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EXTRINSIC k-SYMMETRIC SUBMANIFOLDS ARE TIGHT

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INTRODUCTION

In [2] D. Ferus introduced the notion of *extrinsic symmetric submanifold* of an euclidean space obtaining, at the same time, a classification of this important family of submanifolds. On the other hand in [3] he gave a proof, independent of the classification, of the fact that these submanifolds of an euclidean space are tight. In [5] the notion of extrinsic k-symmetric submanifold of R^N was introduced generalizing Ferus' definition to the case of the so called *regular s-manifolds of order* k (see [4]). [5] contains the classification of these submanifolds *for the case of odd* k as well as a proof of their tightness. That proof depends strongly on the nature of the order k and does not extend to the case of even order. It should be pointed out that the classification of these subma-

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nifolds for k even is still an open problem although some partial results have been obtained.

In this paper we give a proof of the tightness of these submanifolds for k even, complementing the results in [5]. The proof that we give here is a generalization of the one given by Ferus in [3]. But we must point out that this generalization is not straightforward since the case $k > 2$ requires special methods in several points (see for instance the proof of 0-tightness (2.1)). This fact, besides the interest of the result itself is, in our opinion, what justifies the present paper.

The content of the paper is the following. In the next section we recall the definition of extrinsic k-symmetric submanifold and prove that the imbedding is 0-tight while in section 3 we complete the proof of the tightness.

SECTION 2

In [5] the following definition is given. Let M^n be a compact connected Riemannian manifold and let $i:M^n \hookrightarrow R^{n+q}$ be an isometric imbedding which has the following properties.

i) For each $p \in M$ there is an isometry $\sigma_p : R^{n+q} \to R^{n+q}$ such that $\sigma_p^k = id$, $\sigma_p(P) = P$, $\sigma_p(M_p^\perp)$ = identity on $M_p^\perp$ ($M_p^\perp$ denotes the normal space at $p \in M$).

ii) $\sigma_p(M) \subset M$.

iii) Let $\theta_p = (\sigma_p|M)$. The collection $\{\theta_p : p \in M\}$ defines on M a Riemannian regular s-structure of order k ([4]p.4-6) i.e. for every pair of points $x,y \in M$ $\theta_y \circ \theta_x = \theta_z \circ \theta_y$ where $z = \theta_y(x)$.

If conditions (i), (ii) and (iii) are satisfied by our imbedding, we say that M^n is an *extrinsic* k-*symmetric submanifold of* R^{n+q}.

The objective of this section is to prove the following

(2.1) THEOREM. *The imbedding* i: $M^n \to R^{n+q}$ *is* 0-*tight.*

Proof. Let V denote the space R^{n+q} and $S(V) = \{v \in V: \|v\| = 1\}$ be the unit sphere in V. For each $v \in S(V)$ let δ_v be the height function in the direction of v. For almost every $v \in S(V)$ the function δ_v is "*stable*" i.e. *it is a Morse function with only one critical point at each critical level.*

For each critical point p of δ_v, the isometry σ_p leaves δ_v invariant i.e.

$$\delta_v(\sigma_p(x)) = \langle v,\sigma_p(x)\rangle = \langle\sigma_p(v),\sigma_p(x)\rangle = \delta_v(x)$$

since $v \in M_p^{\perp}$.

Since δ_v is stable, we see that each one of its critical points must be left fixed by σ_p (p a critical point of δ_v). In fact, if q is a critical point of δ_v and $s = \sigma_p(q)$

$d\,\delta_v|_s\,\sigma_{p\,*}|_q X = d(\delta_v \circ \sigma_p)|_q\, X = d\delta_v|_q\, X = 0$ so s is also a critical point of δ_v and since $\delta_v(s) = \delta_v(q)$ then s=q.

In order to prove (2.1) we need to show that δ_v has only one critical point of index zero. To that end we shall see that by assuming that δ_v has more than one critical point of index zero, we reach a contradiction.

Let p_1 and p_2 be the *first* two critical points of index zero of δ_v that we find by moving in the direction of v. Since M is connected we must have a critical point p of index one connecting p_1 and p_2. Clearly the level r_0 of p_0 must be higher than the levels r_1 and r_2 of each one of the points p_1 and p_2. For $r_0 > r > \mathrm{Max}\{r_1,r_2\}$, in $M^r = \{x \in M: \delta_v(x) \leqslant r\}$, we have at least two connected components namely $(M^r)_{p_1}$ and $(M^r)_{p_2}$ which became one at the level r_0.

Let $(U,x_1,\ldots,x_n)$ be a Morse coordinate system for δ_v around p_0. Then we may write

$$\delta_v(x_1,\ldots,x_n) = r_0 - x_1^2 + x_2^2 + \ldots + x_n^2.$$

Let $\beta(t) = (t,0,\dots,0)$ the first coordinate curve in $(U,x_1,\dots,x_n)$. To fix our ideas we can say that $\beta(t) \in$ $\in (M^{r_0} - \{p_0\})_{p_1}$ (component of p_1) for $t > 0$. Let now H be an open set such that $p_0 \in H \subset U$ and $\sigma^j_{p_0}(H) \subset U$ for each $j=1,\dots,k$. For $t > 0$ the curve $\beta(t)$ is *descending* i.e. $t < t' \Rightarrow$ $\Rightarrow \delta_v(\beta(t)) > \delta_v(\beta(t'))$. Let $\varepsilon > 0$ be a real number such that $t \in (0,\varepsilon) \Rightarrow \beta(t) \in H$. Then, on $(0,\varepsilon)$, the curve $\beta(t)$ is *descending* and because of the invariance of δ_v all the curves $\sigma^j_{p_0}(\beta(t))$, $j = 1,\dots,k$, are descending.

Let $W_i = U \cap [M^{r_0}-\{p_0\}]_{p_i}$ $i = 1,2$ (component of p_i).

Then

$$W_1 = \{(x_1,\dots,x_n) \in U: x_1^2 > \sum_{j=2}^{n} x_i^2 \ , \ x_1 > 0\}$$

$$W_2 = \{(x_1,\dots,x_n) \in U: x_1^2 > \sum_{j=2}^{n} x_i^2 \ , \ x_1 < 0\}.$$

Let us consider now the sets $Q_i = H \cap W_i$ $i=1,2$. They satisfy $\sigma^j_{p_0}(Q_i) \subset W_i$ $i=1,2$; $j=1,\dots,k$ because $\sigma^j_{p_0}(H) \subset U$, for each j, by definition of H and $\sigma^j_{p_0}([M^{r_0}-\{p_0\}]_{p_i}) \subset [M^{r_0}-\{p_0\}]_{p_i}$ $i=1,2$ $\forall j$. (This is true because $[M^{r_0}-\{p_0\}]_{p_i}$ must be invariant by $\sigma^j_{p_0}$, due to the invariance of δ_v and to the fact that p_i is fixed by $\sigma^j_{p_0}$). Then, since by the definition of $\varepsilon > 0$ $\beta(t) \in Q_1$ for $t \in (0,\varepsilon)$, we can conclude that

The curves $\sigma^j_{p_0}(\beta(t))$ *, for* $t \in [0,\varepsilon]$*, all lie in* $W_1 \cup \{p_0\}$.

Let us study now the curves $\sigma^j_{p_0}(\beta(t))$ in coordinates. For $j = 1,\dots,k$ and for $t \in [0,\varepsilon)$ write $\sigma^j_{p_0}(\beta(t)) = (y_{1j}(t),\dots,y_{nj}(t))$.

We clearly have

$$\delta_v(\beta(t)) = r_0 - t^2$$

$$\delta_v(\sigma^j_{p_0}(\beta(t))) = r_0 - (y_{1j}(t))^2 + \sum_{i=2}^{n}(y_{ij}(t))^2.$$

Now, due to the invariance of δ_v , we get

$$(y_{1j}(t))^2 - t^2 = \sum_{i=2}^{n}(y_{ij}(t))^2 = R$$

and then, since $y_{1j}(t) \geqslant 0$ for $t \in (0,\varepsilon)$ (The curves $\sigma^j_{p_0}(\beta(t))$ lie in $W_1 \dot{\cup} \{p_0\}$), we have

$$y_{1j} - t = \frac{R}{y_{1t} + t} \geqslant 0 \text{ for } t \in (0,\varepsilon).$$

Therefore,

$$\frac{y_{1j}}{t} - 1 \geqslant 0 \quad \text{for} \quad t \in (0,\varepsilon).$$

Let us take now limit for $t \to 0^+$ $(t \to 0,\ t > 0)$ then we get

$$\lim_{t\to 0^+}\frac{y_{1j}}{t} \geqslant 1 \quad \text{i.e.} \quad \dot{y}^+_{1j}(0) \geqslant 1.$$

Since the curves $\sigma^j_{p_0}(\beta(t))$ are differentiable we see that the *right hand side* derivatives $\dot{y}^+_{1t}(0)$ that we have computed are in fact the derivatives and then we have

$$(2.2) \qquad \dot{y}_{1j}(0) \geqslant 1 \quad j = 1,\dots,k.$$

Now $\sigma^j_{p_0}(\dot{\beta}(0)) = (\dot{y}_{1j}(0),\dots,\dot{y}_{nj}(0))$ and since $\sum_{j=1}^{k} \sigma^j_{p_0}(\dot{\beta}(0)) = 0$ we clearly have

$$0 = \sum_{j=1}^{k} \dot{y}_{1j}(0) \geqslant k \quad \text{by} \quad (2.2)$$

which is a contradiction.

This contradiction originated in our assumption of the exist_ ence of more than one critical point of index 0 for our δ_v. This proves (2.1).

SECTION 3

In this section we prove the following

(3.1) THEOREM. *The imbedding* $M^n \to R^{n+q}$ *is tight.*

Proof. As we indicated in the introduction, if k is odd this fact was proved in [5] then we may assume that k=2s.

In order to give a proof of (3.1) we proceed by induction on the dimension n of the manifold. If dim M = 0 then M is just a point and the theorem is trivial in this case. Let us assume that the result is true for every extrinsic k-symmetric submanifold of dimension < n in an euclidean space.

Let now $M^n \to R^{n+q} = V$ be an extrinsic 2s-symmetric submanifold of dimension n. Take $v \in S(V)$ such that δ_v is "*stable*" and let a be a critical point of δ_v.

Let $N = F(\sigma_a^s, M)$ be the fixed point set of σ_a^s in M. Let $N_1, \ldots, N_n$ be the connected components of N. It is well known that each N_i is a closed totally geodesic submanifold of M and all of them are contained in the subspace $W = F(\sigma_a^s, V)$. We prove now the following

(3.2) LEMMA. *Each one of the components of* N *is an extrinsic* 2s-*symmetric submanifold of* W.

Proof. Let N_i be one of the components of N and let b be a point in N_i. It is easy to see that $\sigma_b(W) = W$ and $\sigma_b(N_i) = N_i$. Now let $Nor_b(N_i, W)$ be the normal space of the submanifold N_i in W at the point b. Then we have

$$\mathrm{Nor}_b(N_i,W) = M_b^{\perp} \cap W. \tag{3.3}$$

In fact, it is clear that

$$T_bN_i = \{X \in M_b : \sigma^s_{a*}|_b X = X\} = M_b \cap W$$

and if we call Q the orthogonal complement of T_bN_i in M_b then for $Y \in Q$ we have $\sigma^s_{a*}|_b Y = -Y$. Therefore W is contained in the orthogonal complement of Q in V which is $T_bN_i \oplus M_b^{\perp}$. This implies (3.3).

With these observations if we put $\beta_b = \sigma_b|W$ for each $b \in N_i$ it is easy to see that $N_i \subset W$ and the isometries $\{\beta_b : b \in N_i\}$ satisfy the conditions (i),(ii) and (iii) of the definition of extrinsic 2s-manifold.

REMARK. It is not hard to see that, in fact, the component of the critical point a in N is extrinsic s-symmetric in W but we do not need this fact here.

Since the function δ_v is stable and a is a critical point we see, as in the proof of (2.1), that the critical points of δ_v are all contained in N.

Let U be the gradient field of δ_v in M.
i.e. $\langle U_q, Y\rangle = d\delta_v|_q Y \quad \forall\, q \in M,\ \forall\, Y \in M_q$. The field U satisfies $\sigma^s_{p*}|_q U_q = U_{\sigma^s_p(q)}$ for each p critical point of δ_v because $d\delta_v|_{\sigma^s_p(q)}(\sigma^s_{p*}|_q(Y)) = d(\delta_v \circ \sigma^s_p)|_q Y = d\delta_v|_q Y$.

Then if p=a and $q \in N$ we get $\sigma^s_{a*}|_q U_q = U_q$ and therefore $(U|N)$ *is a tangent field on* N. This means that the restriction of δ_v to $N(\delta_v|N)$ could only have a critical point $q \in N$ if $(U|N)_q = U_q = 0$ and therefore *the critical points of* $(\delta_v|N)$ *are the critical points of* δ_v *on* M. We know, by our choice of δ_v, that these critical points are non-degenerate in M but of course we must check that they are non-degenerate on N.

To that end we study the Hessian of $(\delta_v|N)$ at a critical point q of $(\delta_v|N)$. We want to show that $\mathrm{Hess}(\delta_v|N)|_q$ is non-degenerate in $T_qN \times T_qN$. Let γ be the second fundamental form of N in W. It is well known that for the height function $\delta_v|N$

$$\mathrm{Hess}(\delta_v|N)|_q(X,Y) = <\gamma_q(X,Y),v>.$$

If we call α the second fundamental form of M in V, ω the second fundamental form of N in M and ε the one for N in V we have $\varepsilon = (\alpha|N) + \omega$. But, since N is totally geodesic in M, we have $\omega=0$ and so $\varepsilon = (\alpha|N)$. Now we have, for $X,Y \in T_qN$,

$$\sigma_a^s(\varepsilon(X,Y)) = \varepsilon(\sigma_a^s X, \sigma_a^s Y) = \varepsilon(X,Y)$$

and therefore $\varepsilon(X,Y) = \gamma(X,Y)$ i.e.

$$(3.4) \qquad \gamma_q(X,Y) = \alpha_q(X,Y) \quad \forall\ X,Y \in T_qN .$$

If the critical point q of $(\delta_v|N)$ were a degenerate critical point then $\exists\ X \in T_qN$ such that $<\gamma_q(X,Y),v> = 0 \quad \forall\ Y \in T_qN$ and by (3.4) $<\alpha_q(X,Y),v> = 0 \quad \forall\ Y \in T_qN$.

Let Q, as before, denote the orthogonal complement of T_qN in M_q and take $Z \in Q$. Then

$$<\alpha_q(X,Z),v> = <\sigma_a^s(\alpha_q(X,Z)),\sigma_q^s v> =$$
$$= <\alpha_q(X,-Z),v>.$$

This clearly means that $<\alpha_q(X,Z),v> = 0 \ \forall\ Z \in Q$ and therefore we get that if $\mathrm{Hess}(\delta_v|N)|_q$ is degenerate then $\mathrm{Hess}(\delta_v)|_q$ is degenerate. It follows that $(\delta_v|N)$ *is a Morse function*. Furthermore, if we denote by $\beta_j(\delta_v)$ the number of critical points of index j of δ_v in M and $\beta_j(\delta_v|N) = \sum_{i=1}^{n} \beta_j(\delta_v|N_i)$ is the total number of critical points of index j of $(\delta_v|N)$ then

(3.5) $$\sum_{j\geq 0} \beta_j(\delta_v) = \sum_{j\geq 0} \sum_{i=1}^{n} \beta_j(\delta_v|N_i).$$

Now from (3.2) and the inductive hypothesis it follows, since $\dim N_i < \dim M$, that

(3.6) $$\sum_j \beta_j(\delta_v|N) = \sum_j \sum_{i=1}^{n} b_j(N_i,Z_2)$$

where $b_j(N_i,Z_2)$ is the j-th Betti number of N_i with Z_2 coeficients.

Now, since $N = F(\sigma_a^s,M)$ and $(\sigma_a^s)^2 = id_M$, we have that N is the set of fixed points in M of an action of the group Z_2. The homological structure of the set of fixed points of a Z_2-action is related to the homology of the original manifold M via the *Smith special homology groups* [1,p.123] and one obtain from [1,p.126, Th.4.1].

(3.7) $$\sum_{k\geq 0} b_k(N,Z_2) \leqslant \sum_{j\geq 0} b_j(M,Z_2).$$

Now from (3.5), (3.6) and (3.7) it follows that

(3.8) $$\sum_{k\geq 0} b_k(M,Z_2) \geqslant \sum_{\ell} \beta_\ell(\delta_v)$$

and since the opposite inequality is given by Morse's inequalities we see that (3.8) is in fact an equality and then Theorem (3.1) is proved.

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