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ON LINKED FIELDS

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Introduction

In these notes we present an exposition of results in the algebraic theory of quadratic forms in the context of the so called *linked fields*. There are many properties that make this kind of fields a very interesting object to deal with. The notion and the name of linked fields is due to R. Elman and T.Y. Lam [EL 3] but the idea of linkage was known to algebraists of the thirties, for instance A.A. Albert [A 1] and E. Witt [W]. To start with, we recall the notion of a quaternion algebra. Given $a, b \in K$. where K denotes a field of characteristic $\neq 2$ and $K^{\cdot} := K \setminus \{0\}$, we associate to this pair the four dimensional K-algebra with basis:
1 , i , j , k , and multiplication table

.	1	i	j	k
1	1	i	j	k
i	i	a .1	k	a .j
j	j	-k	b .1	-b .i
k	k	-a .j	b . i	-a .b .1

We denote it by $(a,b)_K$ or simply (a,b) and this is the quaternion algebra associated to a,b. Clearly (a,b) and (b,a) are K-isomorphic.

0.1. Example: $M_2(K)$: = the algebra of 2 x 2 -matrices over K is a quaternion algebra which can be described by (1,a) for any $a \in K^{\cdot}$

In fact we can choose 1, i , j , k as follows:

$$1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j := \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}$$

This is (up to isomorphism) the only quaternion algebra over K which is not a division algebra.

A quaternion algebra (a,b) over K is a central simple algebra and therefore it determines a class [a,b] in the Brauer group B(K) of K. A natural question to ask is :

When form the totality of classes of quaternion algebras a subgroup of B (K) ?

This is clearly equivalent to ask:

When is, the tensor product of two quaternion algebras, similar to a quaternion algebra?
In symbols, similar means to have an algebra isomorphism

$$(a, b) \otimes (c, d) \cong (e, f) \otimes M_2(K)$$

It is well known that the tensor product of two quaternion algebras (a,b), (c,d) is similar to a quaternion algebra *if* both algebras admit a "*common slot*" i.e. there exist $x, y, z \in K^{\cdot}$ such that $(a, b) \approx (x, y)$ and $(c, d) \cong (x, y)$.

If this is the case we have

$$(a, b) \otimes (c, d) \approx (x, y, z) \otimes M_2(K).$$

That the common slot property is also necessary was proved by Pfister [Pf], later by Elman-Lam [E - L 2] and also in a posthumous paper by A. A. Albert [A 2] by using the theory of algebras.

A field is linked if any pair of quaternion algebras admit representations with a common slot. Or equivalently, when the classes of quaternion algebras in the Brauer group of K form a subgroup.

It follows easily from the classical Wedderburn's theorem that the tensor product of two quaternion algebras is either a division algebra or similar to a quaternion algebra.

Therefore a field K is linked if and only if the tensor product of two quaternion algebras *never* is a division algebra.

Let us say that linked fields abund. In fact, any algebraic extension of the rational field and any of its completions are linked fields. These are facts, of course, of a purely arithmetic nature.

Linked fields are interesting fields to be studied in the context of the Witt ring W(K) of regular quadratic forms over K. Recall that in W(K) we have the distinguished ideal, denoted by IK (or simply I) of all (classes of) even dimensional regular quadratic forms. We also have the powers of this ideal, namely I^nK. Every I^nK is generated as an abelian group by the 2^n-dimensional n-fold Pfister forms

$$<< a_1, \ldots, a_n >> := \bigotimes_{i=1}^{n} < 1, a_i >$$

and give rise to a filtration of W (K) :

$$W(K) \supset IK \supset I^2K \supset \ldots \supset I^nK \supset \ldots$$

Let us mention in passing that the common feeling among specialists in quadratic forms is that the quotients

$$\bar{I}^rK := I^rK / I^{r+1}K$$

should provide the invariants needed to characterize isometry classes of quadratic forms over K. But so far there are not many important results in this direction.

An element $q \in I^nK$ can be represented by a sum "in W (K) "

$$q = < a_1 > . \tau_1 + \ldots + < a_r > . \tau_r$$

where τ_i are n - Pfister forms and $a_i \in K^{\cdot}$.

In a linked field K we have a representation of q which is actually "an isometry":

$$q \cong < a_1 > . \tau_1 \perp \ldots \perp < a_r > . \tau_r$$

(The reader has noticed our abuse of notation in using the same symbol to denote a quadratic form and its class in W (K) , O. K .).

This is the so called "simple decomposition" of q, a fundamental result due to Elman-Lam. In this notes we intend to give another proof of this results which avoids the use of the Arason - Pfister Hauptsatz.

In terms of the quotients $\bar{I}^{n}$ K we can also mention the very useful property of linked fields that any element of $\bar{I}^{n}K$ can be repesented in I^{n} K by an n - fold Pfister form i.e. for any $q \in I^{n}K$ there is an n-fold Pfister form φ such that

$$q \equiv \varphi \ (I^{n+1}K).$$

From this one can prove easily that the quotient group $\bar{I}^{2}$ K is isomorphic to the subgroup of B (K) of all classes represented by quaternion algebras. The word *linked* is due also to Elman - Lam and was used originally to express a sort of linkage of n - fold Pfister forms .

Namely, let for $n \in N$, $n > 1$, φ_1, φ_2 be n - fold Pfister forms.

It is said that φ_1 and φ_2 are *linked* if there exists an (n - 1) - fold Pfister form τ such that

$$\varphi_i = \tau \cdot \tau_i$$

with τ_i, $i := 1, 2$, 1 - fold Pfister forms .

Elman-Lam defined a field to be *linked* if for any $n \in N$, $n > 1$, any pair of n - fold Pfister forms are linked.

In general linked fields behave so well that specialists first check properties in this kind of fields . Some also say that linked fields are the "easy" fields. This is partially justified since two main problems in the algebraic theory of quadratic forms are answered positively for linked fields . Those two problems are the conjecture that the u-invariant is a power of 2 and the famous Pfister problem about whether forms in I^2K of Clifford invariant 1 are elements in I^3K. In 1981 the russian mathematician A. S. Merkujev solved Pfister's conjecture. In fact, Merkurjev proved that the Clifford invariant map $c : I^2K / I^3K \to Br_2(K)$ is an *isomorphism,* which amounts to proving Pfister's conjecture and furthermore the long standing problem about $Br_2(K)$ being generated by the classes of quaternion algebras. In 1988, again Markurjev constructed various non real fields with u - invariant equal 6 and in 1989 proved the existence of fields u - invariant equal 2 n , for all natural n .

However, what can be said about the ring structure of W(K) of a linked field?

This paper is expository in nature and most results are well-known to the people working in the field, and can be found mostly in papers by Elman-Lam and Elman. However, proofs given here are to some extend easier and more revealing of the structure of linked fields.

We assume the reader acquainted with the basic facts on Witt rings as can be found in Lam's book [L] and Lorentz [LO].

1. Quaternion algebras. (See Lam [L], O'Meara [O'M]).

Let $a,b \in K^{\cdot}$. we associate with the pair a,b the 4-dimensional K-algebra (quaternion algebra), q.a. defined by a basis 1 , i , j , k , and multiplication table:

$$i^2 = a, \quad j^2 = b, \quad i.j = -j.i = k$$
$$1.x = x.1 = x, \quad \forall x$$

(As usual we identify a .1 with a, for all $a \in K$.)

It is denoted by,

$(a,b)_K$ or simply (a,b).

This algebra is *central* (i.e. its center is K .1 = K) and simple (i.e. it has only 2 two-sided ideals).

Notice that in general there are many ways to represent $(a,b)_K$. For instance

$$(a, b)_K \cong (a.x^2, b.y^2)_K \text{ for all } x, y \in K^{\cdot}.$$

The matrix algebra $M_2(K)$ is a quaternion algebra represented by (1,x) for any $x \in K^{\cdot}$. We shall see that it is (up to isomorphism) the unique split quaternion algebra, i.e. non-division quaternion algebra.

Let $z \in (a, b)_K$, $z = x_0 + x_1.i + x_2.j + x_3.k$, $x_i \in K$.

Denote with $\bar{z}$ the *conjugate* of z:

$$\bar{z} := x_0 - x_1.i - x_2.j + x_3.k.$$

We have on $(a,b)_K$ the quadratic map, the *norm* of $(a, b)_K$:

$$N(z) := z.\bar{z} = x_0^2 - a.x_1^2 - b.x_2^2 + ab.x_3^2$$

The norm is a 2-Pfister form and as usual we write

$$N_{(a,b)} = N = \langle 1, -a, -b, ab \rangle = \langle\langle -a, -b \rangle\rangle.$$

For instance if $(a,b) = (1,x) \cong M_2(K)$ the norm form is *the hyperbolic form*

$$\langle 1, -1, -x, x \rangle = \langle\langle 1, -1 \rangle\rangle.$$

The most striking and important elementary result about quaternion algebras is that a class of algebraic isomorphism of q.a is uniquely determined by a class of isometry of norm maps. That is

$$\text{(Q)} \qquad (a, b) \cong (c, d) \Leftrightarrow N_{(a,b)} \approx N_{(c,d)}$$

where $\cong$: denotes isomorphism of K-algebras
$\approx$: denotes isometry of quadratic spaces

It is easy to see that the quaternion algebra (a,b) is a division algebra if and only if the quadratic form $N_{(a,b)}$ is anisotropic (i.e. $N(z) = 0$ if and only if $z = 0$). Moreover, a Pfister form is isotropic if and only if it is hyperbolic. It follows from these observations and from (Q) that there is (up to isomorphism) a unique q.a. which is not a division algebra, namely the matrix algebra $M_2(K)$, whose norm form is the hyperbolic 2-fold Pfister form. An important problem is to study the structure of the tensor product of quaternion algebras. Recent work by Amitsur, Tignol, Rowen show very unsuspected results. If we recall a well-known isomorphism

$$(a, b) \otimes (a, c) \cong (a, b.c) \otimes M_2(K)$$

we see that the tensor product of q.a. is easy to determine if both algebras (a,b) and (c,d) admit a "common slot".

That is, there exist

$$x, y, z \in K^{\cdot} \text{ such that } (a, b) \cong (x, y), \ (c, d) \cong (x, z)$$

The existence of a common slot can be better analyzed by means of the quadratic space structure. Let $(a,b)^0$ denote the space of *pure quaternions*, i.e. those quaternions

$$z = x_0 + x_1 .i + x_2 .j + x_3 .k \text{ with } x_0 = 0.$$

Then $$z \in (a, b)^0 \Leftrightarrow z^2 = -N(z).$$

So, if $z \in (a, b)^0$ and $c = N(z) \neq 0$ we have that the quadratic form $< -a, -b, ab >$ represents c, that is $<-a, -b, ab> \approx <c, d, cd>$ for some $d \in K^{\bullet}$. If we choose

$i_0 \in (a, b)^0$ with $-c = -N(i_0) = i_0^2$, $j_0 \in (a, b)^0$ with

$i_0 \perp j_0$, $k_0 = i_0 . j_0$ we get the basis $1, i_0, j_0, k_0$ of (a, b) such that

$(a, b) = (-c, -d)$ if $-d = N(j_0)$.

In conclusion given a q.a. $(a, b)_K$ we have $(a, b)_K \cong (c, d)_K$ if and only if c is a value in $K^{\bullet}$ represented by the "pure" form $< -a, -b, ab >$. The next proposition is clearly a corollary of our dicussion.

1.1. Proposition. *Let* (a,b), (c,d) *be* q.a. *over* K. *Then these admit a representation with a common slot if and only if the quadratic form*

$$<a, b, -ab, -c, -d, cd>$$

is isotropic.

We can digress at this point to recall a classical result by A.A. Albert [A1] where he considers the case of Proposition 1.1. for two quaternion algebras over the *rational field.* His Theorem 1 states that "*by finding a single solution of a solvable diophantine equation we may represent any pair of generalized quaternion division algebras in the canonical form*

a) $$A = (e, i, j, ij), \quad i^2 = ae, \quad j^2 = be, \quad ji = -ij,$$

b) $$B = (E, I, J, IJ), \quad I^2 = aE, \quad J^2 = cE, \quad JI = -IJ,$$

with e and E respectively the module (identities) of A and B, where a,b and c are multiplication constants expressed in terms of the original multiplication constants of A and B and the above solution, and where without loss of generality, a,b,c may be taken to be product of distinct rational primes".

If we assume that originally A and B were given by $A = (a_1, b_1)$, $B = (a_2, b_2)$ the above diophantine equation refers to solving equations of the following sort:

$$a_1 X_1^2 + b_1 X_2^2 - a_1 b_1 X_3^2 - a_2 X_4^2 - b_2 X_5^2 = 0,$$

that is , to find a zero of the quadratic form $< a_1, b_1, -a_1 b_1, -a_2, -b_2 >$.

To solve it he invokes *Meyer's Theorem* : An indefinite regular form f over Q in $n \geq 5$ variables is isotropic, (A. Meyer,(1881), Zur Theorie der indefiniten quadratischen Formen. J. reine angew. Math. 108, 125-139).

1.2 Exercise. Let A and B be quaternion algebras as above a) and b). Prove that A and B are isomorphic if and only if b and c are congruent modulo the norm of the quadratic extension $Q(\sqrt{a})$, i.e.

$c = (x_1^2 - x_2^2 a) \cdot b$, for some $x_1, x_2 \in \mathbf{Q}$.

For instance $(-1,b) \equiv (-1,c)$ if and only if $c = (x_1^2 + x_2^2).b$. In particular a quaternion algebra (a,b) is split if and only if b is a norm $b = N_{\mathbf{Q}(\sqrt{a})/\mathbf{Q}}(x)$ for some $x \in \mathbf{Q}(\sqrt{d})$.

2. Linkage results.

2.1.Definitions.: Let $n \in N$ denote with $Pf_n(K) :=$ totality of n-fold Pfister form over K.

i) $\varphi_1, \varphi_2 \in Pf_n(K)$, $n > 1$ are said to be *linked* if there exist $\tau \in Pf_{n-1}(K)$ and $\sigma_1, \sigma_2 \in Pf_1(K)$ such that $\varphi_i \approx \tau \cdot \sigma_i$, $i = 1, 2$.

ii) K is said to be *n-linked* if any pair of form in $Pf_n(K)$ is linked.

iii) K is said to be a *linked field* if it is n-linked for all n, n > 1.

2.2.Proposition. *The following equivalent conditions hold*

i) K *is* 2 - *linked*

ii) K *is linked*

iii) $\forall n \in N, \forall q \in I^nK, \exists \varphi \in Pf_n(K)$ *such that* $q \equiv \varphi\ (I^{n+1})$.

iv) *Any 6-dimensional form of type* $< a,b,ab,-c,-d,-cd >$ *over K is isotropic.*

v) *The classes of quaternion algebras in* Br (K) *form a subgroup* .

Proof.

i) $\Rightarrow$ ii). Proceed step by step using 2-linkages.

ii) $\Rightarrow$ iii). Write $q \in I^nK$ as $q = <a_1> . \tau_1 + <a_2> . \tau_2 + \ldots + <a_r> . \tau_r$ with $\tau_i \in Pf_n(K)$.

It is enough to prove the case r = 2. Let $\varphi \in Pf_{n-1}(K)$ such that $\tau_1 = \varphi . <<x_1>>$, $\tau_2 = \varphi . <<x_2>>$. Then

$$<a_1> . \tau_1 + <a_2> . \tau_2 = \varphi . (<a_1> . <<x_1>> + <a_2> . <<x_2>>)$$

$$= \varphi . (<a_1> . <<x_1>> + <a_2> . <<x_1>> - <a_2> . <<x_1>> + <a_2> . <<x_2>>$$

$$= \varphi . <<x>> . <a_1, a_2> + \varphi . <a_2> . <x_2, -x_1>$$

$$= <a_1> . \varphi . <<x, a_1 a_2>> + \varphi . <a_2> . <x_2, -x_1>$$

$$\equiv <x_2> . <a_2> . \varphi . <1, -x_1 x_2>\ (I^{n+1}K)$$

$$\equiv \varphi . <1, -x_1 x_2>\ (I^{n+1}K).$$

iii) $\Rightarrow$ iv). Let $q = <a, b, ab, -c, -d, -cd> = <<a, b>>' \perp <-1> . <<c, d>>'$.
$= <<a, b>> - <<c, d>> \in I^2K$
and by iii) $q \equiv <<e, f>>\ (I^3K)$.

Assume $e \neq -1$.
If we apply the Clifford invariant $c : I_2K \rightarrow Br(K)$ we have that

$$(<<a, b>>) . c(<<c, d>> = c(<<e, f>>)$$

that is,

$$[(-a, -b)] . [(-c, -d)] = [(-e, -f)] \text{ in } Br(K).$$

Therefore by taking the quadratic extension $K \hookrightarrow K(\sqrt{-e})$ we have

$$(-a , -b)_{K(\sqrt{-e})} \cong (-c , -d)_{K(\sqrt{-e})}$$

Therefore

$$<< a , b >> \approx << c , d >> \text{ over } K (\sqrt{-e})$$

This is equivalent to saying that the form $q = < a, b, ab, -c , -d, -cd >$ goes to zero in the morphism $W(K) \to W(K (\sqrt{-e}))$. If q were anisotropic we could write $q \approx < 1 , e > . h$, where h is a ternary form. But this is contradictory because by taking determinant we would get

$$-1 = e .$$

If $e = -1$, then $<< a, b >> \equiv << c, d >>$ (mod I^3K) and hence $(-a, -b) \approx (-c, -d)$ therefore $<< a, b >> \approx << c, d >>$ and clearly $< a, b, ab, -c, -d, -cd >$ is hyperbolic.

iv) $\Rightarrow$ v) is clear.

v) $\Rightarrow$ i) uses the same ideas as in iii) $\Rightarrow$ iv).

2.3. Exercises.

1) Prove the equivalence of the following conditions on K.

i) K is linked.

ii) Every 5-dimensional form represents its determinant.

iii) Every 6-dimensional form of discriminant 1 is isotropic.

2) Let $(a,b)_K$, $(c,d)_K$ be quaternion algebras. Prove that if $(a,b)\otimes_K (c,d)$ is not a division algebra, then $(a,b)_k$ and $(c,d)_k$ contain a common quadratic extension of K.

2.4. Definition. *We say that an anisotropic form* $q \in I^nK$ *has a simple decomposition (of length* r) *if there exists an isometry*

$$q \approx < a_1 > . \varphi_1 \perp \ldots \perp < a_r > . \varphi_r$$

where $a_i \in \dot{K}$ and $\varphi_i \in Pf_n (K)$.

Notice that if $q \in I^nK$ has a simple decomposition then

$$2^n \mid \dim q.$$

Let $\varphi_1, \varphi_2 \in Pf_n (K)$ and r be a non-negative integer. We say that φ_1 φ_2 are *r - linked* if there exist, $\tau \in Pf_r (K)$, $\tau_i \in Pf_{n-r} (K)$,$i = 1,2$ such that $\varphi_i = \tau . \tau_i$, $i = 1,2$.

We shall need the following proposition that expresses the linkage property in terms of the Witt index.

2.5. Proposition ([EL4] Prop.4.4). *Let* φ *and* γ *be n-fold Pfister forms, and* r *a non-negative integer. Let* $q = \varphi \perp < -1 > \gamma$. *Then* φ *and* γ *are r-linked if and only if the Witt index of* q *is* $\geq 2^r$

2.6. Theorem. *Let* $n \in N$, $n > 1$. *Then* K *is n-linked if and only if* every $q \in I^nK$ *has a simple decomposition.*

Proof. Before we give the proof of this theorem we prove a useful lemma due to Shapiro-Wadsworth.

2.7.Lemma [SW] . *Let φ be a Pfister form, γ a form such that $\varphi \mid \gamma$. Let $c \in D_K(\gamma)$. Then there exists a form γ_0 such that $\gamma \approx c\,\varphi \perp \gamma_0$ and $\varphi \mid \gamma_0$.*

Proof. Recall that a Pfister form is a round form, that is it is hyperbolic or anisotropic satisfying:

$$a \in D_k(\varphi) \Rightarrow \langle a \rangle . \varphi \approx \varphi .$$

Let $\beta = \langle a_1, \ldots, a_m \rangle$ be such that

$$\gamma = \varphi \otimes \beta = \langle a_1 \rangle . \varphi \perp \langle a_2 \rangle . \varphi \perp \ldots \perp \langle a_m \rangle . \varphi .$$

If $c \in D_k(\gamma)$, there exist $t_i \in D(\varphi) \cup \{0\}, i = 1, \ldots, m$ such that

$$c = a_1 t_1 + \ldots + a_{1m} t_m.$$

Let $\delta = \langle x_1, \ldots, x_m \rangle$ with $x_i = t_i a_i$ if $t_i \neq 0$ and $x_i = 1$ if $t_i = 0$

Then it is clear that

$$\gamma = \varphi \otimes \langle x_1, \ldots, x_m \rangle$$

and moreover $c \in D_k(\delta)$ since

$$c = \sum_{t_i \neq 0} 1^2 . x_i + \sum_{t_i = 0} 0 . x_i$$

Therefore $\delta = \langle c \rangle \perp \delta'$ and $\gamma = \langle c \rangle . \varphi \perp \varphi \otimes \delta'$.

We now return to the proof of the theorem.

Let $q \in I^n K$ be anisotropic and assume that q *has no simple decomposition.* We can choose q with the property that it admits a representation in W(K):

$$q = \langle a_1 \rangle \tau_1 + \ldots + \langle a_r \rangle \tau_r \ , \ \tau_i \in Pf_n(k)$$

with r minimum.

Clearly is $r > 1$. Suppose $r = 2$.

There exist $\varphi \in Pf_{n-1}(K)$ and $a_1, a_2 \in \dot{k}$ such that

$$q = \langle a_1 \rangle \tau_1 + \langle a_2 \rangle \tau_2 \text{ and } \varphi \mid \tau_i, \ i = 1, 2.$$

Since q is anisotropic $\langle a_1 \rangle \tau_1 \perp \langle a_2 \rangle \tau_2$ must be isotropic. Hence

$$0 = a + (-a), \text{ with } a \in D(\langle a_1 \rangle \tau_1) \text{ and } -a \in D(\langle a_2 \rangle \tau_2).$$

By applying Lemma 2.6 we can write

$$q = \langle a \rangle . \varphi \perp \langle x \rangle . \varphi \perp \langle -a \rangle . \varphi \perp \langle y \rangle . \varphi$$

that is,

$$q = \langle x, y\rangle . \varphi = \langle x\rangle \langle\langle x y\rangle\rangle . \varphi$$

But this contradicts the minimality of r. So we are done if $r = 2$.
Assume then $r > 2$. The form $\langle a_1\rangle \tau_1 \perp \ldots \perp \langle a_r\rangle \tau_r$ must be isotropic, so let

$$0 = a_1x_1 + a_2x_2 + \ldots + a_rx_r$$

with $x_i \in D(\tau_i) \cup \{0\}$, $i = 1, \ldots, r$. We can assume that all x_i are $\neq 0$. In fact, let $x_r = 0$, say, then the form

$$\langle a_1\rangle . \tau_1 \perp \ldots \perp \langle a_{r-1}\rangle . \tau_{r-1}$$

has a simple decomposition, but it is isotropic, and not hyperbolic. Therefore its kernel form admits a representation of lower length (in the number of n-fold Pfister forms needed). Consequently q would have a lower length, a contradiction. Let then $\varphi \in Pf_{n-1}(K)$ satisfy: $\varphi \mid \tau_1$, $\varphi \mid \tau_2$. We can then apply Lemma 2.6 to

$$\gamma := \langle a_1\rangle . \tau_1 \perp \langle a_2\rangle . \tau_2 \text{ and } c := a_1 x_1 + a_2x_2 \neq 0 \text{ (recall } r > 2),$$

to have

$$\gamma \approx \langle c\rangle . \varphi \perp \downarrow_0 \text{ and } \varphi \mid \gamma_0 .$$

By counting dimensions we must have

$$\gamma_0 = \varphi \otimes \langle x, y, z\rangle , \text{ for some } x, y, z \in \dot{K}$$

and therefore

$$q = \varphi \otimes \langle x, y\rangle + \varphi \otimes \langle c, z\rangle + \langle a_3\rangle \tau_3 + \ldots + \langle a_r\rangle \tau_r .$$

But the form

$$\varphi \otimes \langle c, z\rangle \perp \langle a_3\rangle \tau_3 \perp \ldots \perp \langle a_r\rangle \cdot \tau_r$$

is isotropic and it has length $< r$. So its kernel has a simple decomposition of lower length. But this implies that q itself has a representation with *less* than r n-fold Pfister forms, a contradiction. This proves the necessity in the theorem. Let us see sufficiency. Let ψ and χ be n-fold Pfister forms. Notice that a consequence of the existence of simple decompositions in I^nK is that any nonzero anisotropic form in I^nK has dimension a multiple of 2^n.

Therefore clearly

$$\ker (\psi' \perp -\chi') \approx \langle c\rangle \cdot \delta,$$

for some $\delta \in Pf_n(K)$, $c \in \dot{K}$. But this implies that $\psi' \perp -\chi'$ is isotropic. Hence ψ' and χ' represent a common value a, say. From this follows that $\langle\langle a\rangle\rangle \mid \psi$ and $\langle\langle a\rangle\rangle \mid \chi$. Let then τ be a Pfister form of the highest dimension among those dividing ψ and χ. If $\dim \tau = 2^{n-1}$ we are clearly done. Assume then that $\dim \tau = 2^r < 2^{n-1}$ and write

$$\psi \approx \tau \perp \rho , \chi \approx \tau \perp \sigma, \quad \tau \mid \rho \quad , \quad \tau \mid \sigma$$

for some forms ρ and σ. We have $\dim \rho > 2^{n-1}$ and $\dim \sigma > 2^{n-1}$ and so

$\dim(\rho \perp -\sigma) > 2^n$. From

$$\rho \perp -\sigma = \psi \perp -\chi = \langle c \rangle \, . \, \delta \, , \quad \delta \in Pf_n(K)$$

it follows that $\rho \perp -\sigma$ D must be isotropic. Choose then $e \in \dot{D}(\rho) \cap \dot{D}(\sigma)$. By Lemma 2.6 we can write

$$\psi \approx \tau \perp \langle e \rangle \cdot \tau \perp \rho' \approx \langle\langle e \rangle\rangle \cdot \tau \perp \rho' \, ,$$

$$\chi \approx \tau \perp \langle e \rangle \cdot \tau \perp \sigma' \approx \langle\langle e \rangle\rangle \cdot \tau \perp \sigma' .$$

Then $\psi \perp \langle -1 \rangle \chi$ has Witt index $\geq \dim \tau + 1 = r + 1$ and so, by Prop. 2.5, ψ and χ have an r+1-linkage, a contradiction. Theorem 2.6 is completely proven. This proof does not use the Arason-Pfister Hauptsatz.

Remark. If K is a so called C_n-field then $I^n K$ is linked. A simple proof of this result is obtained by using the following known result:

Let ψ and ρ be Pfister forms with ρ being a v-fold Pfister form. Assume that $\psi \otimes \rho'$ represents $c \in K$. Then there exists a (v-1)-fold Pfister form τ such that

$$\psi \otimes \rho \approx \psi \otimes \langle 1, c \rangle \otimes \tau .$$

3. A theorem by R. Elman.

In this section we give a simple proof of an interesting theorem by R. Elman which allows to determine for a given field K, when is the field K ((t)) of power series over k, in one undeterminate, a linked field.

3.1. Theorem [E]. F : = K ((t)) is a linked field if and only if every 4-dimensional quadratic form over K with determinant $\neq 1$ is isotropic. (In other words, anisotropic 4-dimensional quadratic forms over K have determinant 1).

Proof. ($\Leftarrow$). Let q be a 6-dimensional form over F with discriminant 1, that is, $\det(q) = -1$.

We can write

$$q \approx q_1 \perp \langle t \rangle \cdot q_2$$

where q_i are forms over K.

Now, $\det q = -1 \Rightarrow \dim q_2$ is even. By symmetry is enough to consider the case:

$\dim q_1 = 4$ and $\dim q_2 = 2$.

Therefore

$$\det q_1 = 1 \Rightarrow \det q_2 = -1 \Rightarrow q_2 \text{ isotropic}$$

or

$$\det q_1 \neq 1 \Rightarrow q_1 \text{ is isotropic by hypothesis.}$$

Therefore K ((t)) is a linked field.

($\Rightarrow$) Let $\langle a_1, a_2, a_3, a_4 \rangle$ be an anisotropic form over K.

The 5 - dimensional form over K ((t)) ,

$$< a_1, a_2, a_3, a_4 , t >$$

represents $\Pi a_i \cdot t$. Write then

$$\begin{aligned} \Pi a_i \cdot t = \; & a_1 \cdot (x_0 + x_1 . t + \ldots)^2 + \\ & a_2 \cdot (y_0 + y_1 . t + \ldots)^2 + \\ & a_3 \cdot (z_0 + z_1 . t + \ldots)^2 + \\ & a_4 \cdot (u_0 + u_1 . t + \ldots)^2 + \\ & t \cdot (l_0 + l_1 . t + \ldots)^2 . \end{aligned}$$

The anisotropy of $< a_1, a_2, a_3, a_4 >$ implies $x_0 = z_0 = u_0 = 0$ therefore the first four summands contribute nothing to the first degree term and so

$$\Pi a_i \cdot t = t \cdot l_0^2$$

which means $\Pi a_i = 1 \pmod{K^{\cdot 2}}$.

Remark 1. $\Rightarrow$ holds for K(t).

Remark 2. We leave as an exercises (or else look in [ELW]) to prove the following expansions of 2.1.

The following statements are all equivalent:

1) K((t)) is linked.
2) Any anisotropic 4-dimensional form over K has determinant 1.
3) Every quadratic extension of K is a splitting field for every quaternion algebra over K.
   Moreover if K is a formally real field the above conditions are equivalent to
4) K is an euclidean field (i.e. formally real and $|K/_K{}^{\cdot 2}| = 2$), and also equivalent to
5) K ((t)) is a ED - field (see [P-W, Th. 2]).

4. A theorem by Jacob-Tignol.

The following beautiful result due to Jacob and Tignol was communicated to me by A.Wadsworth.

4.1. Theorem Let K be a field with valuation v with value group Γ_K and residue field $\bar{K}$. Suppose:

i. $char(\bar{K}) \neq 2$,

ii. K is not quadratically closed,

iii. Γ_K is not 2 - divisible.

Then the rational function field K(t) is not linked.

Proof. Pick a,s $\in$ K' with $v(a) = v(s) = 0$ and $\bar{a} \in \bar{K} \setminus \bar{K}^{\cdot 2}$, and pick $\pi \in K^{\cdot}$ with

$$v(\pi) \notin 2\,\Gamma_K. \quad \text{Let } b = t + s^2.$$

Claim: $<< -a , -t >>$ and $<< -\pi , -b >>$ are not linked.

We must prove that $< -a, -t , a t, \pi , b, -\pi b >$ is anisotropic over K(t).

The valuation v on K has a standard extension, also denoted by v, on F(t) with residue field $\bar{K}(t)$ (t trascendental over $\bar{K}$) and value group $\Gamma_{K(t)} = \Gamma_K$. By passing to the henselization of K(t) respect to v and applying Springer's theorem, it suffices to see that

$< -\bar{a}, -t, \bar{a}t , \bar{b} >$ and $<1, -\bar{b}>$ are isotropic over $\bar{K}(t)$

$< 1, -\bar{b} > = < 1 , -(t + \bar{s}^2) >$ is clearly anisotropic.

We have $<-\bar{a},-t,\bar{a}t,\bar{b}> = <\bar{a},-t,\bar{a}t,t+\bar{a}^2>$. To see this is anisotropic, pass to $\bar{K}((t))$, over which the form becomes $<<-\bar{a},-t>>$, which is anisotropic by Spinger's theorem as $<1,\bar{a}>$ is anisotropic over K.

4.2. Application. This theorem applies to the following fields which consequently, are not linked.

$Q(t)$; $Q_p(t)$, $p \neq 2$; $R(X,Y)$; $C(X,Y,Z)$; $K(X,Y)$, K a finite field of char $(K) \neq 2$.

4.3. Main question about non-linked fields.

If K is a non-linked field find an upper bound on the number n of quaternion algebras D_i over K such that $D_1 \otimes \ldots \otimes D_n$ is a division algebra.

5.Examples

5.1. Linked fields.

i) K, a finite field of characteristic $\neq 2$. There exists, up to isomorphism, a unique quaternion algebra, namely: $M_2(K)$. More generally it can be proved that if K is a non-real field with at most 8 squares classes, then the classes of quaternion algebras over K form a subgroup of Br(K).

ii) K, a p-adic field, that is, a complete field respect to a discrete valuation with finite residue class field. These fields are exactly finite extensions of a field Q_p of p-adic numbers or a finite extension of a completion of a field k (X) of rational functions in 1 indeterminate over a finite field k. In this situation there is, up to isomorphism, a unique non-split quaternion algebra.

iii) K, a global field, that is, a finite extension of **Q** or a finite extension of a field k(X) of rational functions in one indeterminate over a finite field k. The completion of K for the various topologies defined by absolute values in K produce a p-adic field for each ultrametric absolute values or else **R** or **C**.

The celebrated Hasse-Minkowski theorem states that over a global field K, a quadratic form is isotropic if and only if it is so in *all* completions of K.

From this theorem it follows immediately that K is a linked field.

iv) K, a function field over the complex number **C** of transcendence degree ≤ 2. According to Lang-Tsen theorem, every quadratic form of dimension greater than 4 is isotropic. Hence K is a linked field.

v) K, a function field over **R** of transcendence degree ≤ 1.

In particular, **R**(X) is a linked field and so is every algebraic extension of it. In other words, **R**(X) is a hereditarily linked field.

Proof. Let $q = <a, b, ab, -c, -d, -cd>$ with coefficients in $K^{\cdot}$. If K contains **C** then K has transcendence degree ≤ 1 over **C** and then by Lang-Tsen, q is isotropic.

In fact, let then K(i)/K be a proper quadratic extension of K. By the same argument as before q is isotropic over K(i). If q is anisotropic we can write $q \approx <c> \cdot <1,1> \perp <a_1, a_2, a_3, a_4>$ over K with $\Pi a_i = -1$. But by Lang-Tsen theorem $<a_1, a_2, a_3, a_4>$ is isotropic over K(i).

Therefore $<a_1, a_2, a_3, a_4> = d.<1,1> \perp <x, y>$, with $\Pi a_i = -1 = x.y$ and so

$$q = <c><1.1> \perp <d><1,1> \perp <1,-1>,$$

a contradiction. Therefore q is isotropic and K is a linked field.

vi) $K := k((X))$, for the following choices of k:

a) finite field of characteristic $\neq 2$,

b) real euclidean field (i.e. formally real and $|K^{\cdot}/K^{\cdot 2}| = 2$),

c) $C((t_1))((t_2))$. (Over $C((t_1))((t_2))$ there is a unique anisotropic form of dimension 4, namely, $<1, t_1, t_2, t_1 . t_2>$).

d) $C((t_1))((t_2))((t_3))$

e) a p-adic field.

5.2. Non-linked fields

a) **Q** (X), **Q** ((X))

b) **R** (X,Y), **R** ((X)) ((Y)).

c) Q_p (X), $p \neq 2$, the function field over the p-adic field.

d) **C** (X, Y, Z).

5.3. Related properties. Formally real fields.

a) Let K be a linked field. Then any odd dimensional universal form over K is isotropic.

Proof Let q be an odd dimensional universal form of dimension > 4.Then we can write

$$q = \perp <a_i> << x_i , y_i >> \perp r$$

with $r = <a>$, or $r = <a, b, c>$.

For $r = <a>$ it follows that $q \perp <-a> \in I^2K$. Now $p \perp <-a>$ is isotropic and its kernel form has dimension a multiple of 4. This implies that

$$q \perp <-a> = 2 . H + q'$$

and from here we conclude that q is isotropic. The other case $r = <a,b,c>$ is treated similarly.

Remark As far as we know the statement is not known for arbitrary fields.

b) Let us consider the following types of fields

1) linked fields ,

2) SAP fields (i.e. fields whose space of ordering satisfies the strong approximation property).

3) ED fields. Property ED (effective diagonalization) is characterized by any of the following equivalent properties

i) K_{pyt} (Pythagorean closure of K) is SAP

ii) K is SAP and every binary torsion form represents a totally positive element of $K^\cdot$

iii) For every real place $v : K^\cdot \to G$ we have

a) $|G / 2G| \leq 2$ and

b) if $|G / 2G| = 2$ then the residue class field $\bar{K}_v$ of v is an euclidean field.

We have the following (in general strict) hierarchy:

$$\text{Linked} \Rightarrow \text{ED} \Rightarrow \text{SAP}$$

If K is a formally real pythagorean field then these properties are equivalent, [EL1].

The fields Q(X) and R(X,Y) are not SAP fields [Pr], therefore are not linked fields. Moreover $Q(X)_{pyt}$ is an example of a non-linked pythagorean field.

We do not know what additional property on ED gives a reciprocal implication above.

SAP fields also admit valuation theoretical characterizations. In fact a field K satisfies SAP if and only if for every valuation $v : K \to G$, with formally real residue field $\bar{K}_v$ we have $|G / 2G| \leq 2$ and if $|G / 2G| = 2$ then $\bar{K}_v$ is uniquely ordered.

We do not know whether there is an analogous result for real linked fields.

6. Remarks

1. It is not known whether given a linked field K, then every quadratic extension is also linked. Even for K non-real and with $u(K) = 4$, this is not known. In this case, if K is linked and $K(\sqrt{a})$ is a quadratic extension then $u(K(\sqrt{a})) = 4$ or 6. If equal 4 then $K(\sqrt{a})$ is linked.

On the other hand, *odd dimensional* extensions of linked fields need not be linked. For instance, let F_0 be the euclidean closure of Q. Let F be a real odd-dimensional galois extension of F_0. Let $K = F_0(\alpha)$. Then $K((t)) = F_0((t))(\alpha)$ is an odd dimensional (galois) extension of the linked field $F_0((t))$. But it is not linked, because otherwise this would imply that F is euclidean, which is not so, for F has infinite classes of squares, according to [L], Cor.3, p.219.

2. In *Algebraic K-theory and quadratic forms*, Invent. Math. 9,318-344 (1970), J.Milnor defined, for every $n > 0$, the groups knK and morphism h_n, s_n

Moreover, he proved that s_n is epimorphism. Now, Elman-Lam (see Journal of Number Theory 5, 367-378 (1973) proved that for a linked field, sn is an isomorphism, for all n. Consequently e_K^n is always defined, for a linked field K. It is also possible to prove that in this case e_K^n is injective. It is not known whether it is also surjective.

3. For the u-invariant of linked fields see [E], [G].

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