', we have

\[
C^\infty(\Omega) \subset D^r_\Omega(\Omega) \subset D_\Omega(\Omega) \subset W^p_0(\Omega).
\]

If \( R > r \), it follows from Theorem 1' that \( W^p_{r,R} \supset W^p_0(\Omega) \). To prove that the inclusion is proper consider the function

\[
k(x) = x_n \phi(x') \psi(x_n) \text{ restricted to } \mathbb{R}^n_+ \text{ where } \phi(x') \in C^\infty(\mathbb{R}^{n-1}), \\
\psi \in C^\infty(\mathbb{R}_1), \phi \text{ and } \psi \text{ equal to one in a neighborhood of zero}.
\]

Then, \( k \) is of bounded support and belongs to \( W^p_0(\mathbb{R}^n_+) \cap W^p_0(\mathbb{R}^n_+) \). If \( k \) belonged to \( W^p_0(\mathbb{R}^n_+) \) then \( k \) should belong to \( W^p(\mathbb{R}^n_+) \). However, \( D^{r+1}_x k \) is not a function, Q.E.D.

By the same argument one gets, for \( r < S < R \), the proper inclusions

\[
(21) \quad W^p_0 \supset W^p_{r,S} \supset W^p_{r,R} \supset W^p_0.
\]

It also holds, since \( \Omega \) is bounded, that the norm

\[
(22) \quad \left( \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \|Du_\alpha; L^p(\Omega)\|^p \right)^{1/p}
\]

is equivalent to the original norm in \( W^p_{r,R}(\Omega) \), (cf. [A], p. 158).

8. **COMMENTS.** The construction of the extension operator (6), \( E_f \), with \( f \) as in paragraph 3, is similar to the one used by Seeley in [Se] however corresponding to entire functions of different nature. In order that \( E_f \) extends \( C^\infty(\mathbb{R}^n_+) \) to \( C^\infty(\mathbb{R}^n) \), Seeley needs \( f(2^h) = (-1)^h \) for \( h = 0, 1, \ldots \) and this is not true for our \( f \) since we have \( f(2^h) = (-1)^{h+1} \) for \( h = 0, \ldots, r-1 \) (on the other hand the coefficients \( a_k \) found by Seeley define
an entire function of exponential type with zeroes and in that case \( g = 1/f \) is not entire). This explains the main difference between our extension operator and that of Seeley and other extension operators, for example, the altogether different one constructed by A.P. Calderón ([C], p.45). It consists in the fact that for the extension \( E_f \) the functions \( D^\alpha E_f u, |\alpha| < r, \) can be discontinuous at the boundary except in the case when they vanish there, and therefore \( E_f \) does not define a continuous operator from \( W^{r,p}(\Omega) \) into \( W^{r,p}(R^*_n) \) (but it does when restricted to \( W^{r,p}(R^*_n, \Omega) \), (cf.Th.4)).

REFERENCES


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