DIFFUSION AND NON LINEAR POPULATION THEORY

CALIXTO P. CALDERON

1. Introduction

Throughout this paper we consider a population evolving in a bounded three dimensional habitat (ocean, rain forest with a height distribution, etcetera).

The function \( u(x_1, x_2, x_3, t) \) will denote the population density at time \( t \) at the point \( x = (x_1, x_2, x_3) \). Our bounded habitat will be denoted by the letter \( G \). Outside \( G \) and defined through the whole space we shall consider the population source per unit of time:

\[
(1.1) \quad f (x_1, x_2, x_3, t).
\]

On \( G \), the population source per unit of time will be given by the expression:

\[
(1.2) \quad c_1(x) \left[ 1 - \beta(x) u \right]
\]

\[
= 0 \quad \text{outside } G.
\]

In the above expression, \( \beta(x) \) stands for a bounded and continuous function defined on \( \mathbb{R}^3 \). The function \( c_1(x) \) is assumed to be continuous. Clearly, (1.2) represents a generalization of the logistic growth.

We may assume a predatorial action per unit of time on our population \( U \), on \( G \), given by the expression:

\[
(1.3) \quad - c_2(x) u^2
\]

\[
= 0 \quad \text{outside } G.
\]

As in the case of \( c_1(x) \), \( c_2(x) \) is a continuous function. Condition (1.3) would indicate that the predatorial action on our population is negligible for small values of the density \( u \). This simply means that the predators switch to alternate preys when the values of \( U \) fall below certain levels.

Finally, we may assume migration in our population \( U \), which is the shift of population from areas of large density to areas of lesser density represented by the scaled laplacian:

\[
(1.4) \quad D_{11} u + D_{22} u + D_{33} u.
\]
In the above expressions $D_u$ denotes the second partial derivative with respect to the variable $x_i$. Likewise, $D_t$ will denote the partial derivative with respect to $x_i$ and $D_t$, that with respect to $t$. The rate of change of the population density $u$ with respect to the time is given by the partial $D_t u$.

The general balance law gives the following equation for the rate of growth of $u$:

$$
(1.5) \quad D_t u - \sum_i D_{x_i} u = c_1 u \left( 1 - \beta (x) u \right) - c_2 u^2 + f.
$$

We may assume that the initial population distribution is known to be:

$$
(1.6) \quad u(x, 0) = g(x).
$$

The aim of this paper is to study the existence of weak global solutions to the initial value problem (1.5), (1.6) in certain $L^p$ classes. Likewise, the paper explores the existence of steady state solutions (solutions independent from $t$) when the source $f$ becomes stable, that is, independent from $t$.

This problem originated in the diffusion equation that governs the spatial patterning of the spruce budworm as studied in [3]. In the case of the Ludwig-Jones-Holling-Aronson-Weinberger equation the term $f$ in (1.5) takes the form:

$$
(1.7) \quad \text{Constant} \cdot u^2 (1 + u^2)^{-1}.
$$

Instead (1.7), I consider here a simpler version of the predatorial action that is not governed by a "logistic" behavior and, as a trade off, one obtains solutions that are not achievable by the "traveling waves" method.

Another important difference is the fact that unlike the setting in [3], this paper presents a three dimensional set up. The reason for that important dimensional difference is the fact that, some times, it is necessary to describe spatial distributions of population densities not only in their surface dispersal, but also in their height or depth variation.

As indicated before, a third dimension is meaningful when describing oceanic distributions of fish populations whose depth range is wide and constitutes a non negligible dimension of the habitat.

Likewise, in the case of the rain forests, the ecological distribution varies with the height range. Many species cover a wide range of altitudes, and in order to describe their interplay is quite natural to consider three dimensional densities.

This paper focuses on the particular problem: "Suppose that the density distribution of a population $U$ is known throughout $\mathbb{R}^3$ and, at an instant $t = 0$, a new bounded habitat $G$ opens up for the species to migrate in. If we assume a logistic growth for the species $U$ as well as a predatorial action within $G$, as described in (1.2) and (1.3), find the density distribution of the population $U$ in $G$ for all time $t > 0$, assuming that the migration is governed by diffusion".

As a simplification, we may consider that the population source outside $G$ is given by the function $f$ as in (1.1) and furthermore, we shall assume

$$
(1.8) \quad f = 0, x \in G, t > 0.
$$

Within $G$, we shall assume the growth of the density $u$ per unit of time as described in (1.2) and the predatorial action as described in (1.3).

The values of $u$ at $t = 0$ will be given by $u_0(x)$. Obviously, we must have:
As a simplification, we neglect the description of any natural barrier beyond \( G \), assuming that any diffusion of the biomass toward infinity can be interpreted as a loss due to inhospitable subhabitats. At any rate, the distortion caused by the "diffusion toward infinity" can be compensated by the selection of an appropriate source function \( f \).

Finally, the problem can be set up as in (1.5) and (1.6) by making the appropriate selection of \( f \) and \( g \) as in (1.8) and (1.9).

**Classes Of Functions And Statement Of The Main Result.**

\( E(x,t) \) will denote the fundamental solution of the heat equation in \( \mathbb{R}^3 \), namely:

\[
(1.10) \quad E(x,t) = \left(4 \pi t\right)^{-3/2} \exp \left\{ - \frac{1}{4t} \right\} \quad \text{where} \quad \left| x \right| = \left( x_1^2 + x_2^2 + x_3^2 \right)^{1/2}.
\]

\( L(u) \) will denote the Heat differential operator applied to \( u \), (left hand side of equation (1.5)). Hence, the equation (1.5) can be written as:

\[
(1.11) \quad L(u) = c_1 u (1 - \beta u) - c_2 u^2 + f.
\]

\( E(v) \) will denote the convolution:

\[
(1.12) \quad E(v) = \int_0^t \int_{\mathbb{R}^3} E(x-y, t-s) v(y,s) \, dy \, ds.
\]

\( W(g) \) will denote the convolution on the spatial variables:

\[
(1.13) \quad \int_{\mathbb{R}^3} E(x-y, t) g(y) \, dy.
\]

The equation (1.10) is going to be rewritten as:

\[
(1.14) \quad L(u) = -a^2 u^2 + b^3 u + f
\]

\[
\text{where} \quad a^2 = c_1 b + c_2, \quad b = (c_3)^{1/3}, \quad c_1 \geq 0, c_2 \geq 0, \quad \beta \geq 0.
\]

Solving the equation (1.5) with initial data (1.6), if one assumes enough regularity on \( u, a, \) \( b, g \) and \( f \), is equivalent with solving the integral equation:

\[
(1.15) \quad u = E(-a^2 u^2 + b^3 u + f) + W(g).
\]

We shall call any solution \( u \) for all \( t > 0 \) of (1.15) a weak global solution of the equation (1.5) with initial data (1.6) whenever the integrals that are involved exist in the Lebesgue sense for all value \( t > 0 \).

Since the local behavior of solutions of the problem (1.5), (1.6) have no biological meaning, we will consider in our discussion only properties of weak solutions that are global in nature.

\( \| g \|_p \) will denote the usual \( L^p \) norm of the function \( g \) in \( \mathbb{R}^3 \).
\( \| \mathbf{v} \|^*_p \) will denote the \( L^p \) norm in \( \mathbb{R}^3 \) of the function:

\[
(1.16) \quad \mathbf{v}^*(x) = \sup_{t > 0} |\mathbf{v}(x, t)|.
\]

The main result of this paper is contained in the following:

**Theorem A.**

There exist two small constants \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) such that whenever

\[
(1.17) \quad \| W(g) \|_{\mathcal{H}_2} + \| f \|_{\mathcal{H}_8} < \varepsilon_0,
\]

\[
(\| b \|_{\mathcal{H}_2})^3 < \delta_0.
\]

The problem (1.5) with initial data (1.6) possesses a global weak solution \( u(x,t) \) that satisfies:

i) \( \| u \|_{\mathcal{H}_2} < C_0 \).

Here, \( C_0 \) depends on \( \varepsilon_0 \) and on \( \delta_0 \).

Concerning steady state solutions, if the source function \( f \) does not depend on \( t \) and satisfies

\[
(1.18) \quad \| f \|_{\mathcal{H}_8} < \varepsilon_1
\]

and \( b \) satisfies

\[
(\| b \|_{\mathcal{H}_2})^3 < \delta_1.
\]

Here \( \varepsilon_1 \) has the same meaning as \( \varepsilon_0 \) above, although its numerical value is not necessarily the same. The same for \( \delta_1 \).

Then, there exists a steady state solution \( u \) that satisfies:

ii) \( \| u \|_{\mathcal{H}_2} < C_1 \).

Here, \( C_1 \) depends on \( \varepsilon_1 \).

2. Proof of Theorem A.

A Potential Inequality.

**Lemma 1.**

Let \( T(v_1, v_2, v_3, v_4) \) be the multilinear operator defined by:

\[
(2.1) \quad \|x\|^{-1} \ast v_1 \ast v_2 \ast v_3 \ast v_4
\]

Here, the functions \( v_i \) are measurable and defined on \( \mathbb{R}^3 \), \( \|x\| \) is the distance from the origin, and \( \ast \) is the convolution symbol. Then:

i) \( \| T \|_{\mathcal{H}_2} \leq C \| v_1 \|_{\mathcal{H}_2} \| v_2 \|_{\mathcal{H}_2} \| v_3 \|_{\mathcal{H}_8} \| v_4 \|_{\mathcal{H}_8} \)
Proof.
The Hardy-Littlewood-Sobolev inequality gives, see ref [5] p.119:

\[(2.2) \| T \|_q < C_p \| v_1 \cdot v_2 \cdot v_3 \cdot v_4 \|_p 1/q = 1/p - 2/3 .\]

Take \( p = 9/8 \) and apply Hölder's inequality to the right hand side of (2.2) above, \( p_i = 9/2 \), \( i = 1, 2, 3, 4 \).

**Lemma 2.**
Consider the convolution \( E(v) \), where \( v = v(x, t) \) is a measurable function in \( \mathbb{R}^3 \times \mathbb{R}_+ \). If \( v^* \) belongs to some \( L^p \) class, we have:

i) \( |E(v)| \leq C_0 |x|^{-1} \cdot v^* \).

**Proof.**
The above estimate is a consequence of:

\[(2.3) \| E(x, t) \| \leq C(|x| + t^{1/2})^{-3} .\]

Taking in the convolution \( E(v) \) the integral with respect to the time as the inner integral and using the estimate (1.16) and (2.3), we obtain i) for the particular value of \( C_0 \)

\[(2.4) \quad C \int_0^\infty (1 + t^{1/2})^{-3} dt .\]

This observation concludes the proof.

**Estimates for the integral equation.**
Calling \( F = E(f) \) and observing that as a consequence of Lemma 2 and the Hardy-Littlewood-Sobolev potential inequality we have:

\[(2.5) \| F \|_{L^2} \leq C \| f \|_{L^2} .\]

Denoting by \( \| \| \) the norm \( \| (\cdot^* \|_{L^2} \), we obtain for the operator:

\[(2.6) \quad T(u,v) = F(-a^2 u v + b^3 u + f) + W(g)\]

the estimate:

\[(2.7) \quad \| T(u,v) \| \leq C \{ \| a \| \| u \| \| v \| \| b \| \| u \| + \| F \| + \| W(g) \| \} \]

which is a consequence of lemmata 1 and 2.

On the other hand, the integral equation can be written as:

\[(2.8) \quad u = T(u, u) .\]

**Lemma 3.**
Let \( T(u,v) \) be a general operator of the type (2.6), mapping the cartesian product \( X \times X \)
into \( X \), where \( X \) denotes a Banach space, such that:
Suppose that $C_1, C_2$ and $\| F \|$ satisfy:

$$\| T(u, v) \| \leq C_1 \| u \| \| v \| + C_2 \| u \| + \| F \|$$

Then, the quadratic operator $T(u, u)$ maps the ball $\{ \| u \| \leq s_1 \}$ into itself if $s_1$ is the smallest root of the equation:

$$C_1 s_1^2 + (C_2 - 1) s + \| F \| = 0.$$ 

If $2 s_1 C_1 + C_2 < 1$, $T(u, u)$ is a contraction mapping in the ball of radius $s_1$. Finally, we have for $s_1$, the estimates:

$$0 < s_1 < \| F \| \left( (1 - C_2)^2 - 4 C_1 \| F \| \right)^{-1/2}.$$ 

For the proof of this lemma I refer the reader to ref [4] (lemma (2.2) there).

Lemma 3 applied to the (2.8) gives immediately part i) of theorem A.

Part ii) follows by using the same arguments as in i), and it reduces to solving the integral equation:

$$u = C |x|^\alpha * (-a^2 u^2 + b^3 u + f).$$

Here, the norms we use are the usual $L^{9/2}(\mathbb{R}^3)$ and $L^{9/8}(\mathbb{R}^3)$ norms. The symbol $*$ denotes convolution on the spatial variables. This concludes the proof of Theorem A.

Final remarks.

If the initial density $g$ of the population $U$ as well as the source function remain below certain levels, the population in $G$ will not experience an outbreak, even if one suppresses the predatorial action.

From the method that we have employed, it follows that the steady state solution is stable, this however will not be done here.

References.


