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GENERALIZED COMPLEX BESSEL TRANSFORMATIONS

J. J. BETANCOR

ABSTRACT. In this paper we consider two variants of the Hankel transform (they are called Bessel-transformations) which are defined by:

$$B_\mu(f)(y) = \int_0^\infty x^{2\mu+1} b_\mu(xy) f(x)\,dx$$

$$_tB_\mu(f)(y) = y^{2\mu+1} \int_0^\infty b_\mu(xy) f(x)\,dx$$

where $b_\mu(z) = z^{-\mu} J_\mu(z)$, $J_\mu(z)$ being the Bessel function of the first kind of order μ. We extend these transformations to certain spaces of generalized functions and prove several results on inversion, uniqueness, boundedness and analyticity. The theory developed here is applied to solve certain Dirichlet problems.

1. INTRODUCTION

After L.Schwartz [10] extended the Fourier transform to certain spaces of distributions (distributions of slow growth), the extension of classical integral transforms to generalized functions constitutes an interesting and active area of study.

There are two main approaches used to extend the classical transform associated to the Kernel K(x,y)

$$(1.1) \qquad F(y) = T(f)(y) = \int_{-\infty}^{\infty} K(x,y)f(x)dx$$

to generalized functions. The first method consists in constructing a space A of testing functions defined on $(-\infty,+\infty)$, which is closed with respect to the classical transform (1.1). The transformation of the functional $f \in A'$ (where A' stands for the dual space) is defined as the adjoint transformation of T:

$$(Tf)(\psi) = f(T\psi) \quad , \quad f \in A' \quad , \quad \psi \in A.$$

This approach has been followed by L.Schwartz [10], A.H.Zemanian [12], J.M.Mêndez [5] and G.Altenburg [1].

In the second procedure a space B of testing functions is constructed such that the kernel function $K(x,y)$ is in B for each real or complex y and then the transform Tf of the generalized function f is defined by the relation

$$(Tf)(y) = f(K(x,y))$$

This method has been followed by A.H.Zemanian [13], E.L.Koh and A.H.Zemanian [3], L.S.Dube and J.N.Pandey [2], and others.

The Bessel transform

$$(1.2) \qquad B_\mu(f)(y) = \int_0^{\infty} x^{2\mu+1} b_\mu(xy)f(x)dx$$

where $b_\mu(z) = z^{-\mu}J_\mu(z)$, $J_\mu(z)$ being the Bessel function of the first kind of order μ, has been extended to certain spaces of generalized functions by using both approaches. Following the second one, L.S.Dube and J.N.Pandey [2] gave an extension of the Bessel transform; they proved an inversion theorem for a certain class of generalized functions interpreting convergence in the weak distributional sense. Later, W.Y.Lee [4] gave two spaces of testing functions F_μ and G_μ, and proved that B_μ is a continuous imbedding of F_μ in G_μ. Then the dual operator B'_μ defined by

$$(B'_\mu f)(\psi) = f(B_\mu \psi)$$

is a continuous map of G'_μ into F'_μ. More recently, G.Altenburg

[1] and J.M.Méndez [5] investigated the transform (1.1) on certain space H' of generalized functions of slow growth. They used the first of the above mentioned methods.

The purpose of this paper is to extend two different Bessel transforms, namely (1.2), introduced by A.L.Schwartz [9]; and

$$ {}_tB_\mu(f)(y) = y^{2\mu+1}\int_0^\infty b_\mu(xy)f(x)dx $$

studied by J.M.Méndez [6], to a space of generalized functions by using the second procedure. Our study has been suggested by a paper of E.L.Koh and A.H.Zemanian [3] on the complex Hankel transformation and also a work of W.Y.Lee [4], where the extension that we carry out here is presented by him as an open problem. As in [3], the independent variable y is allowed to be a complex, and the Bessel transforms defined turn out to be analytic on a certain strip Ω, in contrast with the restriction in [2] and [1]. We give several theorems on boundedness, inversion and uniqueness, together with an operational transform formula for a Bessel type differential operator. In the last paragraph we study some applications of the transformations in the solution of certain Dirichlet problems.

2. THE TESTING FUNCTION SPACES $I_{\mu,a}$, $I_\mu(\sigma)$ AND THEIR DUAL SPACES

Let I be the interval $(0,\infty)$, a be a positive real number and μ any real number. Then, for each pair (a,μ) we define $I_{\mu,a}$, as the collection of the infinitely differentiable complex valued functions ψ on I such that the next inequality holds

$$ \eta^m_{\mu,a}(\psi) = \sup_{x\in I}|e^{-ax}\, x^{\max(0,\mu)+\frac{1}{2}}\, \Delta^m_\mu\psi(x)| < \infty $$

for $m \in N$, where $\Delta_{\mu,x} = x^{-2\mu-1}Dx^{2\mu+1}D$.

We assign to $I_{\mu,a}$ the topology generated by the countable multinorm $\{\eta^m_{\mu,a}\}$. Hence, $I_{\mu,a}$ is a countable multinormed space.

The dual space $I'_{\mu,a}$ consists of all continuous linear functionals on $I_{\mu,a}$. The dual is a linear space endowed with the weak topology.

We now present some properties of these spaces.

Property 1: $I_{\mu,a}$ is a sequentially complete topological vector space. $I_{\mu,a}$ is a Frêchet space.

Property 2: $I_{\mu,a} \subset E(I)$, for every choice of μ and a, and the topology of $I_{\mu,a}$ is stronger than the one induced on it by $E(I)$. Hence, the restriction of any $f \in E'(I)$ to $I_{\mu,a}$ is in $I'_{\mu,a}$, and the convergence in $E'(I)$ implies weak convergence in $I'_{\mu,a}$.

Property 3: $D(I)$ is contained in $I_{\mu,a}$, the inclusion being continuous.

Property 4: If $0 < a < b$, then $I_{\mu,a} \subset I_{\mu,b}$ and the topology of $I_{\mu,a}$ is stronger than that induced on it by $I_{\mu,b}$.

Property 5: The operation $\psi \longrightarrow \Delta_\mu \psi$ is a continuous linear mapping of $I_{\mu,a}$ into itself. The operator

$$\begin{aligned} \Delta'_\mu : I'_{\mu,a} &\longrightarrow I'_{\mu,a} \\ f &\longrightarrow \Delta'_\mu f : I_{\mu,a} \longrightarrow C \\ &\qquad\qquad \psi \longrightarrow (\Delta'_\mu f)(\psi) = f(\Delta_\mu \psi) \end{aligned}$$

is a continuous linear mapping of $I'_{\mu,a}$ into itself.

Property 6: Let f be a locally integrable function defined for $x > 0$ and satisfying

$$\int_0^\infty |f(x)| \; e^{ax} \; x^{-\max(0,\mu)-\frac{1}{2}} \, dx < \infty.$$

Then f generates a regular generalized function in $I'_{\mu,a}$.

Property 7: For each $f \in I'_{\mu,a}$, there exists a nonnegative integer r and a positive constant C such that, for all $\psi \in I_{\mu,a}$,

$$|f(\psi)| \leqslant C \max_{0\leq k\leq r} \eta^{m}_{\mu,a}(\psi).$$

In view of property 4, we can define the following countable union space. Let $\{a_n\}_{n\in N}$ be a monotonically increasing sequence of positive numbers tending to σ (possibly $\sigma=+\infty$). Then we introduce the union space

$$I_\mu(\sigma) = \bigcup_{1}^{\infty} I_{\mu,a_n}$$

which is equiped with the usual topology.

Property 8: The mappings

$$\Delta_\mu: I_\mu(\sigma) \longrightarrow I_\mu(\sigma) \quad , \quad \Delta'_\mu: I'_\mu(\sigma) \longrightarrow I'_\mu(\sigma)$$

are continuous and linear.

We consider the function $b_\mu(z)$. The following Lemma (E.L.Koh and A.H.Zemanian [3]) will be useful in the sequel.

LEMMA 1. *Let* a *be a fixed real number such that* $0 < a < \infty$. *One then has*

$$|e^{-ax}\, b_\mu(xy)| < A_\mu$$

for every y *in the region* $\Omega_a = \{y \in \mathbf{C}: |\mathrm{Im}\ y| < a,\ \ y \notin (-\infty,0]\}$, *for* $0 < x < \infty$, *and for* $\mu \geqslant -\frac{1}{2}$, *where* A_μ *does not depend on* x *and* y.

An application of this Lemma leads to:

Property 9: $$\frac{\partial^m}{\partial y^m}\{y^{2\mu+1} b_\mu(xy)\} \in I_{\mu,a}$$

if $\mu \geqslant -\frac{1}{2}$, for every y in the complex region Ω_a, $x \in (0,\infty)$ and $m \in N$.

The proof follows from the equality

$$\Delta^m_{\mu,x}\{y^{2\mu+1}b_\mu(xy)\} = (-1)^m y^{2\mu+1+2m}b_\mu(xy) \quad \text{for} \quad m \in N.$$

G.Altenburg [1] and J.M.Méndez [5] introduce as a space H of testing functions, the space of the infinitely differentiable complex valued functions ψ defined on I and such that

$$\gamma_{m,n}(\psi) = \sup_{x\in I} |x^m(\frac{1}{x}D)^n\psi(x)| < \infty$$

for each pair of nonnegative integers m and n.

The classical Bessel transform B_μ is an automorphism in H. This allowed both authors to define the generalized transform B'_μ in the dual space of H, H', as the adjoint of the classical transform, namely:

$$(B'_\mu f)(\psi) = f(B_\mu\psi) \quad , \quad \text{for } f \in H' \text{ and } \psi \in H.$$

Providing $\Delta^m_\mu\psi(x) = \sum_{j=0}^{m} b_j x^{2j}(\frac{1}{x}D)^{j+m}\psi(x)$, for every $m \in N$, and

$y^{2\mu+1}b_\mu(xy) \in I_{\mu,a}\setminus H$, for each $\mu \geqslant -\frac{1}{2}$, $a > 0$, one has:

Property 10: $H \subsetneq I_{\mu,a}$, for every choice of $a > 0$, and the topology of H is stronger than the one induced on it by $I_{\mu,a}$. Also $H \subsetneq I_\mu(\sigma)$ and the convergence in H implies the convergence in $I_\mu(\sigma)$.

3. A GENERALIZED BESSEL TRANSFORM B'_μ

Let μ be a real number such that $\mu \geqslant -\frac{1}{2}$. According to the property 4, if $f \in I'_{\mu,a}$ for some real number a, then there exists a real number σ_f (possibly $\sigma_f=+\infty$) such that $f \in I'_{\mu,b}$ for every $b < \sigma_f$ and $f \notin I'_{\mu,b}$ for every $b > \sigma_f$.

Since $y^{2\mu+1}b_\mu(xy) \in I_{\mu,a}$ for every fixed y such that $y \notin (-\infty,0]$, $|\text{Im } y| < \sigma_f$ it is possible to define the generalized μ-th order B'_μ transform of f by:

(3.1) $$F(y) = (B'_\mu f)(y) = f(y^{2\mu+1} b_\mu(xy)), \quad y \in \Omega_f$$

where $\Omega_f = \{y \in \mathbf{C} / |\operatorname{Im} y| < \sigma_f \ , \ y \notin (-\infty, 0]\}$.

The region Ω_f will be called the region of definition for (3.1). However, if f(x) generates a regular generalized function f in $I'_{\mu,a}$, then:

$$(B'_\mu f)(y) = f(y^{2\mu+1} b_\mu(xy)) = y^{2\mu+1} \int_0^\infty b_\mu(xy) f(x) dx =$$

$$= y^{2\mu+1} B_\mu(x^{-2\mu-1} f(x))(y) \quad , \quad y \in \Omega_f .$$

REMARK 1. The definition given by L.S.Dube and J.N.Pandey [2] is essentially different to (3.1); the first one implies that $B'_\mu = B_\mu$ on the locally integrable functions generating regular distributions. In the paragraph 4 we analyze the definition given in [2].

A B'_μ-transform is analytic on its region of definition. We prove this statement in the following

THEOREM 1. *Let* $F(y) = (B'_\mu f)(y)$ *for* $y \in \Omega_f$. *Then,* $F(y)$ *is an analytic function on* Ω_f *and*

$$D^n_y F(y) = f\left(\frac{\partial^n}{\partial y^n}\{y^{2\mu+1} b_\mu(xy)\}\right) \ , \ \textit{for } n \in \mathbb{N}.$$

Proof. Let y be an arbitrary fixed point in Ω_f. Choose a real positive number a such that $y \in \Omega_a \subset \Omega_f$. Let C and C_1 denote two circles with radius r and r_1 respectively such that $r < r_1$. These circles lie completely within Ω_f. Finally, let Δy be a nonzero complex increment such that $|\Delta y| < r$. Now consider the expression:

$$\frac{F(y+\Delta y) - F(y)}{\Delta y} - f\left(\frac{\partial}{\partial y}\{y^{2\mu+1} b_\mu(xy)\}\right) = f(\psi_{\Delta y})$$

where

$$\psi_{\Delta y}(x) = \frac{(y+\Delta y)^{2\mu+1} b_\mu(x(y+\Delta y)) - y^{2\mu+1} b_\mu(xy)}{\Delta y} - \frac{\partial}{\partial y} y^{2\mu+1} b_\mu(xy)$$

By using Cauchy integral formula and interchanging $\frac{\partial}{\partial y}$ with Δ_μ^m, we may write:

$$\Delta_\mu^m \psi_{\Delta y}(x) = \frac{(-1)^m}{2\pi i} \int_{C_1} \frac{\Delta y}{(\eta-y-\Delta y)(\eta-y)^2} \eta^{2m+2\mu+1} b_\mu(xy)\,d\eta$$

Hence, by Lemma 1, we have:

$$\sup_{x\in I} |e^{-ax} x^{\frac{1}{2}+\max(0,\mu)} \Delta_\mu^m \psi_{\Delta y}(x)| < H.\Delta y \longrightarrow 0 \text{ , as } \Delta y \longrightarrow 0.$$

This proves that $\psi_{\Delta y}(x)$ converges to the null function in $I_{\mu,a}$ as $\Delta y \longrightarrow 0$. Since $f \in I'_{\mu,a}$, it follows that

$$\lim_{y\to 0} \frac{F(y+\Delta y)-F(y)}{\Delta y} = f(\frac{\partial}{\partial y} \{y^{2\mu+1} b_\mu(xy)\}).$$

Reasoning in nearly the same way as before we can prove the preceding statement for $n > 1$.

The following result will be helpful.

THEOREM 2. $F(y)$ *is bounded in every region of the form*

$$\Omega_a = \{y \in \mathbf{C}: |\text{Im } y| < a < \sigma_f \text{ and } y \notin (-\infty,0]\}.$$

More precisely, $|F(y)| < |y|^{2\mu+1} P_a(|y|^2)$, *where* P_a *is a polynomial depending on* a.

The proof follows from Property 5 and Lemma 1.

Moreover, $F(y)$ satisfies the inequality:

$$|F(y)| \leqslant \begin{cases} C|y|^{2\mu+1} & \text{, for } 0 < |y| < 1 \\ C|y|^{2\mu+1+2r} & \text{, for } |y| < 1 \end{cases}$$

when r is a sufficiently large natural number and C a positive constant appropiately chosen.

If $f \in I'_\mu(\sigma)$, then $f \in H'$; therefore we can define the generalized transform of f in two different ways: as an element of $I'_\mu(\sigma)$, by (3.1); and as an element of H', according to G.Altenburg [1] and J.M.Mêndez [5]. The following theorem shows the equivalence of both definitions.

THEOREM 3. *Let* f *be a member of* $I'_\mu(\sigma)$ *and* $\mu \geqslant -\frac{1}{2}$. *If* $\psi \in H$, *then*

$$F(\psi) = f(B_\mu(\psi))$$

where $F(y) = f(y^{2\mu+1} b_\mu(xy))$.

Proof. Let y be a positive real number. By virtue of Theorem 2, F(y) defines a regular generalized function in H', as follows

$$F(\psi) = \int_0^\infty f(y)\psi(y)dy = \int_0^\infty f(y^{2\mu+1} b_\mu(xy)\psi(y))dy$$

for $\psi \in H$.

Moreover,

$$\int_0^\infty f(y^{2\mu+1} b_\mu(xy)\psi(y))dy = f(\int_0^\infty y^{2\mu+1} b_\mu(xy)\psi(y)dy)$$

as can be proved by using the techniques of Riemann sums and the proof is completed.

REMARK 2. As we already pointed out, our definition (3.1) is different from the corresponding one of L.S.Dube and J.N.Pandey in [2]. For instance, the property described in Theorem 3 can't be proved if we adopt the definition given in [2].

Now we give an inversion theorem.

THEOREM 4. *Let* $f \in I'_\mu(\sigma)$ *and let* F(y) *be the* B'_μ *transform of* f. *Then*

$$(\int_0^N F(y) b_\mu(xy) x^{2\mu+1} dy)(\psi) \longrightarrow f(\psi)$$

for each $\psi \in D(I)$, *as* $N \to \infty$.

Proof. The function

$$G_r(x) = \int_0^r F(y)x^{2\mu+1}b_\mu(xy)dy$$

for each $r > 0$ and if $\mu \geqslant -\frac{1}{2}$, generates a regular distribution in $D'(I)$ defined by

$$G_r(\psi) = \int_0^r F(y)\xi(y)dy$$

where $\xi(y) = B_\mu(\psi)$. Moreover, by using the technique of Riemann sums, we obtain

$$G_r(\psi) = f(\int_0^r y^{2\mu+1}b_\mu(xy)\xi(y)dy).$$

We now consider the function:

$$G_N(t,x) = (xt)^{2\mu+1}\int_0^N y^{2\mu+1}b_\mu(ty)b_\mu(xy)dy =$$

$$= \frac{N^{2\mu+2}}{t^2-x^2}(tx)^{2\mu+1}(t^2b_{\mu+1}(tN)b_\mu(xN)-x^2b_\mu(tN)b_{\mu+1}(xN))$$

For $0 < a < b$ we have

$$\lim_{N\to\infty}\int_a^b G_N(t,x)x^{-2\mu-1}dx = \begin{cases} 1, & t \in (a,b) \\ \frac{1}{2}, & t=a \quad \text{or} \quad t=b \\ 0, & t \notin [a,b]. \end{cases}$$

In view that for each $\psi \in D(I)$ with support contained in $[a,b]$ and $m \in N$:

$$\Delta_\mu^m\{t^{-2\mu-1}\int_a^b G_N(t,x)\psi(x)dx - \psi(t)\} =$$

$$= t^{-2\mu-1}\int_a^b x^{-2\mu-1}G_N(t,x)(\psi_m(x)-\psi_m(t))dx$$

where $\psi_m(x) = (Dx^{2\mu+1}Dx^{-2\mu-1})^m(x^{2\mu+1}\psi(x))$, a method analogous to the employed by L.S.Dube and J.N.Pandey [2] leads us to:

$$\lim_{r\to\infty} \int_0^r y^{2\mu+1} b_\mu(ty)\xi(y)dy = \psi(t)$$

in the sense of $I_{\mu,c}$, for $0 < c < \sigma$. This proves the theorem.

An immediate consequence of Theorem 4 is the following weak version of a uniqueness theorem:

THEOREM 5. *Let* $F(y) = (B'_\mu f)(y)$ *for* $y \in \Omega_f$, *let* $G(y) = (B'_\mu g)(y)$ *for* $y \in \Omega_g$, *and assume that* $F(y) = G(y)$ *for* $y \in \Omega_f \cap \Omega_g$. *Then* $f=g$, *in the sense of equality in* $D'(I)$.

4. THE GENERALIZED TRANSFORM ${}_tB'_\mu$

J.M.Mêndez [6] defined a variant of the Bessel transform

$${}_tB_\mu(f)(y) = y^{2\mu+1}\int_0^\infty b_\mu(xy)f(x)dx$$

stating its inversion formula.

J.M.Mêndez [7] also extends this transformation to a space of generalized functions ${}_tH'_\mu$, dual of the space of testing functions

$${}_tH_\mu = \{\psi \in C^\infty(I) : \gamma^\mu_{m,n}(\psi) = \sup_{x\in I} |x^m(\tfrac{1}{x} D)^n(x^{-2\mu-1}\psi(x))| < \infty,\ m,n \in N\}$$

The classical transform ${}_tB_\mu$ is an automorphism in ${}_tH_\mu$. The generalized transformation ${}_tB'_\mu$ in ${}_tH'_\mu$ is defined as the adjoint of the classical transform:

$$({}_tB'_\mu f)(\psi) = f({}_tB_\mu\psi) \ , \quad \text{for } f \in {}_tH'_\mu \text{ and } \psi \in {}_tH_\mu.$$

In this paragraph we introduce a complex generalized transform analogous to that given by L.S.Dube and J.N.Pandey [2], although these authors considered only a real transformation.

For each positive real number a and for each real number μ we define the function space

$$ {}_tI_{\mu,a} = \{\psi \in C^\infty(I) : {}_t\eta^m_{\mu,a}(\psi) = \sup_{x\in I} |e^{-ax} x^{-2\mu-1+\max(0,\mu)} {}_t\Delta^m_\mu \psi(x)|\} $$

$m \in N$.

where ${}_t\Delta_\mu = Dx^{2\mu+1}Dx^{-2\mu-1}$.

Properties similar to those listed in Section 2 can be proved for the space ${}_tI_{\mu,a}$ and its dual, ${}_tI'_{\mu,a}$. Hence, ${}_tI_{\mu,a}$ satisfies the following chain of topological inclusions:

$$ D(I) \subset {}_tH_\mu \subset {}_tI_{\mu,a} \subset E(I) $$

Moreover for every $y \in \Omega_a = \{y \in \mathbf{C} : |\mathrm{Im}\, y| < a,\ y \notin (-\infty, 0]\}$ $\mu \geq -\frac{1}{2}$, $m \in N$, the function

$$ \frac{\partial^m}{\partial y^m} \{x^{2\mu+1} b_\mu(xy)\} $$

is a member of ${}_tI_{\mu,a}$.

Also, if $\{a_\nu\}_{\nu\in N}$ is a monotonically increasing sequence of positive numbers tending to σ (possibly $\sigma=+\infty$), the countable union space can be defined

$$ {}_tI_\mu(\sigma) = \bigcup_1^\infty {}_tI_{\mu,a_\nu} $$

which is equiped with the usual topology.

It can easily be seen that if $f \in {}_tI'_{\mu,a}$ for some $a > 0$, then there exists a positive real number σ_f such that $f \notin {}_tI'_{\mu,b}$ for every $b > \sigma_f$ and $f \in {}_tI'_{\mu,b}$, for every $b < \sigma_f$; hence, $f \in {}_tI'_\mu(\sigma_f)$. We define ${}_tB'_\mu f$, the generalized transform of f, as follows:

$$ F(y) = ({}_tB'_\mu f)(y) = f(x^{2\mu+1} b_\mu(xy)) $$

if $\mu \geq -\frac{1}{2}$ and y is in the region $\Omega_f = \{y \in \mathbf{C} : |\mathrm{Im}\, y| < \sigma_f,\ y \notin (-\infty, 0]\}$.

The proofs of the following theorems are very similar to those in the preceding paragraph and therefore are omitted.

THEOREM 6. *Let* $F(y) = ({}_tB'_\mu f)(y)$ *for* $y \in \Omega_f$. *Then,* $F(y)$ *is an analytic function on* Ω_f, *and:*

$$D_y^n F(y) = f(\frac{\partial^n}{\partial y^n}\{x^{2\mu+1} b_\mu(xy)\}) \quad , \quad n \in N \quad , \quad y \in \Omega_f .$$

THEOREM 7. $F(y)$ *is bounded on any region* Ω_a, *according to*

$$|F(y)| < P_a(|y|^2)$$

where P_a *is a polynomial depending on* a *and* Ω_a *denotes the same region as in former sections.*

THEOREM 8. *Let* f *be a member of* ${}_tI'_\mu(\sigma)$ *and* $\mu \geqslant -\frac{1}{2}$. *If* $\psi \in {}_tH_\mu$, *then:*

$$F(\psi) = f({}_tB_\mu \psi)$$

where $F(y) = f(x^{2\mu+1} b_\mu(xy))$.

THEOREM 9. *Let* $f \in I'_\mu(\sigma)$ *and let* $F(y)$ *be the* ${}_tB'_\mu$*-transform of* f. *Then*

$$\lim_{N\to\infty} (\int_0^N F(y) b_\mu(xy) x^{2\mu+1} dy)(\psi) = f(\psi)$$

for each $\psi \in D(I)$.

THEOREM 10. *Let* $F(y) = ({}_tB'_\mu f)(y)$ *for* $y \in \Omega_f$, *let* $G(y) = ({}_tB'_\mu g)(y)$ *for* $y \in \Omega_g$, *and assume that* $F(y) = G(y)$ *for* $y \in \Omega_f \cap \Omega_g$. *Then,* f=g, *in the sense of equality in* $D'(I)$.

5. APPLICATIONS

In the solution of the differential equations of the kind

$$P(B)u = g$$

where $B = \Delta'_\mu$ or $B = {}_t\Delta'_\mu$, the dual operator of ${}_t\Delta_\mu$, P is a polynomial,u and g are transformable functions, the following operational rule is of interest.

THEOREM 11. *Let* P *be a polynomial, then*

$$(B'_\mu P(\Delta'_\mu)u)(y) = P(-y^2)(B'_\mu u)(y) \quad , \textit{for every } u \in I'_\mu(\sigma)$$

$$({}_tB'_\mu P({}_t\Delta'_\mu)u)(y) = P(-y^2)({}_tB'_\mu u)(y) \quad , \textit{for every } u \in {}_tI'_\mu(\sigma).$$

These results are inferred without difficulty from the equality $\Delta_{\mu,x} b_\mu(xy) = -y^2 b_\mu(xy)$.

The integral transforms defined in this paper are useful in the solution of some Dirichlet problems which will be now discussed.

Suppose that we wish to find a function v(r,z) defined in the region $\{(r,z): r > 0, z > 0\}$ and satisfying the differential equation

$$\frac{\partial^2 v(r,z)}{\partial z^2} + \Delta_{\mu,r} v(r,z) = 0 \qquad (5.1)$$

with $\mu \geqslant -\frac{1}{2}$, and the following boundary conditions:

(a) v(r,z) converges to the generalized function $f \in {}_tI'_\mu(\sigma)$ in the sense of convergence in D'(I), as $z \to 0^+$.

(b) $v(r,z) \to 0$, as $z \to \infty$, uniformly on $r \in (0,\infty)$.

(c) $r^\mu v(r,z) = o(r^{-\frac{1}{2}})$, as $r \to \infty$.

(d) v(r,z) is bounded on $r \in (0,\infty)$, for $z > 0$.

A formal application of the B_μ-transform boundary conditions (a) and (b) allow to obtain

$$v(r,z) = \int_0^\infty \rho^{2\mu+1} b_\mu(x\rho) e^{-\rho z} f(x^{2\mu+1}\, b_\mu(x\rho))d\rho$$

This function satisfies (5.1) and (a)-(d), and is a solution of the problem. In the proof of (a) we have to use Lebesgue's dominated convergence theorem and (c) follows as an application of Riemann-Lebesgue Lemma.

We can tackle in a similar way applying the ${}_tB_\mu$-transform the following Dirichlet problem. The following differential equation is considered

$$\frac{\partial^2 v(r,z)}{\partial z^2} + {}_t\Delta_{\mu,r} v(r,z) = 0 \quad , \quad \text{for} \quad \mu \geqslant -\frac{1}{2}$$

with the conditions:

(a) $v(r,z)$ converges to the generalized function $f \in I'_\mu(\sigma)$ in the sense of the convergence in $D'(I)$.

(b) if $\mu = -\frac{1}{2}$, $v(r,z) \to 0$, as $z \to \infty$, uniformly on $r \in (0,\infty)$

if $\mu > -\frac{1}{2}$, $v(r,z) \to 0$, as $z \to \infty$, uniformly on $r \in (0,c)$, for $c > 0$.

(c) $r^{-\mu-1} v(r,z) = o(r^{-\frac{1}{2}})$, as $r \to \infty$.

(d) if $\mu = -\frac{1}{2}$, $v(r,z)$ is bounded on $0 < r < \infty$, for $z > 0$,

if $\mu > -\frac{1}{2}$, $v(r,z) \to 0$, as $r \to 0$, for $z > 0$.

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Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de La Laguna
La Laguna (Tenerife)
Canary Isles-Spain.

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