A regular quadratic form $\psi$ over a field $K$ of characteristic $\neq 2$ is called round if either it is hyperbolic or it is anisotropic, satisfying the following similarity conditions: for any $x \in K$ represented by $\psi$, the isometry $<x> \psi \cong \psi$, holds.

We shall study in this Note, round forms over a linked field mainly with $u$-invariant $u(K) \leq 4$. If $\dim \psi = 2v \ell$, $v \geq 2$, $\ell$ odd, this study was made by M. Marshall [M]. We here complete it to forms of dimensions $2\ell$, $\ell$ odd. We rely on results in [M].

1. PRELIMINARIES.

$K$ will denote a field of characteristic $\neq 2$. Quadratic forms over $K$ will be regular (i.e. non-degenerate) and written in diagonal form $<a_1, \ldots, a_n>$, $a_i \in K$. If $\psi$ and $\varphi$ are quadratic forms, $\psi \perp \varphi$ denotes orthogonal sum and $\psi \otimes \varphi$, tensor product. If $\psi$ is a quadratic form, with $D(\psi)$ we denote the set of all elements of $K$ represented by $\psi$ and $\tilde{D}(\psi) := D(\psi) \setminus \{0\}$.

For $a_1, \ldots, a_n \in K$, the $n$-fold Pfister form $<1, a_1> <1, a_2> \ldots <1, a_n>$ will be denoted by $<<a_1, a_2, \ldots, a_n>>$. A non-empty subset $T$ of $K$ will be called a preordering if it is closed respect to sums and products, i.e. if $T + T \subseteq T$ and $T \cdot T \subseteq T$. 
A (quadratic) form $\psi$ over $K$ will be called round if either,
i) $\psi$ is hyperbolic or ii) $\psi$ is anisotropic and for all $x \in D(\psi)$, $x \neq 0$, $\langle x \rangle . \psi \simeq \psi$, i.e. the similarity factors of $\psi$ coincide with $D(\psi)$.

A field $K$ is called linked (or, a linked field) if the classes of quaternion algebras over $K$ form a subgroup in the Brauer group of $K$. We shall use results on linked fields contained in $[G_1]$ or $[G_2]$.

The $u$-invariant of a field $K$ is by definition: $u(K) = \max\{\dim q\}$ where $q$ runs over the anisotropic torsion forms over $K$. If $K$ is a linked field then it is well-known that $u(K) \in \{0,1,2,4,8\}$.

With $W(K)$ we shall denote the Witt ring of $K$, consisting of the Witt classes of all quadratic forms.

Next, we recall some basic results that will be needed in this paper. Both are from [M]. We give a proof of Proposition 1 which avoids the use of the Hasse-invariant.

1.1. Proposition ([M], Prop. 1.1 (ii)). Let $\psi$ be a round form of dimension $2\ell$, $\ell$ odd. Then

$$D(\psi) \subset D(\langle 1, \det \psi \rangle).$$

Proof. Let $a \in D(\psi)$, so $\langle a \rangle . \psi \simeq \psi$, and hence $\langle 1, -a \rangle . \varphi = 0$ in the ring of $K$. This means that $\varphi \in \text{Ann}(\langle 1, -a \rangle) =: \text{annihilator ideal in } W(K)$ of $\langle 1, -a \rangle$. Now it is well known (see [EL], Cor. 2.3) that we have an isometry

$$\psi \simeq \beta_1 \ldots \beta_\ell,$$

where $\beta_i = \langle c_i \rangle . \langle 1, -b_i \rangle$, $b_i \in D(\langle 1, -a \rangle)$. Therefore $-\det \psi = b_1 \ldots b_\ell \in D(\langle 1, -a \rangle)$.

Thus $\langle 1, -a \rangle = \langle -\det \psi, \det \psi . a \rangle$
or $\langle 1, \det \psi \rangle = \langle a, \det \psi . a \rangle$,

and so $a \in D(\langle 1, \det \psi \rangle)$. 
1.2 PROPOSITION ([M], Prop.2.7). Let $K$ be a linked field with $u(K) \leq 4$. Let $\psi$ be a round form over $K$ of dimension $2^\ell \ell$, $\ell$ odd. Then

\begin{enumerate}
  \item If $\nu = 2$, there exists a unique $\nu$-fold Pfister form $\psi_0$ defined over $K$ such that $\psi = \ell \times (\det \psi) \psi_0') (\psi_0 \approx <1> \perp \psi_0')$.
  \item If $\nu > 3$, there exists a unique $\nu$-fold Pfister form $\psi_0$ and a unique universal 2-fold Pfister form $\rho$ defined over $K$ such that $\psi \approx \ell \times (\det \psi) \psi_1')$ where $\psi_1$ is defined by $\psi_0 \rho \approx \psi \perp \rho H$. (H denotes a hyperbolic plane).
\end{enumerate}

2. ROUND FORMS OVER LINKED FIELDS.

2.1. PROPOSITION. Let $K$ be a linked field and $\psi$ a round form of dimension $2\ell$, $\ell > 1$, odd. Then

\begin{enumerate}
  \item $\psi \approx l <x_i, \varphi_i > l <1, \det \psi>$ with $\varphi_i$, 2-fold Pfister forms, $x_i \in K$;
  \item $q := l <x_i, \varphi_i >$ is a round form and $D(\psi) = D(q)$;
  \item $D(\psi) = D(<1, \det \psi>) = D(q)$;
  \item $D(\psi)$ is a preordering.
\end{enumerate}

Proof. If $\ell = 1$ then $\psi = <1, a>$, and so we can assume $\ell > 1$.

i) Being $K$ a linked field we can write (see [G.1])

\[ \psi \approx l <y_i, \varphi_i > l <a, b> \]

with $\varphi_i$, 2-fold Pfister forms.

Clearly, $\det \psi = a^b$. If we multiply $\psi$ by $<a>$ we get

\[ \psi \approx <a> \psi \approx l <x_i, \varphi_i > l <1, \det \psi>. \]

ii) From Prop.1.1 we have $D(\psi) \subset D(<1, \det \psi>)$ and then from i) it is clear that

\[ D(\psi) = D(<1, \det \psi>). \]

iii) is consequence of i) and ii).

iv) Let $x, y \in D(\psi)$. Then $x \in D(l <x_i, \varphi_i>)$ and $y \in D(<1, \det \psi>)$.
and so \( x + y \in D(\psi) \). For the product \( x \cdot y \), it is clear that \( x \cdot y \in D(\psi) \).

2.2. PROPOSITION. Let \( K \) be a linked field with \( u(K) \leq 4 \) and let \( \psi \) be a round form of dimension \( 2\ell, \ell = 2k+1, k > 0 \).

1) Assume \( k \) odd. Then, there exists a unique 2-fold Pfister form \( \langle a, b \rangle \) such that
   i) \( \psi = k \langle a, b \rangle \ll \langle 1, \det \psi \rangle \)
   ii) \( D(\langle 1, \det \psi \rangle) = D(\langle a, b \rangle) \)
   iii) \( D(\langle a, b \rangle) \) is a preordering.

2) Assume \( k = 2^r h, r > 1, h \) odd. Then, there exists a unique \((r+1)\)-fold Pfister form \( \psi_0 \) and a unique universal 2-fold Pfister form \( \varrho \) such that
   i) \( \psi = h \psi_1 \ll \langle 1, \det \psi \rangle \) where \( \psi_1 \) is a round form defined by \( \psi_0 \varrho \varrho = \psi_1 \ll 2H. \)
   ii) \( D(\psi_1) \) is a preordering.
   iii) \( D(\psi_1) = D(\langle 1, \det \psi \rangle) \).

Proof. 1) By using Prop. 2.1 i) we can write
   \[ \psi = \ll x_1 \varphi_1 \ll \langle 1, \det \psi \rangle \]
   with: \( \varphi_1 \), 2-fold Pfister form, and \( q := \ll x_1 \varphi_1 \) a round form of dimension \( 4k, k \) odd.

By applying [M], 2.7 (i) it follows the existence of a unique 2-fold Pfister form \( \psi_0 = \langle a, b \rangle \) such that
   \[ q = k \langle a, b \rangle. \]

Therefore
   \[ \psi = k\psi_0 \ll \langle 1, \det \psi \rangle. \]

If \( k = 1 \), then \( D(\langle a, b \rangle) = D(\langle 1, \det \psi \rangle) \) and we know that \( D(\langle 1, \det \psi \rangle) \) is a preordering. Assume then \( k > 1 \). That \( k \langle a, b \rangle \) is a round form implies, by using [M] 1.7, that \( D(\langle a, b \rangle) \) is a preordering. So \( D(\langle a, b \rangle) = D(k \langle a, b \rangle) = D(\langle 1, \det \psi \rangle) \).
2) Assume \( k = 2^r \cdot h \), \( r > 1 \), \( h \) odd. With the notation of Prop. 2.1
i) we have that \( q = 1 < x_1 > \varphi_1 \) is a round form of dimension \( 2^{r+1} \cdot h \), \( h \) odd. It follows from [M] 2.7, the existence of a unique \( \nu+1 \)-fold Pfister form \( \psi_0 \) and a unique universal 2-fold Pfister form \( \rho \) such that

\[
\rho \sim h. \varphi_1
\]

where \( \varphi_1 \) is defined by

\[
\varphi_0 \perp \rho \sim \varphi_1 \perp 2H.
\]

Therefore

\[
\psi \sim h\varphi_1 \perp 1, \det \psi >.
\]

By Prop. 2.1 we have that

\[
D(h. \varphi_1) = D(<1, \det \psi>).
\]

If \( h = 1 \), then \( \varphi_1 \) is round and \( D(\varphi_1) \) is a preordering. If \( h > 1 \), then by [M], 1.7 it follows that \( D(\varphi_1) \) is a preordering and \( \varphi_1 \) is round.

2.3. PROPOSITION. Let \( K \) be any field and let \( \psi \) be an anisotropic form over \( K \), and \( \varphi_1 \) a round form. Then if \( \psi \) can be written as

\[
\psi \sim k. \varphi_1 \perp 1, \det \psi >
\]

with \( k \in \mathbb{N} \), \( k \) odd, and

\[
D(\varphi_1) = D(<1, \det \psi>) \quad \text{a preordering,}
\]

then \( \psi \) is a round form.

Proof. Since \( D(\varphi_1) \) is a preordering \( \neq K \), it follows from [M], 1.7 that \( k\varphi_1 \) is a round form if \( k > 1 \). If \( k = 1 \), the same is clearly true. Let \( x \in D(\psi) \), write

\[
x = x_1 + x_2,
\]

\[
x_1 \in D(k\varphi_1), \quad x_2 \in D(<1, \det \psi>).
\]

Then, \( x_1 + x_2 \in D(<1, \det \psi>) = D(\varphi_1) \). Therefore

\[
<x_1 + x_2>: \varphi_1 \sim \psi, \quad (x_1 + x_2).k\varphi_1 \sim k\varphi_1 \quad \text{and}
\]
\[<x_1 + x_2>, <1, \det \psi > = <1, \det \psi >.\]

Consequently
\[<x>, \psi = <x_1 + x_2>, \psi = \psi,\]

and \(\psi\) is round.

2.4. REMARK.

If \(K\) is a global field and \(\psi\) is an anisotropic round form over \(K\) then \(\dim K \equiv 0 \pmod{4}\) and \(\det \psi = 1\) (see [HJ]). In fact, since \(D(<1, \det \psi>)\) is a preordering, it follows that \(<1, \det \psi>\) represents all sum of squares. Therefore, for every discrete prime \(p\) in \(K\) we have that \(<1, \det \psi>_p\) is universal in the completion \(K_p\) of \(K\). Now, according to [OM], 63:15 (ii) if \(\varphi\) is a two-dimensional anisotropic form over a local field \(K\) and if \(\varphi\) represents 1, then \(D(\varphi)\) is a subgroup of \(K\) of index 2. Therefore our form \(<1, \det \psi>\) is isotropic for all but a finite number of spots (the real ones). Equivalently \(-\det \psi\) is a square in all, but a finite number of \(K_p\). By [OM], 65:15, we conclude that \(-\det \psi\) is a square in \(K\), i.e. the form \(<1, \det \psi>\) is isotropic. This is a contradiction, since it is a subform of an anisotropic form.

After this paper was finished I received a preprint on Round Quadratic Forms by Burkhard Alpers (University of Saskatoon, Canada).
REFERENCES


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